2. The nucleon spin decomposition of QCD

2.1. Introduction to the problem

Although one might think it a little “academic problem”, to get a complete decomposition of nucleon spin is a fundamentally important task of QCD.

In fact, if our research ends up without accomplishing this task, a tremendous efforts since the first discovery of nucleon spin crisis would go up in smoke.

Unfortunately, this is an extremely delicate problem, which has rejected a clear answer for more than 20 years since the first seminal the paper by

- R.L. Jaffe and A.V. Manohar, Nucl. Phys. B337, 509 (1990).

Recently, two reviews appeared to overview controversial status of the problem :

- E. Leader and C. Lorcé, Phys. Rept. 541, 163 (2014) [arXiv : 1309.4235].
- M. Wakamatsu, Int. J. Mod. Phys. A29, 1430012 (2014) [arXiv:1402.4193].

Remember first the **Ji sum rule**, which gives $J^q + J^G = 1/2$ with

$$J^q = \frac{1}{2}\int x\,[\,H^q(x,0,0) + E^q(x,0,0)\,]\,dx$$
$$J^G = \frac{1}{2}\int x\left[H^G(x,0,0) + E^G(x,0,0)\right]dx$$

At first sight, further decomposition seems easy, since in pQCD framework

$$\Delta\Sigma^q = \int \Delta q(x)\,dx \quad : \quad \texttt{quark spin fraction}$$
$$\Delta G = \int \Delta g(x)\,dx \quad : \quad \texttt{gluon spin fraction}$$

We would therefore get the decomposition

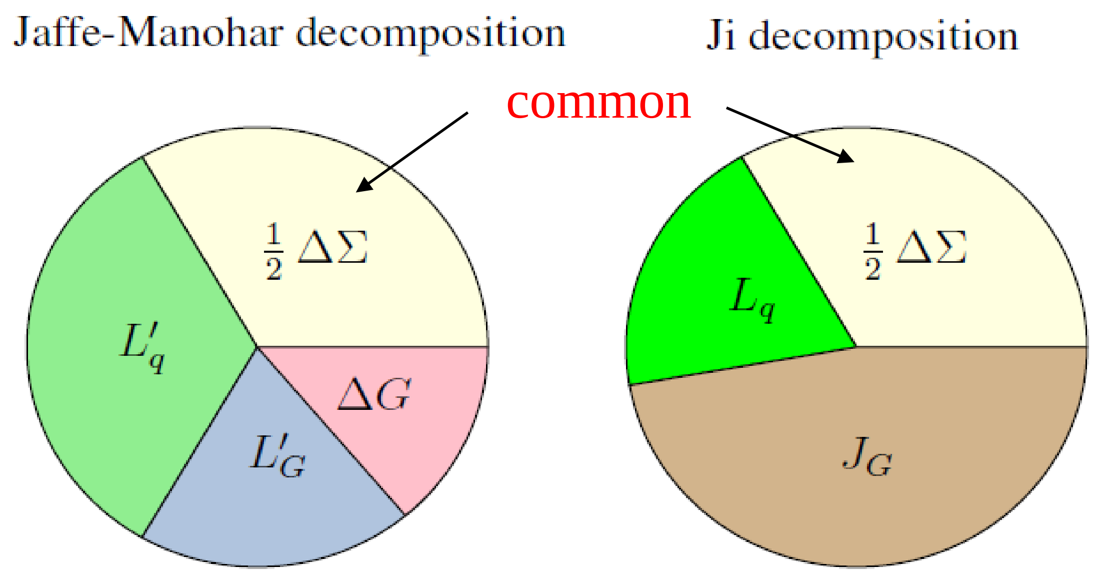
$$\frac{1}{2} = J^q + J^G = \left(L^q + \frac{1}{2}L^q\right) + \left(L^G + \Delta G\right)$$

with

$$L^q \equiv J^q - \frac{1}{2}\Delta\Sigma^q, \qquad L^G \equiv J^G - \Delta G$$

Life is not so easy, however, because of color gauge-invariance.

two popular decompositions of the nucleon spin



$$
\begin{aligned}
\boldsymbol{J}_{QCD} \;=\; & \int \psi^\dagger \frac{1}{2} \boldsymbol{\Sigma} \, \psi \, d^3x \\
+ & \int \psi^\dagger \, \boldsymbol{x} \times \frac{1}{i} \nabla \, \psi \, d^3x \\
+ & \int \boldsymbol{E}^a \times \boldsymbol{A}^a \, d^3x \\
+ & \int E^{ai} \, \boldsymbol{x} \times \nabla \, A^{ai} \, d^3x
\end{aligned}
$$

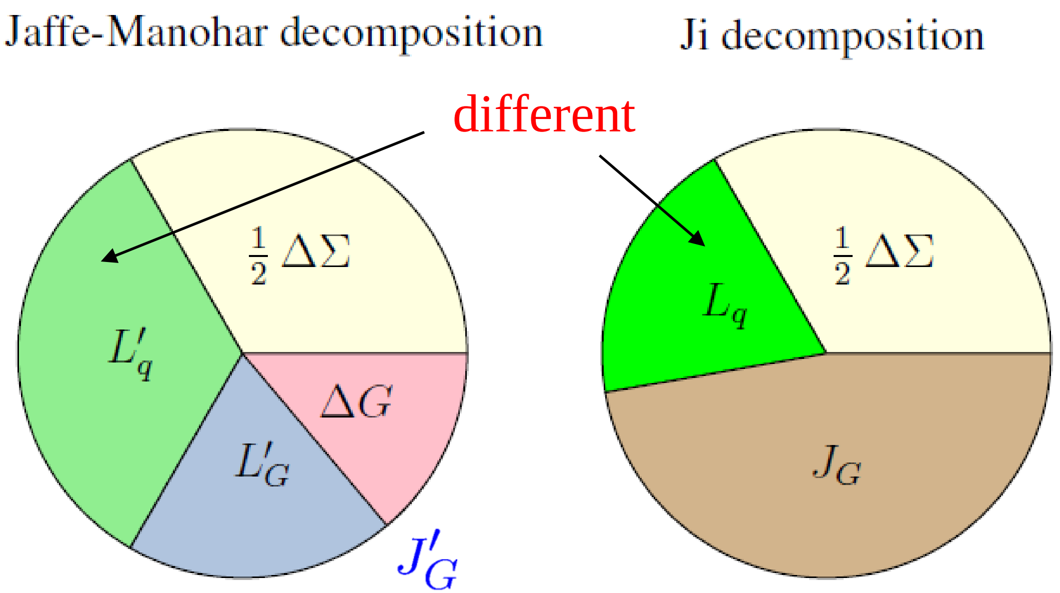
$$
\begin{aligned}
\boldsymbol{J}_{QCD} \;=\; & \int \psi^\dagger \frac{1}{2} \boldsymbol{\Sigma} \, \psi \, d^3x \\
+ & \int \psi^\dagger \, \boldsymbol{x} \times \frac{1}{i} \boldsymbol{D} \, \psi \, d^3x \\
+ & \int \boldsymbol{x} \times (\boldsymbol{E}^a \times \boldsymbol{B}^a) \, d^3x
\end{aligned}
$$

J_G

Each term is not separately gauge-invariant !

