Perturbative QCD and Hard Processes

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The plan for my four lectures

- **The Goal:**

  To understand the strong interaction dynamics, and hadron structure, in terms of Quantum Chromo-dynamics (QCD)

- **The Plan (approximately):**

  Fundamentals of perturbative QCD, factorization, evolution,

  and single scale hard processes

  Two lecture

  Hard processes – observables with multiple momentum scales

  One lecture

  Hard processes – with identified polarization(s)

  One lecture

Quantum Chromo-dynamics (QCD)

= A quantum field theory of quarks and gluons =

- **Fields:**

$\psi_i^f(x)$ — **Quark fields: spin-½ Dirac fermion (like electron)**

Color triplet: $i = 1, 2, 3 = N_c$

Flavor: $f = u, d, s, c, b, t$

$A_{\mu,a}(x)$ — **Gluon fields: spin-1 vector field (like photon)**

Color octet: $a = 1, 2, ..., 8 = N_c^2 - 1$

- **QCD Lagrangian density:**

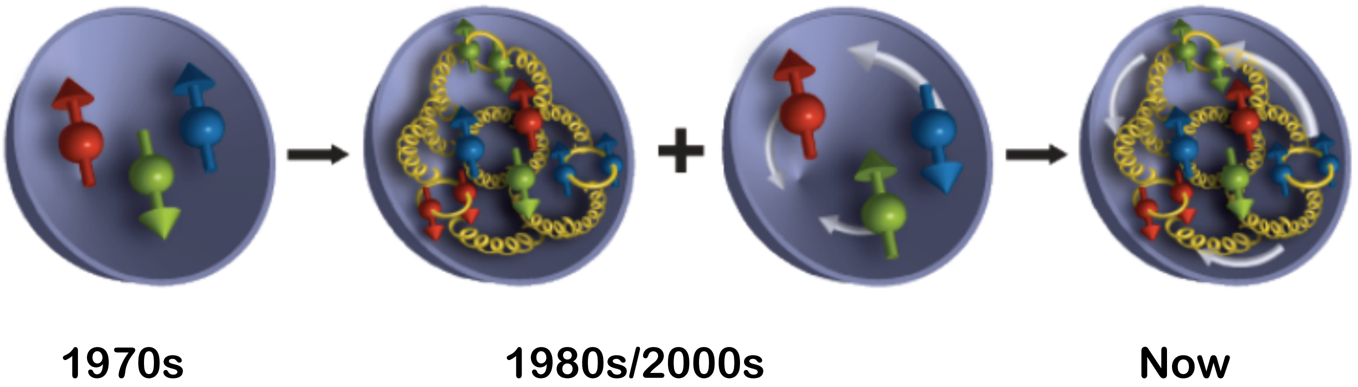
$$\mathcal{L}_{QCD}(\psi, A) = \sum_f \overline{\psi}_i^f \left[(i\partial_\mu \delta_{ij} - g A_{\mu,a}(t_a)_{ij})\gamma^\mu - m_f \delta_{ij} \right] \psi_j^f$$
$$- \frac{1}{4} \left[\partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} - g C_{abc} A_{\mu,b} A_{\nu,c} \right]^2$$
$$+ \text{ gauge fixing} + \text{ghost terms}$$

- **QCD Confinement:**

No free quarks or gluons ever been detected

The hadron and its internal structure

❑ Our understanding of the proton evolves



Hadron is a strongly interacting, relativistic bound state of quarks and gluons

❑ QCD bound states:

- ✧ Neither quarks nor gluons appear in isolation!
- ✧ Understanding such systems completely is still beyond the capability of the best minds in the world

❑ The great intellectual challenge:

Probe nucleon structure without "seeing" quarks and gluons?

Foundation of perturbative QCD

- Renormalization
  - QCD is renormalizable

Nobel Prize, 1999
't Hooft, Veltman

- Asymptotic freedom
  - weaker interaction at a shorter distance

Nobel Prize, 2004
Gross, Politzer, Welczek

- Infrared safety and factorization
  - calculable short distance dynamics
  - pQCD factorization – connect the partons to physical cross sections

J. J. Sakurai Prize, 2003
Mueller, Sterman

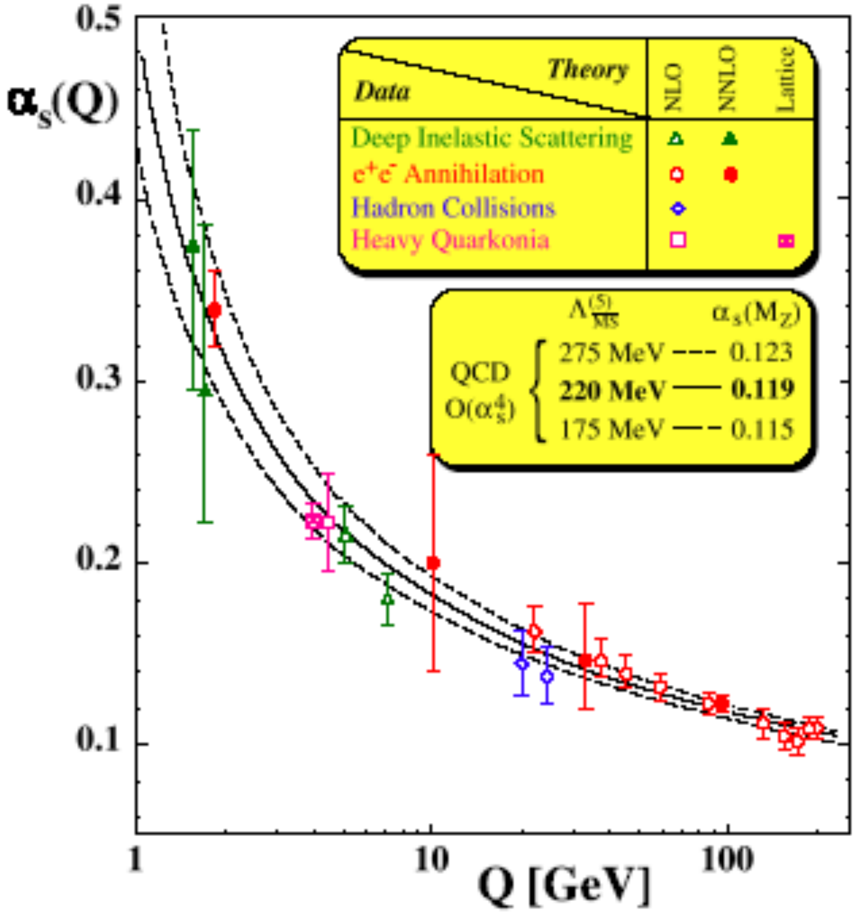
Look for infrared safe and factorizable observables!

QCD Asymptotic Freedom

- ❑ **QCD is a renormalizable theory**

- ❑ **Running coupling:**

$$\alpha_s(\mu_2) = \frac{\alpha_s(\mu_1)}{1 - \frac{\beta_1}{4\pi}\alpha_s(\mu_1)\ln\left(\frac{\mu_2^2}{\mu_1^2}\right)} \equiv \frac{4\pi}{-\beta_1 \ln\left(\frac{\mu_2^2}{\Lambda_{\rm QCD}^2}\right)}$$



μ_2 **and** μ_1 **not independent**

Asymptotic Freedom ⇔ antiscreening

$$\text{QCD:}\ \frac{\partial \alpha_s(Q^2)}{\partial \ln Q^2} = \beta(\alpha_s) < 0$$

Compare

$$\text{QED:}\ \frac{\partial \alpha_{EM}(Q^2)}{\partial \ln Q^2} = \beta(\alpha_{EM}) > 0$$

D.Gross, F.Willczek, Phys.Rev.Lett 30, (1973)
H.Politzer, Phys.Rev.Lett. 30, (1973)

2004 Nobel Prize in Physics

Effective quark mass

- **Running quark mass:**

$$m(\mu_2) = m(\mu_1)\exp\left[-\int_{\mu_1}^{\mu_2}\frac{d\lambda}{\lambda}[1+\gamma_m(g(\lambda))]\right]$$

Quark mass depend on the renormalization scale!