No further GI decomposition !

two popular decompositions of the nucleon spin - continued -



An especially annoying observation here was that, since

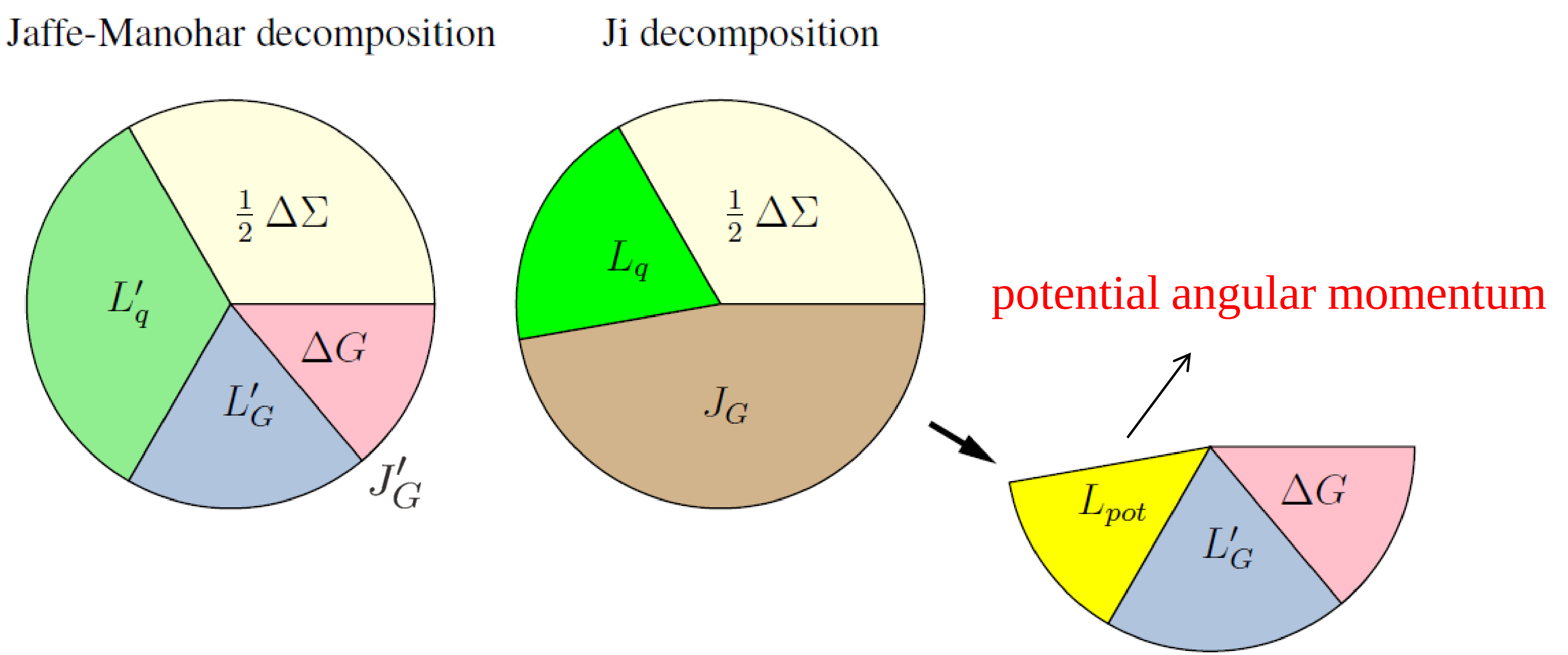
$$L'_q \neq L_q$$

one must inevitably conclude that

$$J'_G = \Delta G + L'_G \neq J_G \ !$$

Now we know the answer of this puzzle.

- M.W. , Phys. Rev. D81 (2010) 114010.



L_{pot} characterizes the difference between $J_G(\texttt{Ji})$ and $J'_G(\texttt{JM})$.

$$J_G(\texttt{Ji}) = J'_G(\texttt{JM}) + L_{pot}$$

Pay attention to the **difference** of **quark OAMs** in the two decompositions.

$$L_Q(\mathrm{JM}) \;\sim\; \psi^\dagger \boldsymbol{x} \times \boldsymbol{p}\, \psi \qquad\qquad L_Q(\mathrm{Ji}) \;\sim\; \psi^\dagger \boldsymbol{x} \times (\boldsymbol{p} - g\, \boldsymbol{A})\, \psi$$

canonical OAM	**mechanical OAM** (or **kinetic OAM**)
not gauge invariant !	**gauge invariant** !

gauge principle

observables must be gauge-invariant !

- Observability of canonical OAM has long been questioned ?
- On the other hand, it has been shown that the mechanical quark OAM can be related to observables through GPDs. (X. Ji, 1997)

The recent intensive dispute began with Chen et al.'s papers.

- X.-S. Chen et al., Phys. Rev. Lett. 103, 062001 (2009) ; 100, 232002 (2008).

basic idea

$$A^{\mu} = A^{\mu}_{phys} + A^{\mu}_{pure}$$

which is a generalization of the familiar decomposition of photon field in QED into the transverse and longitudinal components :

$$\boldsymbol{A}_{phys} \Leftrightarrow \boldsymbol{A}_{\perp} \,(\texttt{gauge-invariant}), \quad \boldsymbol{A}_{pure} \Leftrightarrow \boldsymbol{A}_{\parallel}$$

Their decomposition is given in the following form :

$$\begin{aligned} \boldsymbol{J}_{QCD} &= \boldsymbol{S}'_q + \boldsymbol{L}'_q + \boldsymbol{S}'_G + \boldsymbol{L}'_G \\ &= \int \psi^{\dagger} \frac{1}{2} \Sigma \psi \, d^3x + \int \psi^{\dagger} \boldsymbol{x} \times \left(\frac{1}{i} \nabla - g \, \boldsymbol{A}_{pure} \right) \psi \, d^3x \\ &+ \int \boldsymbol{E}^a \times \boldsymbol{A}^a_{phys} \, d^3x + \int E^{aj} \, (\boldsymbol{x} \times \boldsymbol{D}_{pure}) \, A^{aj}_{phys} \, d^3x \end{aligned}$$

It can be shown that each term is separately gauge-invariant !

- GI version of Jaffe-Manohar decomposition ? -

Soon after, we noticed that the way of gauge-invariant decomposition of nucleon spin is not necessarily unique, and proposed another G.I. decomposition :

- M.W. , Phys. Rev. D81 (2010) 114010.

$$\boldsymbol{J}_{QCD} = \boldsymbol{S}_q + \boldsymbol{L}_q + \boldsymbol{S}_G + \boldsymbol{L}_G$$

where

$$\boldsymbol{S}_q = \boldsymbol{S}'_q, \qquad \boldsymbol{S}_G = \boldsymbol{S}'_G$$

$$\boldsymbol{L}_q = \int \psi \, \boldsymbol{x} \times \left(\frac{1}{i} \nabla - g \, \boldsymbol{A} \right) \psi^\dagger \, d^3 x = \boldsymbol{L}_q \, (\texttt{Ji})$$

$$\boldsymbol{L}_G = \boldsymbol{L}'_G + \boxed{\int \rho^a \, (\boldsymbol{x} \times \boldsymbol{A}^a_{phys}) \, d^3 x} \longrightarrow \boldsymbol{L}_{pot}$$

"**potential angular momentum**"

The QED correspondent of $\boldsymbol{L}_{pot}$ is the orbital angular momentum carried by electromagnetic potential, appearing in the famous Feynman paradox.