- **QCD running quark mass:**

$$m(\mu_2) \Rightarrow 0 \qquad \text{as } \mu_2 \to \infty \qquad\qquad \text{since} \;\; \gamma_m(g(\lambda)) > 0$$

- **Choice of renormalization scale:**

$\mu \sim Q$ **for small logarithms in the perturbative coefficients**

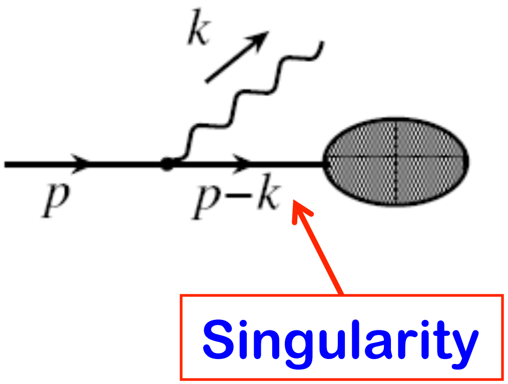
- **Light quark mass:** $m_f(\mu) \ll \Lambda_{\text{QCD}} \quad \text{for } f = u, d, \text{ even } s$

QCD perturbation theory (Q>>Λ_{QCD})
is effectively a massless theory

Infrared and collinear divergences

- Consider a general diagram:

$p^2 = 0, \quad k^2 = 0$ **for a massless theory**



✧ $k^\mu \to 0 \;\Rightarrow\; (p-k)^2 \to p^2 = 0$

⟹ **Infrared (IR) divergence**

✧ $k^\mu \,||\, p^\mu \;\Rightarrow\; k^\mu = \lambda p^\mu \quad \text{with} \;\; 0 < \lambda < 1$

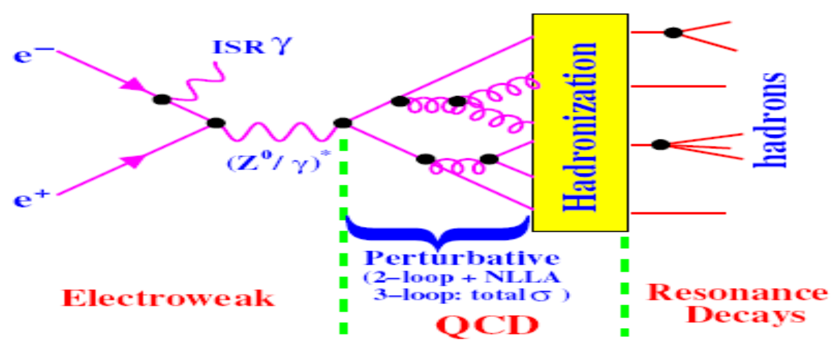
$\Rightarrow\; (p-k)^2 \to (1-\lambda)^2\, p^2 = 0$

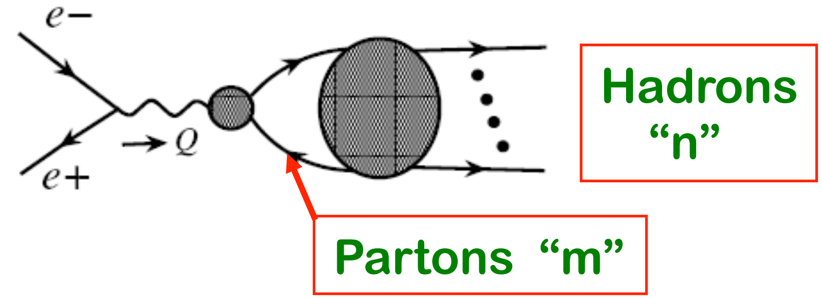
⟹ **Collinear (CO) divergence**

IR and CO divergences are generic problems of a massless perturbation theory

Observables not sensitive to hadronization

❑ e^+e^- ➔ hadron total cross section – no identified hadron!





If there is no quantum interference between partons and hadrons,

$$\sigma^{tot}_{e^+e^-\to\text{hadrons}} \propto \sum_n \mathrm{P}_{e^+e^-\to n} = \sum_n \sum_m \mathrm{P}_{e^+e^-\to m}\mathrm{P}_{m\to n} = \sum_m \mathrm{P}_{e^+e^-\to m} \sum_n \mathrm{P}_{m\to n} = 1$$

Unitarity

$$\sigma^{tot}_{e^+e^-\to\text{partons}} \propto \sum_m \mathrm{P}_{e^+e^-\to m}$$

$$\Rightarrow \quad \sigma^{tot}_{e^+e^-\to\text{hadrons}} = \sigma^{tot}_{e^+e^-\to\text{partons}}$$

Finite in perturbation theory – KLN theorem

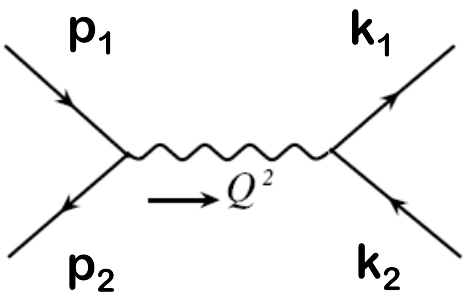
❑ e^+e^- ➔ parton total cross section:

$$\sigma^{tot}_{e^+e^-\to\text{partons}}(s=Q^2) = \sum_n \sigma^{(n)}(Q^2,\mu^2)\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n$$

Calculable in pQCD

Lowest order (LO) perturbative calculation

- **Lowest order Feynman diagram:**



- **Invariant amplitude square:**

$$|\bar{M}_{e^+e^-\to Q\bar{Q}}|^2 = e^4 e_Q^2 N_c \frac{1}{s^2}\frac{1}{2^2}\mathrm{Tr}\left[\gamma\cdot p_2\gamma^\mu\gamma\cdot p_1\gamma^\nu\right]$$
$$\times\mathrm{Tr}\left[\left(\gamma\cdot k_1+m_Q\right)\gamma_\mu\left(\gamma\cdot k_2-m_Q\right)\gamma_\nu\right]$$
$$= e^4 e_Q^2 N_c\frac{2}{s^2}\left[(m_Q^2-t)^2+(m_Q^2-u)^2+2m_Q^2 s\right]$$

$$s=(p_1+p_2)^2$$
$$t=(p_1-k_1)^2$$
$$u=(p_2-k_1)^2$$

- **Lowest order cross section:**

$$\frac{d\sigma_{e^+e^-\to Q\bar{Q}}}{dt}=\frac{1}{16\pi s^2}|\bar{M}_{e^+e^-\to Q\bar{Q}}|^2 \quad \text{where } s=Q^2$$

Threshold constraint

$$\sigma_2^{(0)}=\sum_Q\sigma_{e^+e^-\to Q\bar{Q}}=\sum_Q e_Q^2 N_c\frac{4\pi\alpha_{em}^2}{3s}\left[1+\frac{2m_Q^2}{s}\right]\sqrt{1-\frac{4m_Q^2}{s}}$$

One of the best tests for the number of colors

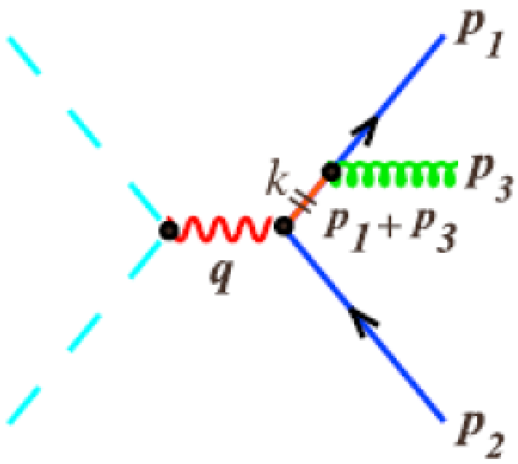
Next-to-leading order (NLO) contribution

❑ Real Feynman diagram:

$$x_i = \frac{E_i}{\sqrt{s}/2} = \frac{2p_i.q}{s} \qquad \text{with } i = 1,2,3$$

$$\sum_i x_i = \frac{2\left(\sum_i p_i\right).q}{s} = 2$$

$$2(1-x_1) = x_2 x_3 (1-\cos\theta_{23}), \quad cycl.$$



+ crossing

❑ Contribution to the cross section:

$$\frac{1}{\sigma_0}\frac{d\sigma_{e^+e^-\to Q\bar{Q}g}}{dx_1 dx_2} = \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