An **arbitrariness** of the spin decomposition arises, because this potential angular momentum term is solely gauge-invariant ! Shifting it to the quark OAM part

$$\left.\begin{aligned} \boldsymbol{L}_q + \boldsymbol{L}_{pot} &= \boldsymbol{L}'_q \ (\text{Chen}) \\ \boldsymbol{L}_G - \boldsymbol{L}_{pot} &= \boldsymbol{L}'_G \ (\text{Chen}) \end{aligned}\right\} \Longrightarrow \boldsymbol{J}_G = \boldsymbol{J}'_G + \boldsymbol{L}_{pot}$$

Ji   J-M or Chen

We are thus left with two gauge-invariant decompositions of the nucleon spin :

"canonical" decomposition

$$J_{QCD} = \boldsymbol{S}'_q + \boldsymbol{L}'_q + \boldsymbol{S}'_G + \boldsymbol{L}'_G$$

with

$$\boldsymbol{S}'_q = \int \psi^\dagger \frac{1}{2} \Sigma \, \psi \, d^3x$$

$$\boldsymbol{L}'_q = \int \psi^\dagger \, \boldsymbol{x} \times \left(\frac{1}{i} \nabla - g \, \boldsymbol{A}_{pure} \right) \psi \, d^3x$$

$$\boldsymbol{S}'_G = \int \boldsymbol{E}^a \times \boldsymbol{A}^a_{phys} \, d^3x$$

$$\boldsymbol{L}'_G = \int E^{aj} \left(\boldsymbol{x} \times \boldsymbol{\mathcal{D}}_{pure} \, A^{aj}_{phys} \right) d^3x$$

"mechanical" decomposition

$$J_{QCD} = \boldsymbol{S}_q + \boldsymbol{L}_q + \boldsymbol{S}_G + \boldsymbol{L}_G$$

with

$$\boldsymbol{S}_q = \boldsymbol{S}'_q$$

$$\boldsymbol{L}_q = \int \psi^\dagger \left(\frac{1}{i} \nabla - g \, \boldsymbol{A} \right) \psi \, d^3x$$

$$\boldsymbol{S}_G = \boldsymbol{S}'_G$$

$$\boldsymbol{L}_G = \boldsymbol{L}'_G + \boldsymbol{L}_{pot}$$

[Word of caution]

- These decompositions are based on the familiar transverse-longitudinal decomposition of the gauge field.
- However, the transverse-longitudinal decomposition is given only after fixing the Lorentz-frame of reference.

- breaks **Lorentz-covariance** -

“Seemingly” covariant decomposition of the angular momentum operator

The most general forms of gauge-invariant complete decomposition of the nucleon spin, which have “seemingly” covariant appearances, was given in

- M.W. , Phys. Rev. D83, 014012 (2011)

“canonical” decomposition

$$M^{\lambda\mu\nu}_{QCD} = M'^{\lambda\mu\nu}_{q-spin} + M'^{\lambda\mu\nu}_{q-OAM} + M'^{\lambda\mu\nu}_{G-spin} + M'^{\lambda\mu\nu}_{G-OAM} + \text{boost} + \text{total divergence}$$

where

$$M'^{\lambda\mu\nu}_{q-spin} = \frac{1}{2}\epsilon^{\lambda\mu\nu\sigma}\,\bar{\psi}\,\gamma_\sigma\,\gamma_5\,\psi$$
$$M'^{\lambda\mu\nu}_{q-OAM} = \bar{\psi}\,\gamma^\lambda\,(x^\mu\,i\,D^\nu_{pure} - x^\nu\,i\,D^\mu_{pure})\,\psi$$

$$M'^{\lambda\mu\nu}_{G-spin} = 2\,\mathrm{Tr}\,\{F^{\lambda\nu}\,A^\mu_{phys} - F^{\lambda\mu}\,A^\nu_{phys}\}$$
$$M'^{\lambda\mu\nu}_{G-OAM} = 2\,\mathrm{Tr}\,\{F^{\lambda\alpha}\,(x^\mu\,D^\nu_{pure} - x^\nu\,D^\mu_{pure})\,A^{phys}_\alpha\}$$

“mechanical” decomposition

$$M^{\lambda\mu\nu}_{QCD} = M^{\lambda\mu\nu}_{q-spin} + M^{\lambda\mu\nu}_{q-OAM} + M^{\mu\nu\lambda}_{G-spin} + M^{\lambda\mu\nu}_{G-OAM} + \text{boost} + \text{total divergence}$$

where

$$M^{\lambda\mu\nu}_{q-spin} = M'^{\lambda\mu\nu}$$
$$M^{\lambda\mu\nu}_{q-OAM} = \bar{\psi}\,\gamma^\lambda\,(x^\mu\,i\,D^\nu - x^\nu\,i\,D^\mu)\,\psi$$

$$M^{\lambda\mu\nu}_{G-spin} = M'^{\lambda\mu\nu}_{G-spin}$$
$$M^{\lambda\mu\nu}_{g-OAM} = M'^{\lambda\mu\nu}_{G-OAM} + 2\,\mathrm{Tr}\,[\,(D_\alpha\,F^{\alpha\lambda})\,(x^\mu\,A^\nu_{phys} - x^\nu\,A^\mu_{phys})\,]$$

The startingpoint of these gauge-invariant decompositions of the 4-vector potential

$$A^\mu = A^\mu_{phys} + A^\mu_{pure}$$

To obtain the above two "seemingly" covariant complete decompositions of the QCD angular momentum tensor, we need to impose very general conditions only :

$$F^{\mu\nu}_{pure} \equiv \partial^\mu A^\nu_{pure} - \partial^\nu A^\mu_{pure} - i\, g\, [A^\mu_{pure}, A^\nu_{pure}] = 0$$

and

$$\begin{aligned} A^\mu_{phys}(x) &\rightarrow U(x)\, A^\mu_{phys}(x)\, U^{-1}(x) \\ A^\mu_{pure}(x) &\rightarrow U(x) \left(A^\mu_{pure}(x) + \frac{i}{g}\, \partial^\mu \right) U^{-1}(x) \end{aligned}$$

- Actually, these conditions are not enough to fix the decomposition uniquely !
- It is nevertheless true that one of our decompositions, i.e. the "canonical" type decomposition contains the LC-gauge motivated Bashinsky-Jaffe (or Hatta) decomposition as well as the Coulomb-gauge motivated Chen decomposition, after a suitable choice of the Lorentz frame.

[**critiques to non-uniqueness nature**]

It was criticized by several researchers that our formal decomposition of the gauge field into its physical and pure-gauge components is not unique at all and there are in principle infinitely many such decompositions, which in turn leads to infinitely many decomposition of the nucleon spin.

According to

- X. Ji, Y. Xu, and Y. Zhao, JHEP 08 (2010) 082.

the arbitrariness of the decomposition comes from the path-dependence of the Wilson line, which is necessary for explicitly fixing the decomposition of the gauge field into the physical and pure-gauge components.

Another argument in favor of the existence of infinitely many decomposition of the nucleon spin was advocated by

- C. Lorcé, Phys. Lett. B719, 185 (2013).

based on what-he-call the (hidden) Stueckelberg symmetry of gauge-trans., which changes both of A_{phys} and A_{pure} , while leaving their sum intact.

After long debate, we realize that the remaining issues in the gauge-invariant decomposition problem of the nucleon spin are the following two :

1) Are there infinitely many decompositions of the nucleon spin ? If not, what physical principle favors one particular decomposition among many candidates ?

2) Among the two different decompositions, i.e. the "canonical" type and "mechanical" type decompositions, which can we say is more physical ?

(More "physical" here means that it is closer to direct observation.)

Actually, the 1st question above is closely connected with the long-lasting fundamental question of the nucleon spin decomposition problem.

1') Can the total gluon angular momentum be gauge-invariantly decomposed into its spin and orbital parts without causing conflict with the textbook negative statement on the similar question on the total photon angular momentum ?

We believe that a clear answer to both these questions are given in

- M. Wakamatsu, Eur. Phys. J. A51 (2015) 52 ; arXiv : 1409.4474 [hep-ph]

2.2. Uniqueness problem of the nucleon spin decomposition

- the role of Lorentz-symmetry -

Ji et al. argued that the total gluon helicity in a polarized proton is shown to be large momentum limit of a gauge-invariant operator $\boldsymbol{E} \times \boldsymbol{A}_\perp$, with $\boldsymbol{A}_\perp$ being the usual transverse component of the gauge potential.

- X. Ji, J.-H. Zhang, and Y. Zhao, Phys. Rev. 111 (2013) 112002.

(1) First, they pointed out that, for the abelian case, the gluon spin operator S_G, which corresponds to DIS measurements, can be expressed in the form :

$$S_G = (\boldsymbol{E}(0) \times \boldsymbol{A}_{phys}(0))^3$$

with

$$\boldsymbol{A}_{phys}(0) = \boldsymbol{A}(0) - \frac{1}{\nabla^+} \nabla A^+(\xi^-) \Big|_{\xi^- = 0}$$

(2) Next, they showed that the above operator is just the IMF limit of

$\boldsymbol{E} \times \boldsymbol{A}_\perp$: gluon spin operator of Chen et al.

From this fact, they concluded that, to identify $(\boldsymbol{E} \times \boldsymbol{A}_\perp)^3$ as the gluon helicity, one must have the following conditions :

Infinite Momentum Frame & physical gauge ($\sim$ LC gauge)

The statement is nothing wrong, but it has a danger of causing a misunderstanding.

In fact, the gluon spin, or more generally, the longitudinally polarized gluon distribution, must be a Lorentz-frame independent quantity.

This is clear from the fact that the measurement of these quantities is carried out in the laboratory frame not in the IMF !

This especially means that the gluon spin or the longitudinally polarized gluon distribution should not depend on the magnitude of nucleon momentum P_z .

$$\Delta G(\cancel{P_z}), \qquad \Delta g(x, \cancel{P_z})$$

On the Lorentz-frame independence of PDF (from Collins' textbook)

definition of unpolarized PDF (n being the light-like vector with $n^2 = 0$)

$$\int \frac{d\lambda}{2\pi} e^{i\lambda k\cdot n} \langle P | \bar{\psi}(0) \not{n} \psi(\lambda n) | P \rangle$$

Since the r.h.s is a scalar function, it must be a function of $k \cdot n$ and $P \cdot n$:

$$\tilde{q}(k \cdot n, P \cdot n)$$

The formula is invariant under scaling of n by an arbitrary positive factor, so that only the combination $x \equiv k \cdot n \,/\, P \cdot n$ is allowed :

$$q\left(x \equiv \frac{k \cdot n}{P \cdot n}\right)$$

This gives

$$q(x) = \int \frac{d\lambda}{2\pi} e^{i\lambda x (P\cdot n)} \langle P | \bar{\psi}(0) \not{n} \psi(\lambda n) | P \rangle$$

It is invariant under the boost along the direction of the nucleon momentum.

$$p^0 \rightarrow \gamma (p^0 - v p^3), \quad p^1 \rightarrow p^1, \quad p^2 \rightarrow p^2, \quad p^3 \rightarrow \gamma (p^3 - v p^0)$$

To see the importance of the constraint from Lorentz-frame independence in our decomposition problem, it would be instructive to compare a vital difference between the various definitions of the "physical" component of the gauge field :

- Y. Hatta, X. Ji, and Y. Zhao, Phys. Rev. D89 (2014) 085030.

- LC gauge motivated $(A^+ = 0)$ $\Rightarrow$ $A^\mu_{phys} = \frac{1}{D^+} F^{+\mu}$
- temporal gauge motivated $(A^0 = 0)$ $\Rightarrow$ $A^\mu_{phys} = \frac{1}{D^0} F^{0\mu}$
- spatial axial gauge motivated $(A^3 = 0)$ $\Rightarrow$ $A^\mu_{phys} = \frac{1}{D^3} F^{3\mu}$
- Coulomb gauge motivated $(\nabla \cdot \boldsymbol{A} = 0)$ $\Rightarrow$ $\boldsymbol{A}_{phys} = \boldsymbol{A} - \nabla \frac{1}{\nabla^2} \nabla \cdot \boldsymbol{A}$

A distinguishing feature of the LC gauge motivated choice is that it is invariant under the Lorentz-boost along the 3-direction, i.e. the direction of nucleon momentum !

In fact, under the Lorentz boost along the 3-direction

$$x^1 \rightarrow x^1, \quad x^2 \rightarrow x^2, \quad x^3 \rightarrow \gamma\,(x^3 - v\,x^0), \quad x^0 \rightarrow \gamma\,(x^0 - v\,x^3)$$

one can easily verify that (for $k = 1, 2$)

$$A^k_{phys} \equiv \frac{1}{D^+} F^{+k} = \frac{1}{\partial^+ - i\,g\,A^+} F^{+k}$$
$$\rightarrow \frac{1}{\gamma\,(1-v)\,(\partial^+ - i\,g\,A^+)} \gamma\,(1-v)\,F^{+k} = \frac{1}{D^+} F^{+k} = A^k_{phys}$$

On the contrary, any other definitions of A^k_{phys} is not invariant under the boost.

We therefore conclude that what plays a key role in the uniqueness problem of the GI decomposition of the nucleon spin is the **Lorentz-frame independence**.

Somewhat ironically, then, what selects a particular GI nucleon spin decomposition is not the gauge symmetry but the Lorentz symmetry !

Still noteworthy observation is as follows. In the free field limit with

$$A^0 = A^3 = 0$$

we see that

$$\begin{aligned}
\texttt{LC} \quad &: \quad A^\mu_{phys} = \frac{1}{D^+} F^{+\mu} \rightarrow \frac{1}{\partial^+} \partial^+ A^\mu = A^\mu \\
\texttt{tempolal} \quad &: \quad A^\mu_{phys} = \frac{1}{D^0} F^{0\mu} \rightarrow \frac{1}{\partial^0} \partial^0 A^\mu = A^\mu \\
\texttt{spatial axial} \quad &: \quad A^\mu_{phys} = \frac{1}{D^3} F^{3\mu} \rightarrow \frac{1}{\partial^3} \partial^3 A^\mu = A^\mu
\end{aligned}$$

This indicates perturbative equivalence of these three. In fact, we find that the 1-loop anomalous dimension of the gluon spin operators are just the same.

- M.W., Phys. Rev. D87 (2013) 094035.

$$\begin{aligned}
\langle PS \,|\, \tilde{F}^{+k} A^k_{phys} \, PS \rangle_G \,|_{A^+=0} &= \left[1 + \frac{\alpha_S}{4\pi} \cdot \frac{\beta_0}{\varepsilon} \right] \langle PS \,|\, \tilde{F}^{+k} A^k_{phys} \, PS \rangle_G^{tree} \\
\langle PS \,|\, \tilde{F}^{0k} A^k_{phys} \, PS \rangle_G \,|_{A^0=0} &= \left[1 + \frac{\alpha_S}{4\pi} \cdot \frac{\beta_0}{\varepsilon} \right] \langle PS \,|\, \tilde{F}^{0k} A^k_{phys} \, PS \rangle_G^{tree} \\
\langle PS \,|\, \tilde{F}^{3k} A^k_{phys} \, PS \rangle_G \,|_{A^3=0} &= \left[1 + \frac{\alpha_S}{4\pi} \cdot \frac{\beta_0}{\varepsilon} \right] \langle PS \,|\, \tilde{F}^{3k} A^k_{phys} \, PS \rangle_G^{tree}
\end{aligned}$$

Now, the definition of the gluon spin operator corresponding to DIS measurements seems unique, so that there is only one (or two) nucleon spin decomposition.

$$\Delta G \;=\; \frac{1}{2\,P^+}\,\langle PS\,|\,2\,\mathrm{Tr}\,\left[\epsilon_\perp^{jk}\,\tilde{F}^{j+}(0)\,A^k_{phys}(0)\right]\,|\,PS\rangle$$

where

$$A^k_{phys}(0) \;=\; -\frac{1}{2}\int d\xi^-\,\epsilon(\xi^-)\,\mathcal{L}[0,\xi^-]\,F^{+k}(\xi^-)\,\mathcal{L}[\xi^-,0]$$

It is gauge-invariant as well as Lorentz-boost invariant along the 3-direction.

- Any contradiction with the standard textbook knowledge ?
- Is the lack of full covariance an indication of the fact that the gluon spin is not a gauge-invariant quantity in an ordinary sense ?

$\Downarrow$

A key is the existence of particular spatial direction in the DIS observables !

\- direction of nucleon momentum -

One can convince it if one remembers

decomposition problem of the total photon angular momentum

- S.J. Van Enk and G. Nienhuis, Europhys. Lett. 25, 497 (1994).
- S.J. Van Enk and G. Nienhuis, J. Mod. Optics 41, 963 (1994).