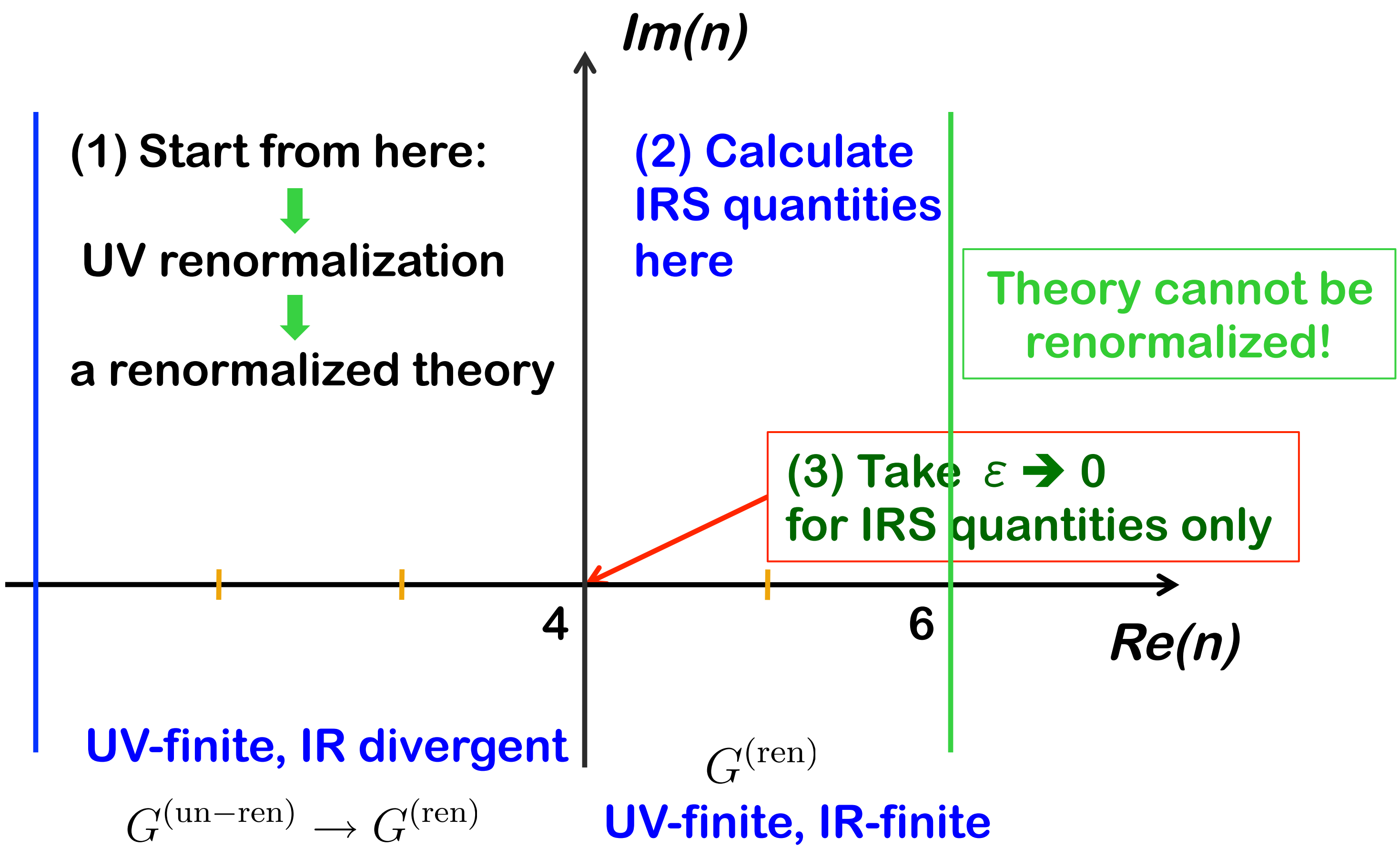
IR as $x3 \to 0$
CO as $\theta_{13} \to 0$
$\theta_{23} \to 0$

Divergent as $x_i \to 1$

Need the virtual contribution and a regulator!

How does dimensional regularization work?

- ❑ **Complex *n*-dimensional space:** $\int d^n k \, F(k,Q)$



Dimensional regularization for both IR and CO

❑ NLO with a dimensional regulator:

✧ **Real:** $$\sigma_{3,\varepsilon}^{(1)} = \sigma_{2,\varepsilon}^{(0)} \frac{4}{3}\left(\frac{\alpha_s}{\pi}\right)\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\left[\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-3\varepsilon)}\right]\left[\frac{1}{\varepsilon^2}+\frac{3}{2\varepsilon}+\frac{19}{4}\right]$$

✧ **Virtual:**

$$\sigma_{2,\varepsilon}^{(1)} = \sigma_{2,\varepsilon}^{(0)} \frac{4}{3}\left(\frac{\alpha_s}{\pi}\right)\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\left[\frac{\Gamma(1-\varepsilon)^2\,\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\right]\left[-\frac{1}{\varepsilon^2}-\frac{3}{2\varepsilon}+\frac{\pi^2}{2}-4\right]$$

✧ **NLO:** $$\sigma_{3,\varepsilon}^{(1)} + \sigma_{2,\varepsilon}^{(1)} = \sigma_2^{(0)}\left[\frac{\alpha_s}{\pi}+O(\varepsilon)\right]$$

No ε dependence!

✧ **Total:** $$\sigma^{\text{tot}} = \sigma_2^{(0)} + \sigma_{3,\varepsilon}^{(1)} + \sigma_{2,\varepsilon}^{(1)} + O\left(\alpha_s^2\right) = \sigma_2^{(0)}\left[1+\frac{\alpha_s}{\pi}\right] + O\left(\alpha_s^2\right)$$

σ^{tot} is Infrared Safe!

σ^{tot} is independent of the choice of IR and CO regularization

Go beyond the inclusive total cross section?

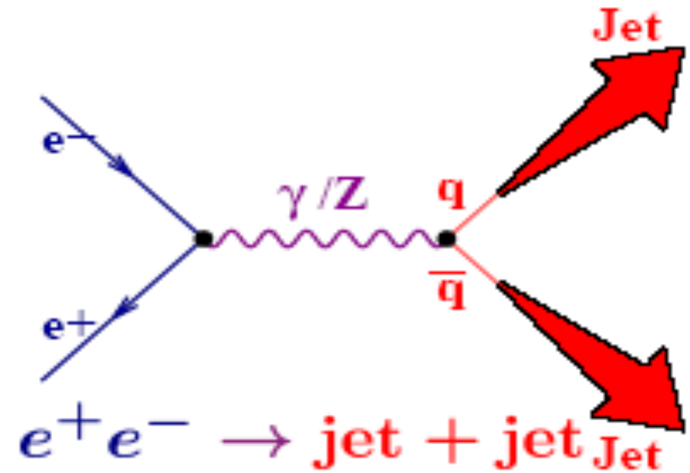
Jets in e^+e^- collisions – trace of partons

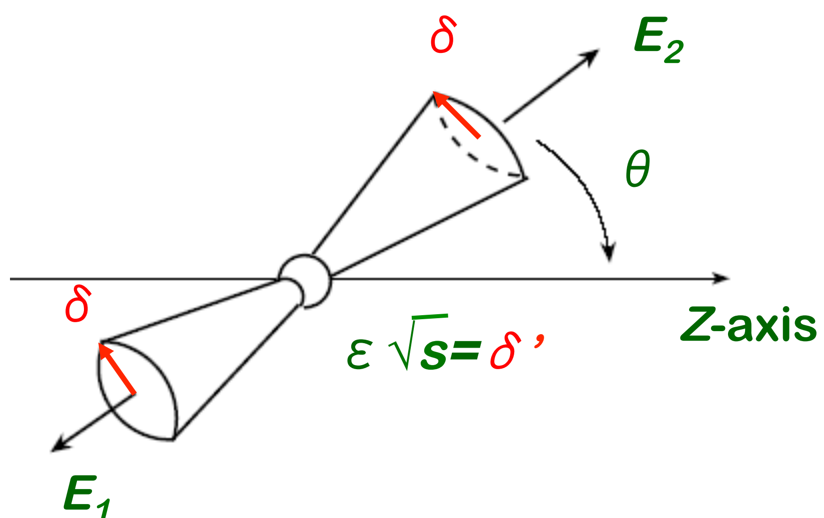
- ❑ **Jets – "total" cross-section with a limited phase-space**

  No identified hadron!

- ❑ **Q: will IR cancellation be completed?**
  - ✧ Leading partons are moving away from each other
  - ✧ Soft gluon interactions should not change the direction of an energetic parton → a "jet" – "trace" of a parton