They argue that the total angular momentum of free electromagnetic field can be gauge-invariantly decomposed into "spin" and "orbital" parts, $\boldsymbol{J}_\gamma = \boldsymbol{S} + \boldsymbol{L}$.

(1) This separation is not Lorentz invariant.

(2) Neither $\boldsymbol{S}$ nor $\boldsymbol{L}$ does obey the SU(2) commutation relation.

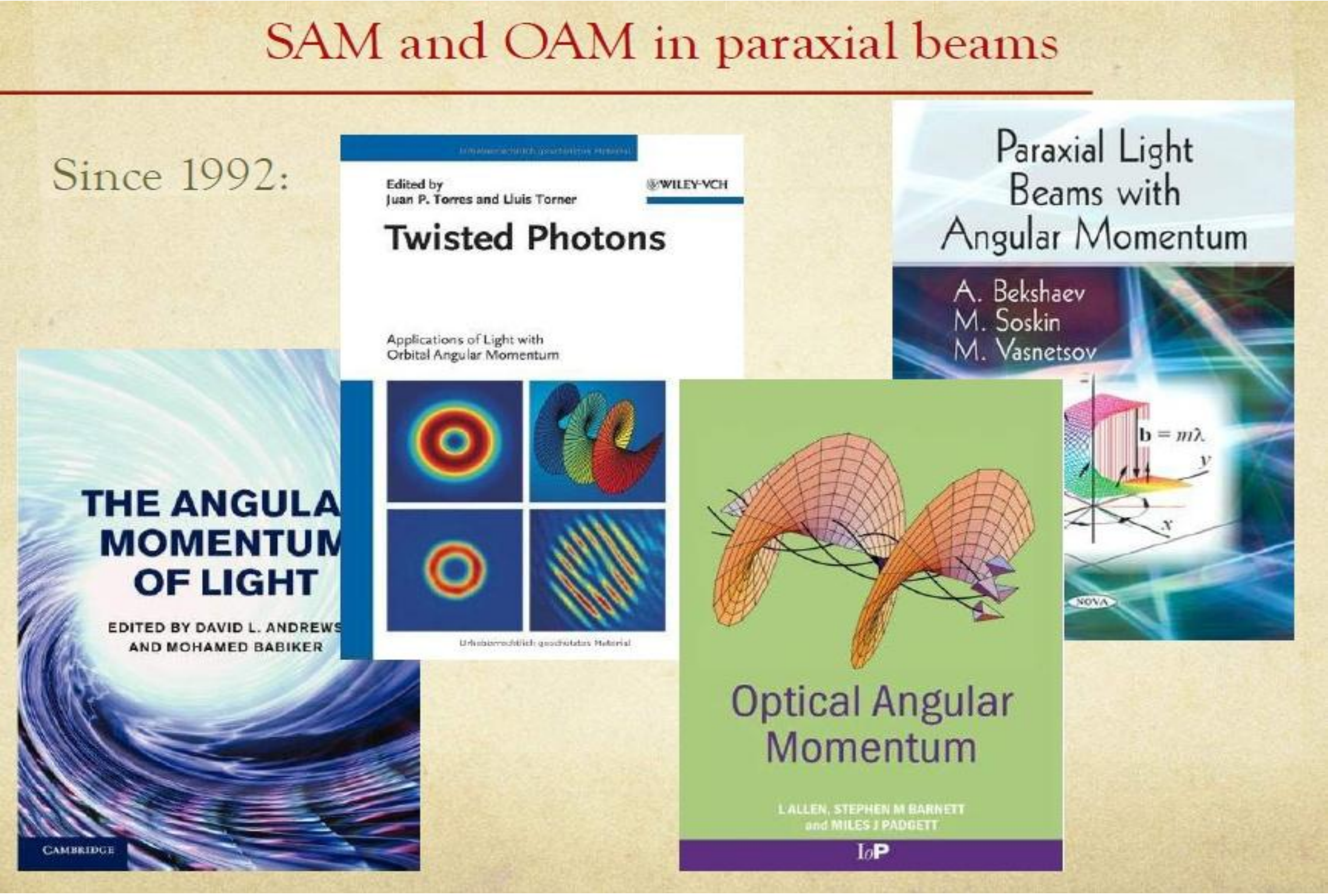
(1) causes no problem, because the photon spin measurement is performed in a fixed laboratory frame by making use of the interaction with atoms.

(2) is not also the problem, because both of $\boldsymbol{S}$ and $\boldsymbol{L}$ (actually their components along the photon beam direction) can be separately measured.

⇩

It appears that the key is again the existence of a particular spatial direction in the measurement, i.e. the direction of paraxial laser beam.

For similarity between optical measurements and TMD physics : Bliokh@ECT*2014



It may be fun to inspect the physical contents of the resultant gluon spin operator.

In the LC gauge ($A^+ = 0$) , it reduces to the following form :

$$\Delta G \;=\; \frac{1}{2\,P^+}\,\langle PS\,|\,2\,\mathrm{Tr}\,\left[\epsilon_\perp^{jk}\,F^{j+}(0)\,A^k_{phys}(0)\right]\,|\,PS\rangle$$
$$\Rightarrow\; \frac{1}{2\,P^+}\,\langle PS\,|\,(\boldsymbol{E}_\perp \times \boldsymbol{A}_\perp)^3 \;+\; \boldsymbol{B}_\perp \cdot \boldsymbol{A}_\perp\,|\,PS\rangle$$

We emphasize that the presence of the 2nd term is essential, because the 1st term alone is not invariant under the Lorentz boost along the 3-direction.

Jaffe once estimated the contributions of both terms in the bag model as well as in the quark model.

- R.L. Jaffe, Phys. Lett. B365 (1996) 359.

Jaffe already recognized that, since the sum is boost-invariant, the above ΔG can be calculated in any Lorentz frame, including the rest frame of the nucleon, provided that the above $\boldsymbol{A}_\perp$ is the gauge potential in the LC gauge.

What is curious here is the physical meaning of the peculiar 2nd term.

Very interestingly, it resembles the quantity :

$$S = \int \boldsymbol{B} \cdot \boldsymbol{A} \, d^3 x$$

except the absence of the 3-component in $\boldsymbol{B}_\perp \cdot \boldsymbol{A}_\perp$.

In the field of space and laboratory plasma physics, the above ***S*** is called the magnetic helicity, which gives a measure of the topological configuration of magnetic field.

magnetic helicity = topological invariant

- M. Berger, Plasma. Phys. Control. Fusion 41 (1999) B167.

This might indicates that, if a topological configuration of the gluon field plays some role in the gluon spin in the nucleon, it is through this 2nd term (?)

Leaving aside such a speculation, a perturbative consideration gives transparent physical meaning of the term $\boldsymbol{B}_\perp \cdot \boldsymbol{A}_\perp$.

Using the free field expansion of the gauge potential

$$\boldsymbol{A}_\perp(\boldsymbol{x}, t) = \int d^3\tilde{k}^3 \sum_{\lambda=\pm 1} [a(\boldsymbol{k},\lambda)\,\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)\,e^{-ik\cdot x} + a^\dagger(\boldsymbol{k},\lambda)\,\boldsymbol{\varepsilon}^*(\boldsymbol{k},\lambda)\,e^{ik\cdot x}]$$

one can easily show that

$$\int \boldsymbol{E}_\perp \times \boldsymbol{A}_\perp\, d^3x = \int d\tilde{k}^3 \sum_{\lambda=\pm 1} \hat{\boldsymbol{k}}\, \lambda\, a^\dagger(\boldsymbol{k},\lambda)\, a(\boldsymbol{k},\lambda)$$

and

$$\int \boldsymbol{B}_\perp \cdot \boldsymbol{A}_\perp\, d^3x = \int d\tilde{k}^3 \sum_{\lambda=\pm 1} \lambda\, a^\dagger(\boldsymbol{k},\lambda)\, a(\boldsymbol{k},\lambda)$$

Thus

$$\frac{1}{2}\int \left[(\boldsymbol{E}_\perp \times \boldsymbol{A}_\perp)^3 + \boldsymbol{B}_\perp \cdot \boldsymbol{A}_\perp \right] d^3x = \sum_{\lambda=\pm 1} \lambda\, a^\dagger(\boldsymbol{k},\lambda)\, a^{(}\boldsymbol{k},\lambda)$$

reduces to the ordinary helicity operator.