- ❑ **Many Jet algorithms**

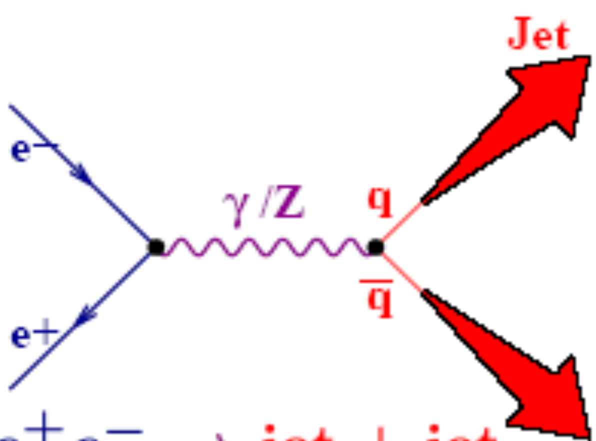




Sterman-Weinberg Jet

An early clean two-jet event

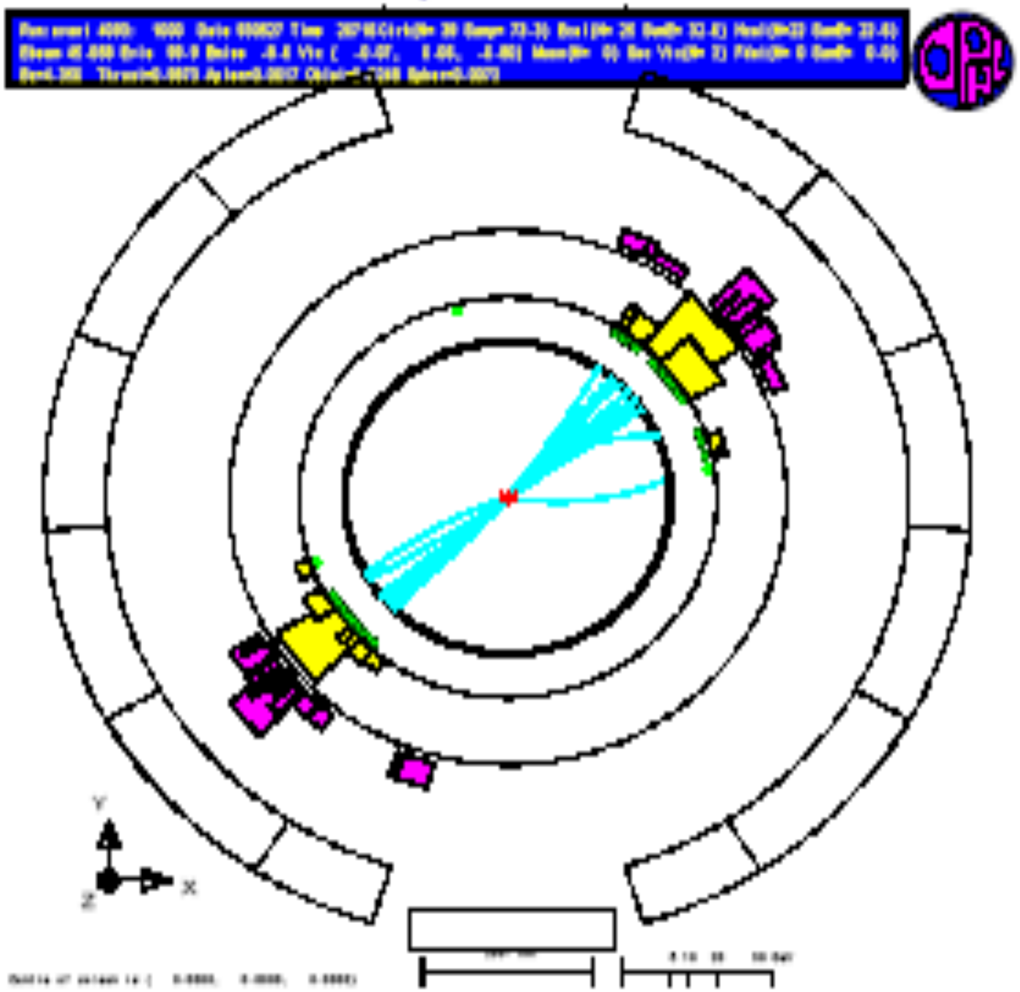
Lowest order ($\mathcal{O}(\alpha^2\alpha_s^0)$):



$e^+e^- \rightarrow \text{jet} + \text{jet}$

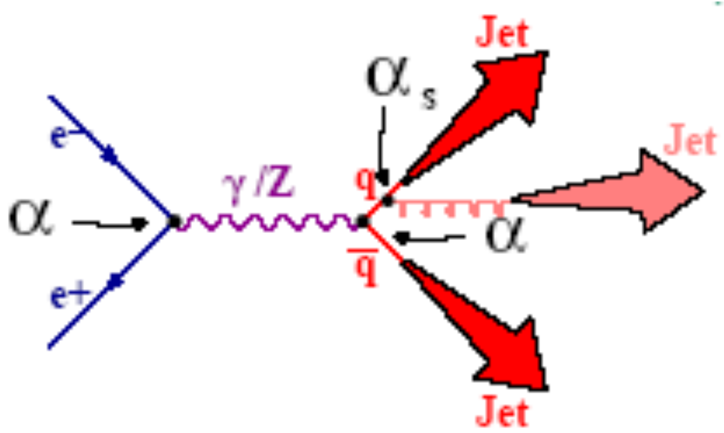
A clean trace of two partons – a pair of quark and antiquark

LEP ($\sqrt{s} = 90 - 205$ GeV)



Discovery of a gluon jet

First order in QCD ($\mathcal{O}(\alpha^2\alpha_s^1)$):



Reputed to be the first three-jet event from TASSO

TASSO Collab., Phys. Lett. B86 (1979) 243

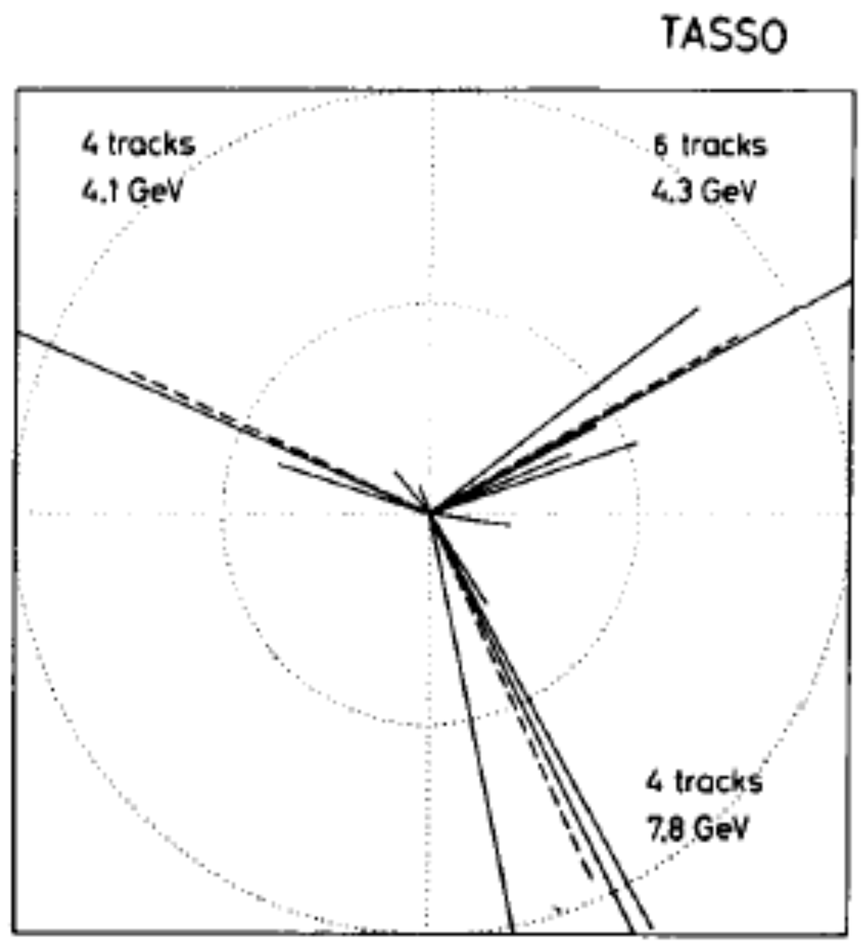
MARK-J Collab., Phys. Rev. Lett. 43 (1979) 830

PLUTO Collab., Phys. Lett. B86 (1979) 418

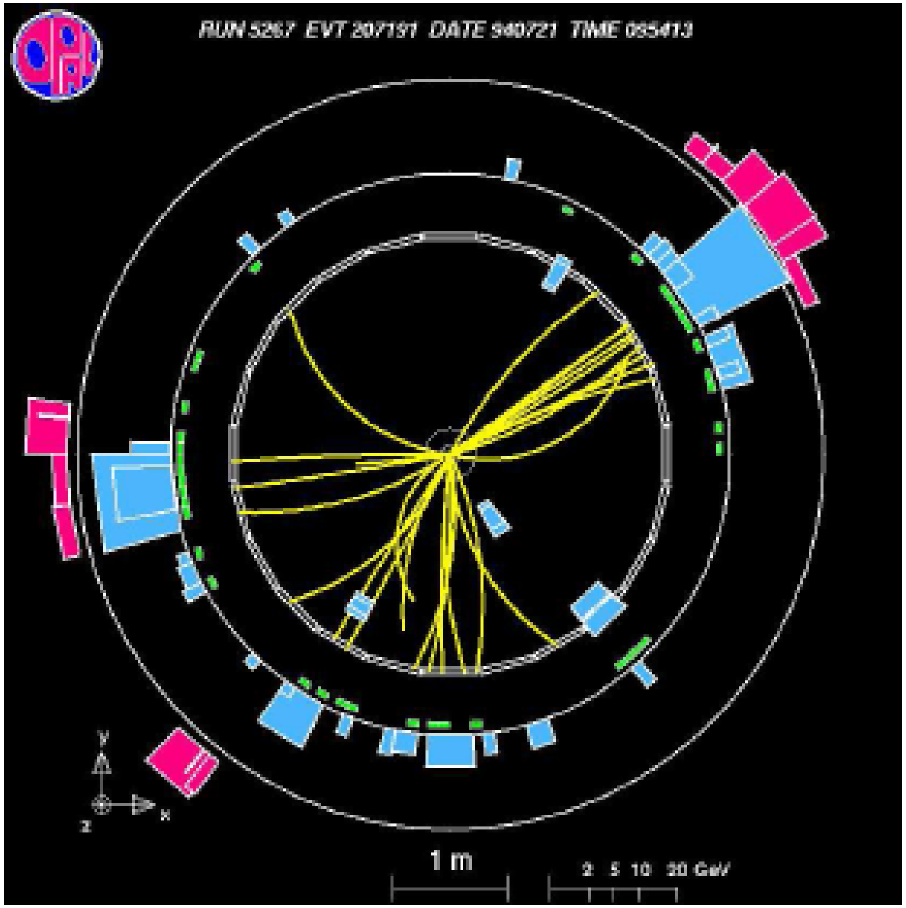
JADE Collab., Phys. Lett. B91 (1980) 142