Now we can make a clear statement on our "seemingly" covariant decompositions of the angular momentum operator (Phys. Rev. D83, 014012 (2011)).

$$M^{\lambda\mu\nu}_{QCD} = M'^{\lambda\mu\nu}_{q-spin} + M'^{\lambda\mu\nu}_{q-OAM} + M'^{\lambda\mu\nu}_{G-spin} + M'^{\lambda\mu\nu}_{G-OAM} + \text{boost} + \text{total divergence}$$

where

$$M'^{\lambda\mu\nu}_{q-spin} = \frac{1}{2}\epsilon^{\lambda\mu\nu\sigma}\,\bar{\psi}\,\gamma_\sigma\,\gamma_5\,\psi$$

$$M'^{\lambda\mu\nu}_{q-OAM} = \bar{\psi}\,\gamma^\lambda\,(\,x^\mu\,i\,D^\nu_{pure} - x^\nu\,i\,D^\mu_{pure}\,)\,\psi$$

$$M'^{\lambda\mu\nu}_{G-spin} = 2\,\mathrm{Tr}\,\{\,F^{\lambda\nu}\,A^\mu_{phys} - F^{\lambda\mu}\,A^\nu_{phys}\,\}$$

$$M'^{\lambda\mu\nu}_{G-OAM} = 2\,\mathrm{Tr}\,\{\,F^{\lambda\alpha}\,(\,x^\mu\,D^\nu_{pure} - x^\nu\,D^\mu_{pure}\,)\,A^{phys}_\alpha\,\}$$

$$M^{\lambda\mu\nu}_{QCD} = M^{\lambda\mu\nu}_{q-spin} + M^{\lambda\mu\nu}_{q-OAM} + M^{\mu\nu\lambda}_{G-spin} + M^{\lambda\mu\nu}_{G-OAM} + \text{boost} + \text{total divergence}$$

where

$$M^{\lambda\mu\nu}_{q-spin} = M'^{\lambda\mu\nu}$$

$$M^{\lambda\mu\nu}_{q-OAM} = \bar{\psi}\,\gamma^\lambda\,(\,x^\mu\,i\,D^\nu - x^\nu\,i\,D^\mu\,)\,\psi$$

$$M^{\lambda\mu\nu}_{G-spin} = M'^{\lambda\mu\nu}_{G-spin}$$

$$M^{\lambda\mu\nu}_{g-OAM} = M'^{\lambda\mu\nu}_{G-OAM} + 2\,\mathrm{Tr}\,[\,(\,D_\alpha\,F^{\alpha\lambda}\,)\,(\,x^\mu\,A^\nu_{phys} - x^\nu\,A^\mu_{phys}\,)\,]$$

Both of these decompositions looks covariant, but it is only seemingly so.

The reason is that the decomposition of the gauge field into its physical and pure-gauge component can be done only in non-covariant manner.

Now the problem (1), the very delicate gauge-invariance issue of the gluon spin, has been essentially resolved, so that what remains is the problem (2), i.e.

relative merits of "canonical" and "mechanical" decompositions

(We recall that the gluon spin part is just common in the two decompositions !)

Often-claimed **advantages (?)** of "canonical" decomposition.

(1) Each piece of the decomposition satisfies the SU(2) commutation relation

$$[L^i_{can}, L^j_{can}] = i \, \epsilon^{ijk} \, L^k_{can}$$

(2) L_{can} is compatible with free partonic picture of constituent orbital motion.

The 1st advantage was already denied for the massless particle.

- M.W., Int. J. Mod. Phys. A29, 1430012 (2014).
- W.-M. Sun, arXiv : 1407.2035 [quant-ph].

The underlying reason is that a massless particle is described by a little group $E(2) \sim ISO(2)$ of the Lorentz group.

- P. M. Zhang and D. G. Pak, Eur. Phys. J. A 48, 91 (2012).

Just a reminder on power balance (?) of "canonical" or "mechanical" party

From the slide of my talk at "Transversity 2011", Veli Losinj, Croatia

canonical OAM party

- Jaffe-Manohar
- Bashinsky-Jaffe
- Chen et al.
- Cho et al.
- Leader

mechanical OAM party

- Ji
- Wakamatsu

Neutral party

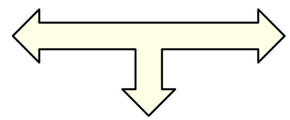
- Burkardt-BC

Current status (2014-2015 ?)

canonical OAM party

- Jaffe-Manohar
- Bashinsky-Jaffe
- Chen et al.
- Leader
- Hatta
- Ji (?)

…



dynamical OAM party

- Wakamatsu

Neutral party

- Burkardt
- Lorce (?)
- Tiwari (?)

Widespread superstition originating from the "appearance" of the two OAMs :

$$L_{mech} = \int \psi^\dagger \, r \times \frac{1}{i} (\nabla - i\, g\, A)\, \psi \, d^3 r \;\xrightarrow{G.F.}\; \int \psi^\dagger \, r \times \frac{1}{i} \left(\nabla - i\, g\, A_{phys}\right) \psi \, d^3 r$$

$$L_{can} = \int \psi^\dagger \, r \times \frac{1}{i} (\nabla - i\, g\, A_{pure})\, \psi \, d^3 r \;\xrightarrow{G.F.}\; \int \psi^\dagger \, r \times \frac{1}{i} \nabla \psi \, d^3 r \quad ?$$

- The "mechanical" OAM appears to contains quark-gluon interaction.
- The "canonical" OAM does not contain quark-gluon interaction, so that it seems compatible with the partonic interpretation.

That this understanding is not necessarily correct was argued in Sect.6 of

- M.W., Int. J. Mod. Phys. A29, 1430012 (2014).

We shall discuss now that this misconception arises, because they are too much accustomed with a weak-coupling treatment of gauge theory as exemplified by

hydrogen atom problem of QED

2.3. What is "potential angular momentum" ?   - Lessons from CED & QED -

The **key quantity**, which distinguishes the two OAMs appearing in the two decompositions, is what-we-call the "**potential angular momentum**" term.

To understand its **physical meaning**, we find it instructive to study easier QED case, especially a system of **charged particles** and **photons**.

$$H \;=\; \sum_i \frac{1}{2} m_i \, \dot{\boldsymbol{r}}_i^2 \;+\; \frac{1}{2} \int d^3 r \, [\, \boldsymbol{E}^2 + \boldsymbol{B}^2 \,]$$

longitudinal-transverse decomposition :

$$\boldsymbol{A} \;=\; \boldsymbol{A}_{||} \;+\; \boldsymbol{A}_{\perp}$$

with the properties

$$\nabla \times \boldsymbol{A}_{||} \;=\; 0, \quad \nabla \cdot \boldsymbol{A}_{\perp} \;=\; 0$$

corresponding decomposition for electric and magnetic fields

$$\boldsymbol{E} \;=\; \boldsymbol{E}_{||} \;+\; \boldsymbol{E}_{\perp}, \qquad \boldsymbol{B} \;=\; \boldsymbol{B}_{\perp}$$

with

$$\boldsymbol{E}_{||} \;=\; -\nabla A^0 \;-\; \frac{\partial}{\partial t} \boldsymbol{A}_{||}, \quad \boldsymbol{E}_{\perp} \;=\; -\frac{\partial}{\partial t} \boldsymbol{A}_{\perp}, \quad \boldsymbol{B}_{\perp} \;=\; \nabla \times \boldsymbol{A}_{\perp}$$