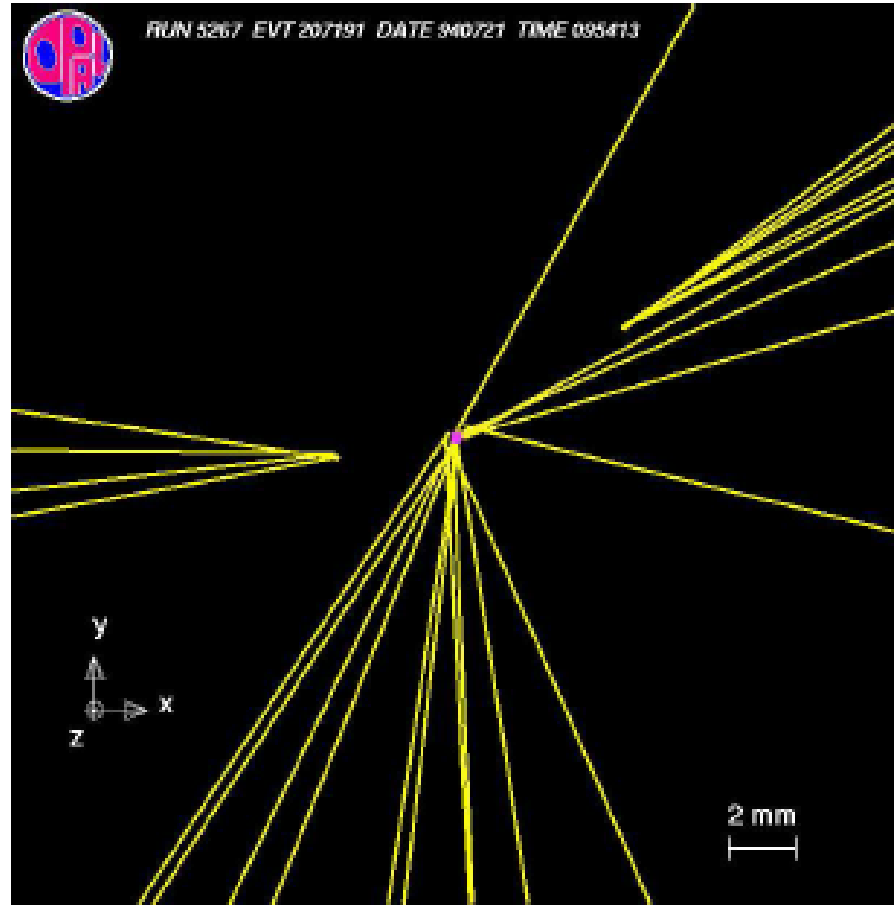
PETRA e^+e^- storage ring at DESY:

$E_{c.m.} \gtrsim 15$ GeV



Tagged three-jet event from LEP





Gluon Jet

Basics of jet finding algorithms

❑ Recombination jet algorithms (almost all e+e- colliders):

Recombination metric: $y_{ij} = \frac{M_{ij}^2}{E_{c.m.}^2}$ $\quad M_{ij}^2 = 2\min(E_i^2, E_j^2)(1-\cos\theta_{ij})$

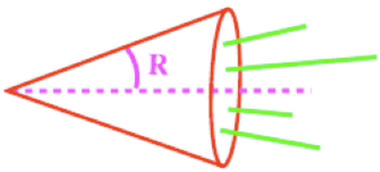
for Durham k_T

- ✧ different algorithm = different choice of M_{ij}^2 :
- ✧ Combine the particle pair (i,j) with the smallest y_{ij} : $(i,j) \to k$

$$e.g.\ \text{E scheme}: \quad p_k = p_i + p_j$$

- ✧ iterate until all remaining pairs satisfy: $y_{ij} > y_{cut}$

❑ Cone jet algorithms (CDF, …, colliders):



- ✧ Cluster all particles into a cone of half angle R into a jet:
- ✧ Require a minimum visible jet energy: $E_{jet} > \epsilon$

Recombination metric: $d_{ij} = \min\left(k_{T_i}^{2p}, k_{T_j}^{2p}\right)\frac{\Delta_{ij}^2}{R^2}$

with $\Delta_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2$

- ✧ Classical choices: ***p=1*** – "***k_T*** algorithm", ***p= -1*** – "anti-***k_T***", …

Infrared safety for restricted cross sections

- For any observable with a phase space constraint, Γ,

$$d\sigma(\Gamma) \equiv \frac{1}{2!}\int d\Omega_2 \frac{d\sigma^{(2)}}{d\Omega_2}\Gamma_2(k_1,k_2)$$
$$+ \frac{1}{3!}\int d\Omega_3 \frac{d\sigma^{(3)}}{d\Omega_3}\Gamma_3(k_1,k_2,k_3)$$
$$+ \ldots$$
$$+ \frac{1}{n!}\int d\Omega_n \frac{d\sigma^{(n)}}{d\Omega_n}\Gamma_n(k_1,k_2,\ldots,k_n) + \ldots$$

Where $\Gamma_n(k_1,k_2,\ldots,k_n)$ are constraint functions and invariant under Interchange of n-particles

- Conditions for IRS of dσ(Γ):

$$\Gamma_{n+1}\left(k_1,k_2,\ldots,(1-\lambda)k_n^\mu,\lambda k_n^\mu\right) = \Gamma_n\left(k_1,k_2,\ldots,k_n^\mu\right) \quad \text{with} \quad 0 \le \lambda \le 1$$

Physical meaning:

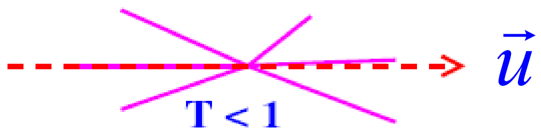
Measurement cannot distinguish a state with a zero/collinear momentum parton from a state without the parton

Special case: $\Gamma_n\left(k_1,k_2,\ldots,k_n\right) = 1 \;\; \text{for all } n \quad \Rightarrow \quad \sigma^{(\text{tot})}$

Another example: Thrust distribution

❑ **Thrust axis:** $\vec{u}$

$$T_n\left(p_1^\mu, p_2^\mu, \ldots, p_n^\mu\right) = \max_{\vec{u}}\left(\frac{\sum_{i=1}^{n} \vec{p}_i \cdot \vec{u}}{\sum_{i=1}^{n} \left|\vec{p}_i\right|}\right)$$



❑ **Phase space constraint:**

$$\Gamma_n\left(p_1^\mu, p_2^\mu, \ldots, p_n^\mu\right) = \delta\left(T - T_n\left(p_1^\mu, p_2^\mu, \ldots, p_n^\mu\right)\right)$$

✧ Contribution from p=0 particles drops out the sum

✧ Replace two collinear particles by one particle does not change the thrust

$$|(1-\lambda)\,\vec{p}_n \cdot \vec{u}| + |\lambda\,\vec{p}_n \cdot \vec{u}| = |\vec{p}_n \cdot \vec{u}|$$