Then we have

$$H = \sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + \frac{1}{2} \int d^3r\, \boldsymbol{E}_{\parallel}^2 + \frac{1}{2} \int d^3r\, [\, \boldsymbol{E}_{\perp}^2 + \boldsymbol{B}_{\perp}^2 \,]$$

Here, by using the Gauss law $\nabla \cdot \boldsymbol{E}_{\parallel} = \rho$, we can show that

$$\frac{1}{2} \int d^3r\, \boldsymbol{E}_{\parallel}^2 = \frac{1}{2} \int d^3r\, d^3r' \, \frac{\rho(\boldsymbol{r})\, \rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} = V_{Coul}$$

V_{Coul} : Coulomb interaction between charged particles

We are thus led to

$$H = \sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + V_{Coul} + \frac{1}{2} \int d^3r\, [\, \boldsymbol{E}_{\perp}^2 + \boldsymbol{B}_{\perp}^2 \,]$$

total momentum (of the electron photon system)

Poynting vector

$$\boldsymbol{P}_{tot} = \sum_i m_i \dot{\boldsymbol{r}}_i + \int d^3r\, \boldsymbol{E} \times \boldsymbol{B}$$

$$= \sum_i m_i \dot{\boldsymbol{r}}_i + \int d^3r\, (\boldsymbol{E}_{\|} + \boldsymbol{E}_{\perp}) \times \boldsymbol{B}_{\perp}$$

$$= \sum_i m_i \dot{\boldsymbol{r}}_i + \boldsymbol{P}_{long} + \boldsymbol{P}_{trans}$$

$$= \sum_i m_i \dot{\boldsymbol{r}}_i + \sum_i q_i\, \boldsymbol{A}_{\perp}(\boldsymbol{r}_i) + \boldsymbol{P}_{trans}$$

mechanical momentum

potential momentum : *a la* Konopinski

Note that it was originally contained in $\int d^3r\, \boldsymbol{E} \times \boldsymbol{B}$.

If we combine it with the **mechanical momentum** $m_i \dot{\boldsymbol{r}}_i$

$$m_i \dot{\boldsymbol{r}}_i + q_i\, \boldsymbol{A}_{\perp}(\boldsymbol{r}_i) = m_i \dot{\boldsymbol{r}}_i + q_i\, (\boldsymbol{A}(\boldsymbol{r}_i) - \boldsymbol{A}_{\|}(\boldsymbol{r}_i))$$

$$= (m_i \dot{\boldsymbol{r}}_i + q_i\, \boldsymbol{A}(\boldsymbol{r}_i)) - q_i\, \boldsymbol{A}_{\|}(\boldsymbol{r}_i)$$

$$= \boldsymbol{p}_i - q_i\, \boldsymbol{A}_{\|}(\boldsymbol{r}_i)$$

with $\boldsymbol{p}_i \equiv m_i \dot{\boldsymbol{r}}_i + q_i\, \boldsymbol{A}(\boldsymbol{r}_i)$ being the standard **canonical momentum**.

We therefore obtain

$$\begin{aligned} \boldsymbol{P}_{tot} &= \sum_i m_i \dot{\boldsymbol{r}}_i + \sum_i q_i \boldsymbol{A}_\perp(\boldsymbol{r}_i) + \boldsymbol{P}_{trans} \\ &= \sum_i (\boldsymbol{p}_i - q_i \boldsymbol{A}_\parallel(\boldsymbol{r}_i)) + \boldsymbol{P}_{trans} = \boldsymbol{P}_{\text{``}can\text{''}} + \boldsymbol{P}_{trans} \end{aligned}$$

with

$$\boldsymbol{P}_{\text{``}can\text{''}} \equiv \sum_i (\boldsymbol{p}_i - q_i \boldsymbol{A}_\parallel(\boldsymbol{r}_i))$$

being the generalized (gauge-invariant) canonical momentum of Chen et al.

In the Coulomb gauge ($\boldsymbol{A}_\parallel = 0$) , it reduces to the usual canonical one.

$$\boldsymbol{P}_{\text{``}can\text{''}} \rightarrow \sum_i \boldsymbol{p}_i = \boldsymbol{P}_{can}$$

In any case, we have two different decomposition of $\boldsymbol{P}_{tot}$.

$$\begin{aligned} \boldsymbol{P}_{tot} &= \boldsymbol{P}_{mech} + \boldsymbol{P}_{pot} + \boldsymbol{P}_{trans} \\ &= \boldsymbol{P}_{\text{``}can\text{''}} + \boldsymbol{P}_{trans} \end{aligned}$$

total angular momentum

$$\begin{aligned} \boldsymbol{J}_{tot} &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i + \int d^3r\, \boldsymbol{r} \times \left[(\boldsymbol{E}_{\parallel} + \boldsymbol{E}_{\perp}) \times \boldsymbol{B}_{\perp} \right] \\ &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i + \boldsymbol{J}_{long} + \boldsymbol{J}_{trans} \\ &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i + \sum_i \boldsymbol{r}_i \times q_i \boldsymbol{A}_{\perp}(\boldsymbol{r}_i) + \boldsymbol{J}_{trans} \end{aligned}$$

↙ mechanical OAM ↘ what-we-call the "**potential angular momentum**"

↓

originally contained in $\int d^3r\, \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B})$.

Again, combining this term with the **mechanical angular momentum**, we get

$$\begin{aligned} \boldsymbol{J}_{tot} &= \boldsymbol{L}_{mech} + \boldsymbol{L}_{pot} + \boldsymbol{J}_{trans} \\ &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i + \sum_i \boldsymbol{r}_i \times q_i \boldsymbol{A}_{\perp}(\boldsymbol{r}_i) + \boldsymbol{J}_{trans} \\ &= \sum_i \boldsymbol{r}_i \times (\boldsymbol{p}_i - q_i \boldsymbol{A}_{\parallel}(\boldsymbol{r}_i)) + \boldsymbol{J}_{trans} \\ &= \boldsymbol{L}_{\text{“}can\text{”}} + \boldsymbol{J}_{trans} \end{aligned}$$

Incidentally, the transverse part can be decomposed into two pieces :

$$\begin{aligned} \boldsymbol{J}_{trans} &\equiv \int d^3r \, \boldsymbol{r} \times (\boldsymbol{E}_\perp \times \boldsymbol{B}) \\ &= \int d^3r \, E_\perp^k \, (\boldsymbol{r} \times \nabla) \, A_\perp^k \; + \; \int d^3r \, \boldsymbol{E}_\perp \times \boldsymbol{A}_\perp \\ &= \quad \boldsymbol{L}'_\gamma \qquad + \qquad \boldsymbol{S}'_\gamma \end{aligned}$$

which respectively correspond to the OAM and intrinsic spin of free photon.

$\Downarrow$

After all these steps, we arrive at two different decompositions of $\boldsymbol{J}_{tot}$.

Two physically different decompositions

$$\boldsymbol{J}_{tot} = \boldsymbol{L}'_p + \boldsymbol{S}'_\gamma + \boldsymbol{L}'_\gamma = \boldsymbol{L}_p + \boldsymbol{S}_\gamma + \boldsymbol{L}_\gamma$$

"canonical" decomposition **"mechanical" decomposition**

where

$$\boldsymbol{L}'_p = \sum_i \boldsymbol{r}_i \times (\boldsymbol{p}_i - q_i \boldsymbol{A}_{||}(\boldsymbol{r}_i))$$

$$\Rightarrow \sum_i \boldsymbol{r}_i \times \frac{1}{i} \boldsymbol{D}_{i,pure}$$

$$\boldsymbol{S}'_\gamma = \int d^3r\, \boldsymbol{E}_\perp \times \boldsymbol{A}_\perp$$

$$\boldsymbol{L}'_\gamma = \int d^3r\, E^k_\perp (\boldsymbol{r} \times \nabla) A^k_\perp$$

⇕

Chen decomposition

$$\boldsymbol{L}_p = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i$$

$$\Rightarrow \boxed{\sum_i \boldsymbol{r}_i \times \frac{1}{i} \boldsymbol{D}_i}$$

$$\boldsymbol{S}_\gamma = \boldsymbol{S}'_\gamma$$

$$\boldsymbol{L}_\gamma = \boldsymbol{L}'_\gamma + \boldsymbol{L}_{pot}$$

⇕

Our decomposition

Important remark

It is a wide-spread belief that, among the following two quantities :

$$\boldsymbol{L}_{\text{“}can\text{''}} = \boldsymbol{r} \times (\boldsymbol{p} - e\,\boldsymbol{A}_{\parallel}) \quad \Longleftrightarrow \quad \boldsymbol{L}_{mech} = \boldsymbol{r} \times (\boldsymbol{p} - e\,\boldsymbol{A}_{\perp})$$

what is closer to physical image of orbital motion is the former, because the latter appears to contain an genuine **interaction term** with the gauge field !

The fact is just opposite !

$$\begin{aligned} \boldsymbol{L}_{\text{“}can\text{”}} &= \boldsymbol{L}_{mech} + \sum_i \boldsymbol{r}_i \times q_i\,\boldsymbol{A}_{\perp}(\boldsymbol{r}_i) \\ &= \sum_i m_i\,\boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i + \int d^3r\,\boldsymbol{r} \times (\boldsymbol{E}_{\parallel} \times \boldsymbol{B}_{\perp}) \end{aligned}$$

orbital motion !

- It is the “mechanical” angular momentum $\boldsymbol{L}_{mech}$ not the “canonical” angular momentum $\boldsymbol{L}_{\text{“}can\text{”}}$ that has a natural physical interpretation as orbital motion of particles under the presence of gauge potential

$\boldsymbol{A}_{\perp}$

Also for the nucleon spin decomposition problem of QCD, there are many who believe that the ``canonical'' OAM rather than the ``mechanical'' OAM matches the idea of partonic orbital motion of quarks.

I would say that this understanding is not correct.

This misconception arises, because they are too much accustomed with a weak-coupling treatment of gauge theory as exemplified by

hydrogen atom problem of QED

In such problems, although the Coulomb force is handled nonperturbatively, the transverse photons are treated only perturbatively.

[Cf.] QCD requires nonperturbative treatment of transverse gluon field.

Hydrogen atom Hamiltonian (in Coulomb gauge)

$$m\,\dot{\boldsymbol{r}} = \boldsymbol{p} - e\,\boldsymbol{A}_{\perp}(\boldsymbol{r})$$

$$H = \frac{1}{2} m\,\dot{\boldsymbol{r}}^2 + V_{Coul} + H_{trans} = H_0 + H_{int} + H_{trans}$$

$$H_0 = \frac{\boldsymbol{p}^2}{2\,m} + V_{Coul}(\boldsymbol{r})$$

$$H_{trans} = \sum_{\boldsymbol{k}} \sum_{\lambda=1,2} \hbar\,\omega_{\boldsymbol{k}}\; a^{\dagger}_{\boldsymbol{k},\lambda}\, a_{\boldsymbol{k},\lambda}$$

interaction term !

$$H_{int} = \frac{e}{2\,m}\left[\boldsymbol{p}\cdot\boldsymbol{A}_{\perp}(\boldsymbol{r}) + \boldsymbol{A}_{\perp}(\boldsymbol{r})\cdot\boldsymbol{p}\right] + \frac{e^2}{2\,m}\boldsymbol{A}_{\perp}(\boldsymbol{r})\cdot\boldsymbol{A}_{\perp}(\boldsymbol{r})$$

general form of eigen-states of ***H*** : $|\,\psi_n\rangle \otimes |\{n_{\boldsymbol{k},\lambda}\}\rangle$

$$H_0\,|\,\psi_n\rangle = E_n\,|\,\psi_n\rangle$$

$$|\,\{n_{\boldsymbol{k},\lambda}\}\rangle = \prod_{\alpha} |\,n_{\boldsymbol{k}_\alpha,\lambda_\alpha}\rangle$$

In the standard description of hydrogen atom, we do not include

Fock components of transverse photons !

$$|\,\{n_{\boldsymbol{k},\lambda}\}\rangle \Rightarrow |\,0\,\rangle_{photon}$$

eigen-equation of hydrogen atom (relativistic Dirac equation)

$$H\,\psi_n = \left(\frac{\boldsymbol{\alpha}\cdot\nabla}{i} + \beta\, m - \frac{\alpha}{r}\right)\psi_n = E_n\,\psi_n$$

eigen wave function

$$\psi_{jm}{}^{l} = \begin{pmatrix} i\,\dfrac{G_{lj}(r)}{r}\,\varphi^{l}_{jm} \\ \dfrac{F_{lj}(r)}{r}\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\,\varphi^{l}_{jm} \end{pmatrix}$$

where

$$\varphi^{l}_{jm} = \begin{cases} \varphi^{(+)}_{jm} & \text{if } j = l + 1/2 \\ \varphi^{(-)}_{jm} & \text{if } j = l - 1/2 \end{cases} \qquad \text{with} \quad \boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\,\varphi^{(+)}_{jm} = \varphi^{(-)}_{jm}$$

spin and orbital angular momentum

$$\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\Sigma = \begin{pmatrix} \boldsymbol{L} & 0 \\ 0 & \boldsymbol{L} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

We know that

$$[\boldsymbol{J}, H] = 0 \quad \text{but} \quad [\boldsymbol{L}, H] \neq 0, \; [\boldsymbol{\Sigma}, H] \neq 0$$

Expectation value

$$\langle O \rangle \;\equiv\; \langle \psi_{jm}{}^{l} \,|\, O \,|\, \psi_{jm}{}^{l} \rangle$$

It holds that

$$\langle L_3 \rangle \;=\; m \left\{ \frac{2j-1}{2j} \int_0^\infty G_{lj}^2 \, dr \;+\; \frac{2j+3}{2(j+1)} \int_0^\infty F_{lj}^2 \, dr \right\}$$

$$\left\langle \frac{1}{2} \Sigma_3 \right\rangle \;=\; m \left\{ \frac{1}{2j} \int_0^\infty G_{lj}^2 \, dr \;-\; \frac{1}{2(j+1)} \int_0^\infty F_{lj}^2 \, dr \right\}$$

$$\langle J_3 \rangle \;=\; m \int_0^\infty \left[G_{lj}^2 + F_{lj}^2 \right] dr \;=\; m$$

Electron alone saturates the spins of hydrogen atom !

In this problem, there is no difference between

$$\boldsymbol{L}_{can} = \boldsymbol{r} \times \boldsymbol{p} \quad \Longleftrightarrow \quad \boldsymbol{L}_{mech} = \boldsymbol{r} \times (\boldsymbol{p} - e\,\boldsymbol{A}_\perp)$$

because $\langle \boldsymbol{A}_\perp \rangle = 0$ in the restricted Fock space of hydrogen w.f.

no transverse photon Fock components !

The situation is absolutely different for the nucleon spin problem of QCD.

strongly-coupled gauge system of quark and gluons !

The nucleon w.f. contains the Fock-components of transverse gluon $\boldsymbol{A}_\perp$.

Otherwise, we would have

$$g(x) \;=\; 0, \qquad \Delta g(x) \;=\; 0$$

In other words, nonzero gluon distributions means the existence of strong vector potential inside the nucleon, and the quarks necessarily undergo this background field so that they are not free partons.

The intrinsic OAM of quarks inside the nucleon is therefore

$$\boldsymbol{L}_{mech} \quad \texttt{not} \quad \boldsymbol{L}_{can}$$

We shall later show more convincing QCD argument to support this statement.