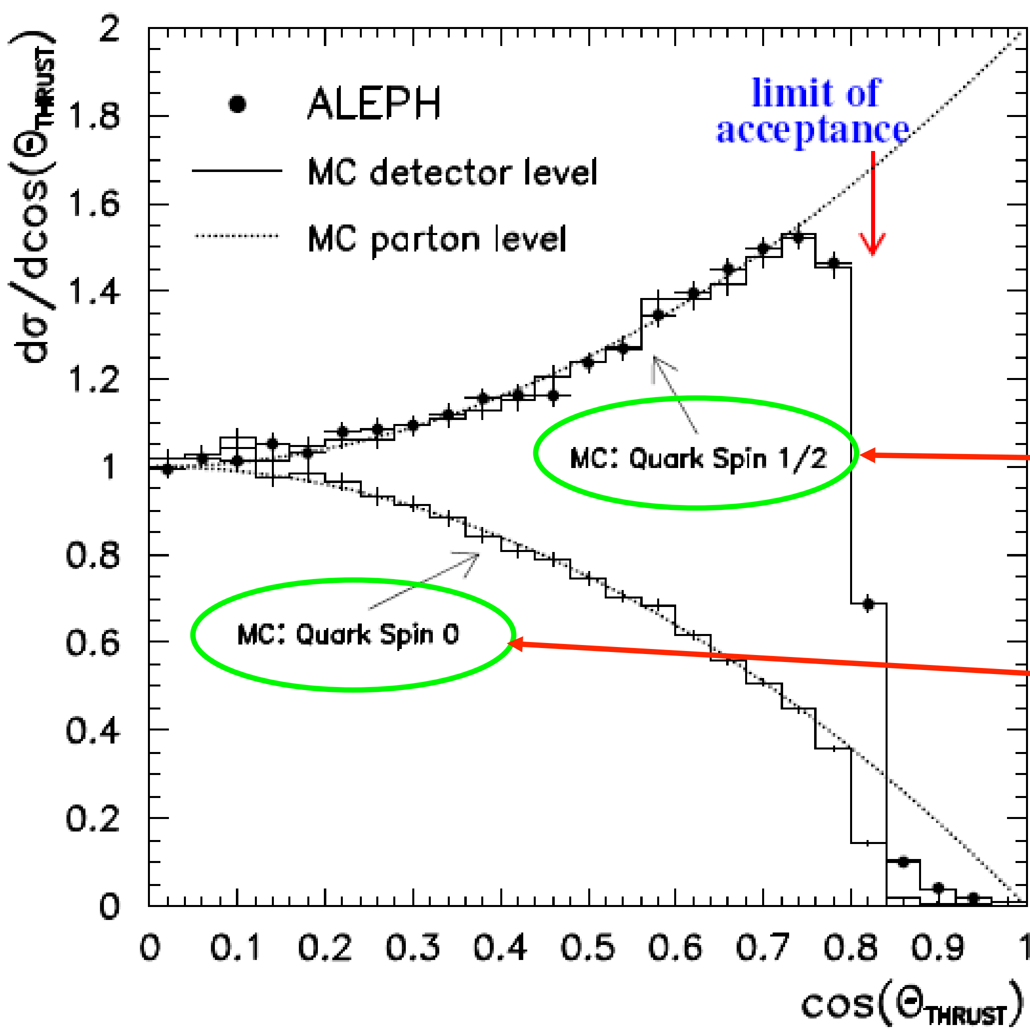
and

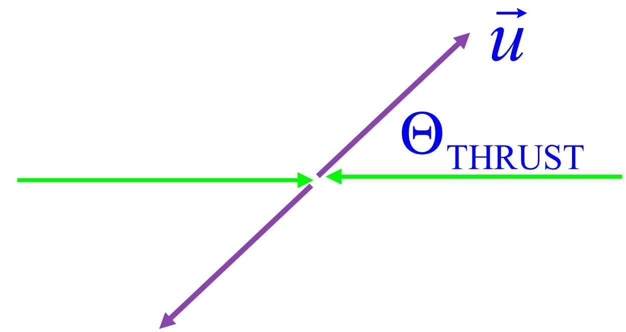
$$|(1-\lambda)\,\vec{p}_n| + |\lambda\,\vec{p}_n| = |\vec{p}_n|$$

Another test of quark spin

- Angle between the thrust axis and the beam axis:

[ALEPH Collab., Phys. Rep. 294 (1998) 1]





At LO:

$1 + \cos^2 \Theta_{THRUST}$

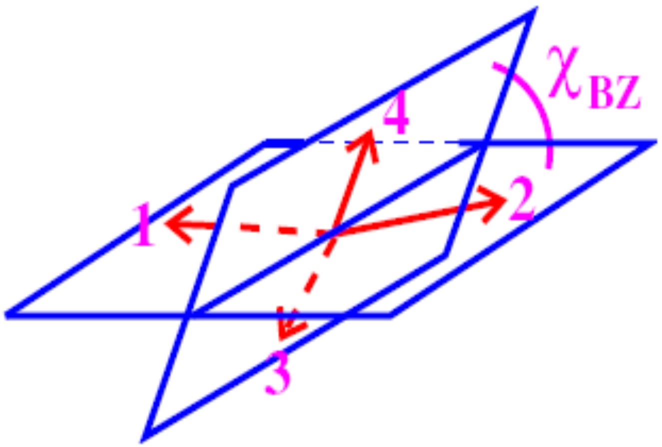
$1 - \cos^2 \Theta_{THRUST}$

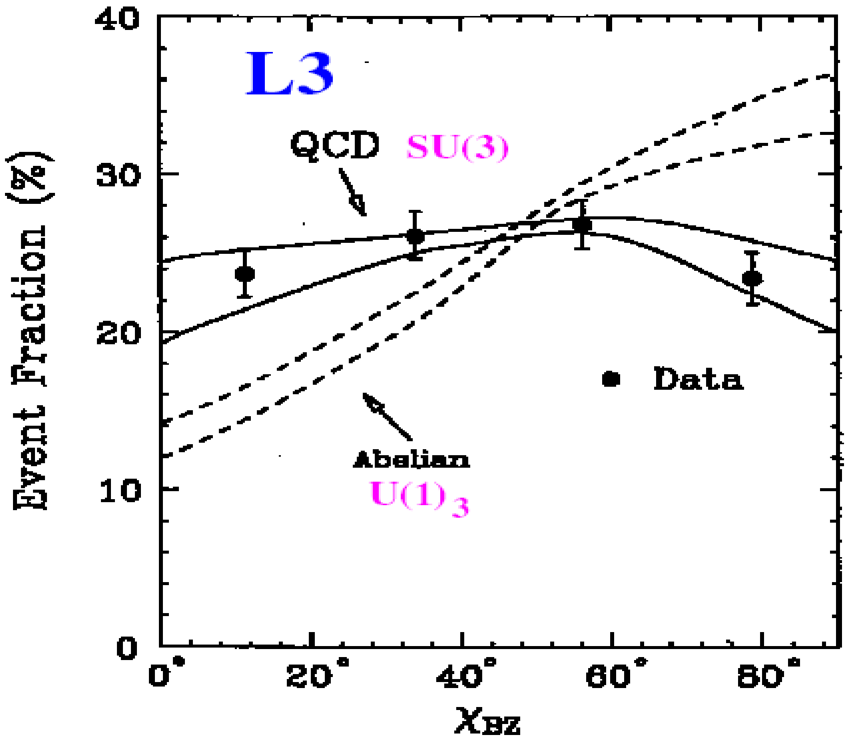
Another test of SU(3) color

- **Select 4-jet events:** $E_1 > E_2 > E_3 > E_4$

  jet-3 and jet-4 are more likely from radiation

- **Bengtsson-Zerwas angle:**





The harder question

❑ **Question:**

How to test QCD in a reaction with identified hadron(s)?
– to probe the quark-gluon structure of the hadron

❑ **Facts:**

Hadronic scale ~ 1/fm ~ Λ_{QCD} is non-perturbative

Cross section involving identified hadron(s) is not IR safe and is not perturbatively calculable!

❑ **Solution – Factorization:**

✧ **Isolate the calculable dynamics of quarks and gluons**

✧ **Connect quarks and gluons to hadrons via non-perturbative but universal distribution functions**

– provide information on the partonic structure of the hadron

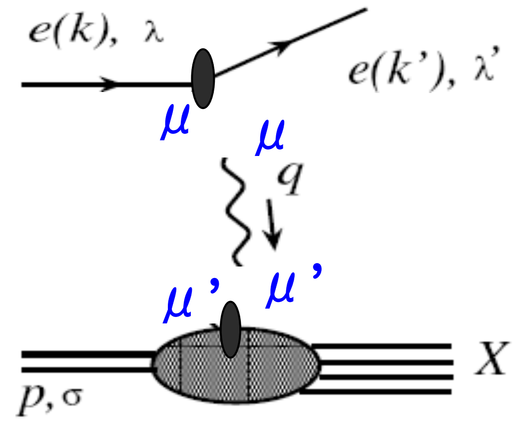
Inclusive lepton-hadron DIS – one hadron

❑ Scattering amplitude:

$$M(\lambda,\lambda';\sigma,q) = \bar{u}_{\lambda'}(k')\left[-ie\gamma_\mu\right]u_\lambda(k)$$

$$* \left(\frac{i}{q^2}\right)\left(-g^{\mu\mu'}\right)$$

$$* \left\langle X \left| eJ^{em}_{\mu'}(0)\right| p,\sigma\right\rangle$$



❑ Cross section:

$$d\sigma^{\rm DIS} = \frac{1}{2s}\left(\frac{1}{2}\right)^2 \sum_X \sum_{\lambda,\lambda',\sigma} \left|M(\lambda,\lambda';\sigma,q)\right|^2 \left[\prod_{i=1}^{X} \frac{d^3 l_i}{(2\pi)^3\, 2E_i}\right] \frac{d^3k'}{(2\pi)^3\, 2E'} (2\pi)^4 \delta^4\left(\sum_{i=1}^{X} l_i + k' - p - k\right)$$

$$\Rightarrow \quad E'\frac{d\sigma^{\rm DIS}}{d^3k'} = \frac{1}{2s}\left(\frac{1}{Q^2}\right)^2 L^{\mu\nu}(k,k')W_{\mu\nu}(q,p)$$

❑ Leptonic tensor:

– known from QED $\quad L^{\mu\nu}(k,k') = \frac{e^2}{2\pi^2}\left(k^\mu k'^\nu + k^\nu k'^\mu - k\cdot k' g^{\mu\nu}\right)$

DIS structure functions

- **Hadronic tensor:**

$$W_{\mu\nu}(q,p,S) = \frac{1}{4\pi}\int d^4z\ e^{iq\cdot z}\ \langle p,S|J_\mu^\dagger(z)J_\nu(0)|p,S\rangle$$

- **Symmetries:**

  - Parity invariance (EM current) → $W_{\mu\nu} = W_{\nu\mu}$ sysmetric for spin avg.
  - Time-reversal invariance → $W_{\mu\nu} = W_{\mu\nu}^*$ real
  - Current conservation → $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$

$$W_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)F_1\left(x_B,Q^2\right) + \frac{1}{p\cdot q}\left(p_\mu - q_\mu\frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu\frac{p\cdot q}{q^2}\right)F_2\left(x_B,Q^2\right)$$

$$+iM_p\varepsilon^{\mu\nu\rho\sigma}q_\rho\left[\frac{S_\sigma}{p.q}g_1\left(x_B,Q^2\right) + \frac{(p.q)S_\sigma - (S.q)p_\sigma}{(p.q)^2}g_2\left(x_B,Q^2\right)\right]$$

$$Q^2 = -q^2 \qquad x_B = \frac{Q^2}{2p\cdot q}$$

- **Structure functions – infrared sensitive:**

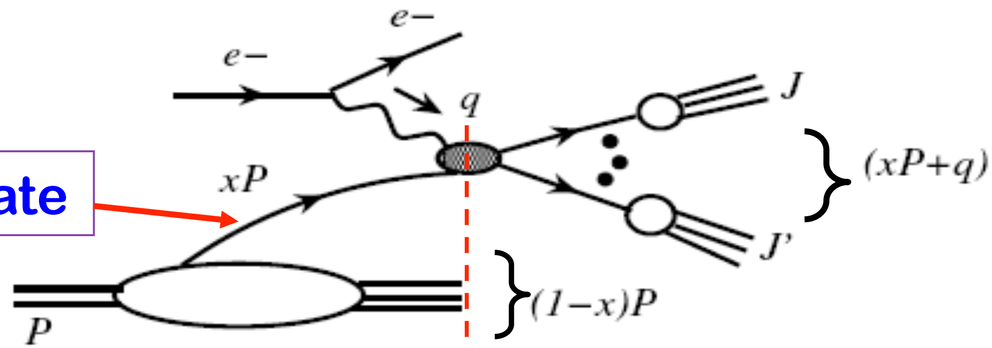
$$F_1\left(x_B,Q^2\right), F_2\left(x_B,Q^2\right), g_1\left(x_B,Q^2\right), g_2\left(x_B,Q^2\right)$$

No QCD parton dynamics used in above derivation!

Picture of factorization for DIS

- **Time evolution:**

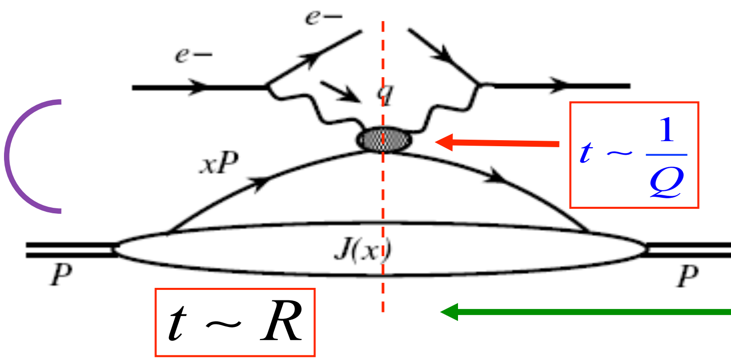
Long-lived parton state



Time: "Past" **Now** "Future"

- **Unitarity – summing over all hard jets:**

$$\sigma_{\rm tot}^{\rm DIS} \propto \mathrm{Im}\Big($$



$$\Big)$$

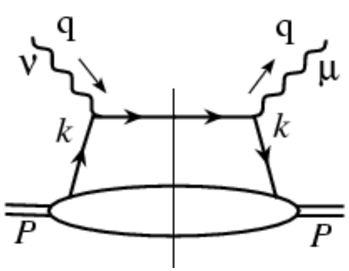
Not IR safe

Interaction between the "past" and "now" are suppressed!

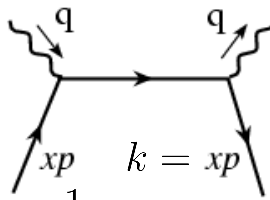
Collinear factorization - approximation

- Collinear approximation, if $Q \sim xp\cdot n \gg k_T, \sqrt{k^2}$

  – Lowest order:

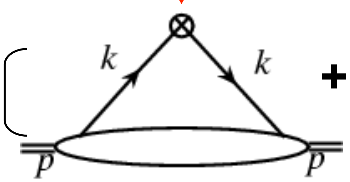


$\approx \Big[$



$\frac{1}{2}\gamma\cdot(xp)$

$+O\left(\frac{k_T^2}{Q^2}\right)\Big] \otimes \Big[$



+UVCT $\Big]$

$\int \frac{dx}{x}$ $\quad \frac{\gamma\cdot n}{2p\cdot n}\,\delta\left(x-\frac{k\cdot n}{p\cdot n}\right)\frac{d^4k}{(2\pi)^4}$

Same as elastic x-section

Scheme dependence

Parton's transverse momentum is integrated into parton distributions, and provides a scale of power corrections

- DIS limit: $\nu, Q^2 \to \infty$, while x_B fixed

  ⇨ **Feynman's parton model and Bjorken scaling**

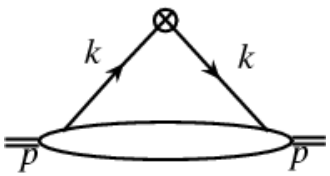
$$F_2(x_B,Q^2) = x_B \sum_f e_f^2\,\phi_f(x_B) = 2x_B F_1(x_B,Q^2)$$

Spin-½ parton!

- Corrections: $\mathcal{O}(\alpha_s) + \mathcal{O}\left(\langle k^2\rangle/Q^2\right)$

Parton distribution functions (PDFs)

- **PDFs as matrix elements of two parton fields:**



– combine the amplitude & its complex-conjugate

$$\phi_{q/h}(x,\mu^2) = \int \frac{p^+ dy^-}{2\pi}\, e^{ixp^+y^-} \langle h(p)|\bar{\psi}_q(0)\, \frac{\gamma^+}{2p^+}\, \psi_q(y^-)|h(p)\rangle\, \mathcal{Z}_{\mathcal{O}}(\mu^2)$$

$|h(p)\rangle$ **can be a hadron, or a nucleus, or a parton state!**

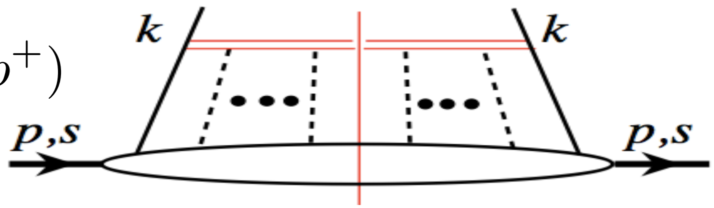
But, it is not gauge invariant! $\psi(x) \to e^{i\alpha_a(x)t_a}\psi(x) \quad \bar{\psi}(x) \to \bar{\psi}(x)e^{-i\alpha_a(x)t_a}$

– need a gauge link:

$$\phi_{q/h}(x,\mu^2) = \int \frac{dy^-}{2\pi}\, e^{ixp^+y^-} \langle h(p)|\bar{\psi}_q(0) \left[\mathcal{P}e^{-ig\int_0^{y^-} d\eta^- A^+(\eta^-)}\right] \frac{\gamma^+}{2p^+}\, \psi_q(y^-)|h(p)\rangle \mathcal{Z}_{\mathcal{O}}(\mu^2)$$

– corresponding diagram in momentum space:

$$\int \frac{d^4k}{(2\pi)^4}\, \delta(x - k^+/p^+)$$



$+\,\mathrm{UVCT}(\mu^2)$

μ -dependence

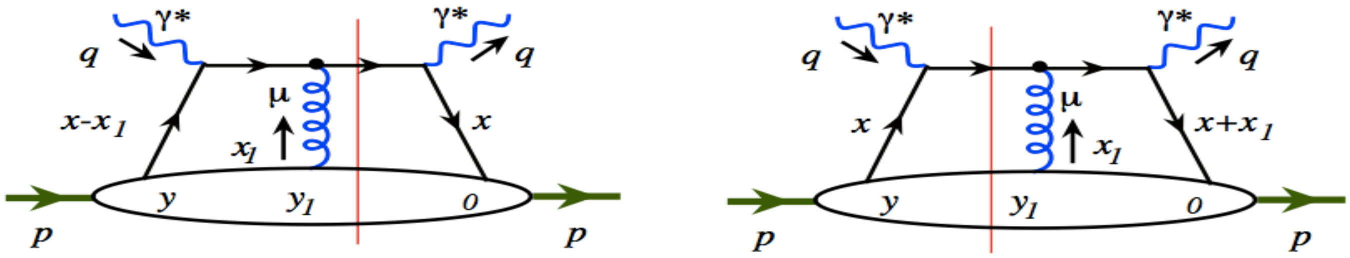
Universality – process independence – predictive power

- **Physics interpretation:**

Probability density to find a quark of momentum fraction x with all k_T

Gauge link – 1st order in coupling "g"

❑ Longitudinal gluon:



❑ Left diagram:

$$\int dx_1 \left[\int \frac{p^+ dy_1^-}{2\pi}\, \mathrm{e}^{ix_1p^+(y_1^- - y^-)} n\cdot A^a(y_1^-)\right] \mathcal{M}(-igt^a)\frac{\gamma\cdot p}{p^+}\frac{i((x-x_1-x_B)\gamma\cdot p + (Q^2/2x_Bp^+)\gamma\cdot n)}{(x-x_1-x_B)Q^2/x_B + i\epsilon}$$

$$= g\int \frac{dy_1^-}{2\pi}\, n\cdot A^a(y_1^-)t^a \left[\int dx_1 \frac{1}{-x_1+i\epsilon}\, \mathrm{e}^{ix_1p^+(y_1^- - y^-)}\right] \mathcal{M} = -ig\int_{y^-}^{\infty} dy_1^-\, n\cdot A(y_1^-)\mathcal{M}$$

❑ Right diagram:

$$\int dx_1 \left[\int \frac{p^+ dy_1^-}{2\pi}\, \mathrm{e}^{ix_1p^+y_1^-} n\cdot A^a(y_1^-)\right] \frac{-i((x+x_1-x_B)\gamma\cdot p + (Q^2/2x_Bp^+)\gamma\cdot n)}{(x+x_1-x_B)Q^2/x_B - i\epsilon}(+igt^a)\,\frac{\gamma\cdot p}{p^+}\,\mathcal{M}$$

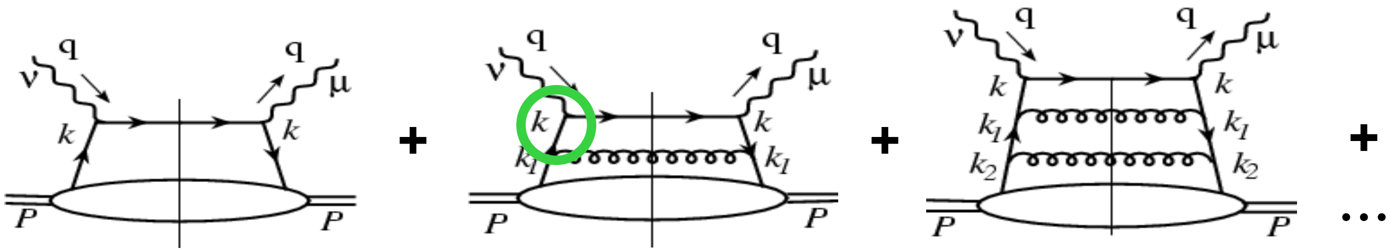
$$= g\int \frac{dy_1^-}{2\pi}\, n\cdot A^a(y_1^-)t^a \left[\int dx_1 \frac{1}{x_1-i\epsilon}\, \mathrm{e}^{ix_1p^+y_1^-}\right]\mathcal{M} \;= ig\int_0^{\infty} dy_1^-\, n\cdot A(y_1^-)\,\mathcal{M}$$

❑ Total contribution:

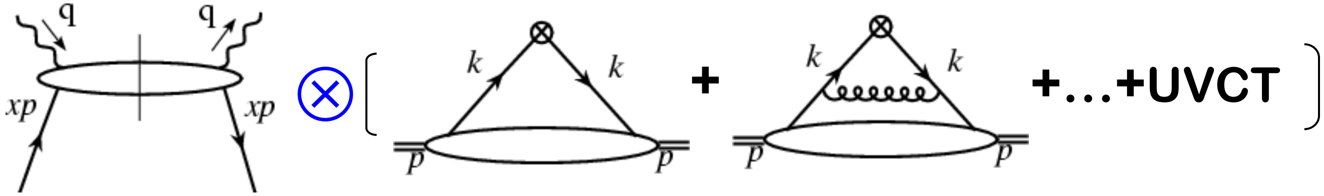
$$-ig\left[\int_0^{\infty} - \int_{y^-}^{\infty}\right] dy_1^-\, n\cdot A(y_1^-)\,\mathcal{M}_{\mathrm{LO}}$$

QCD high order corrections

- QCD corrections: pinch singularities in $\int d^4 k_i$



- Logarithmic contributions into parton distributions:



$$F_2(x_B, Q^2) = \sum_f C_f\left(\frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s\right) \otimes \varphi_f\left(x, \mu_F^2\right) + O\left(\frac{\Lambda_{\rm QCD}^2}{Q^2}\right)$$

- Factorization scale: μ_F^2

→ To separate the collinear from non-collinear contribution

Recall: renormalization scale to separate local from non-local contribution

Dependence on factorization scale

- Physical cross sections should not depend on the factorization scale

$$\mu_F^2 \frac{d}{d\mu_F^2} F_2(x_B, Q^2) = 0$$

→ Evolution (differential-integral) equation for PDFs

$$\sum_f \left[\mu_F^2 \frac{d}{d\mu_F^2} C_f\left(\frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s\right)\right] \otimes \varphi_f\left(x, \mu_F^2\right) + \sum_f C_f\left(\frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s\right) \otimes \mu_F^2 \frac{d}{d\mu_F^2} \varphi_f\left(x, \mu_F^2\right) = 0$$

- PDFs and coefficient functions share the same logarithms

PDFs: $\log\left(\mu_F^2/\mu_0^2\right)$ or $\log\left(\mu_F^2/\Lambda_{\mathrm{QCD}}^2\right)$

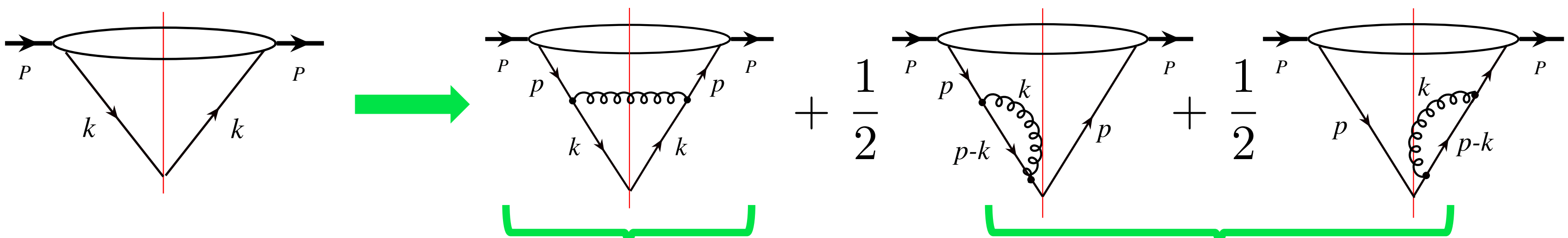
Coefficient functions: $\log\left(Q^2/\mu_F^2\right)$ or $\log\left(Q^2/\mu^2\right)$

→ DGLAP evolution equation:

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} \varphi_i(x, \mu_F^2) = \sum_j P_{i/j}\left(\frac{x}{x'}, \alpha_s\right) \otimes \varphi_j(x', \mu_F^2)$$

Calculation of evolution kernels

- Evolution kernels are process independent
  - ✧ Parton distribution functions are universal
  - ✧ Could be derived in many different ways
- Extract from calculating parton PDFs' scale dependence



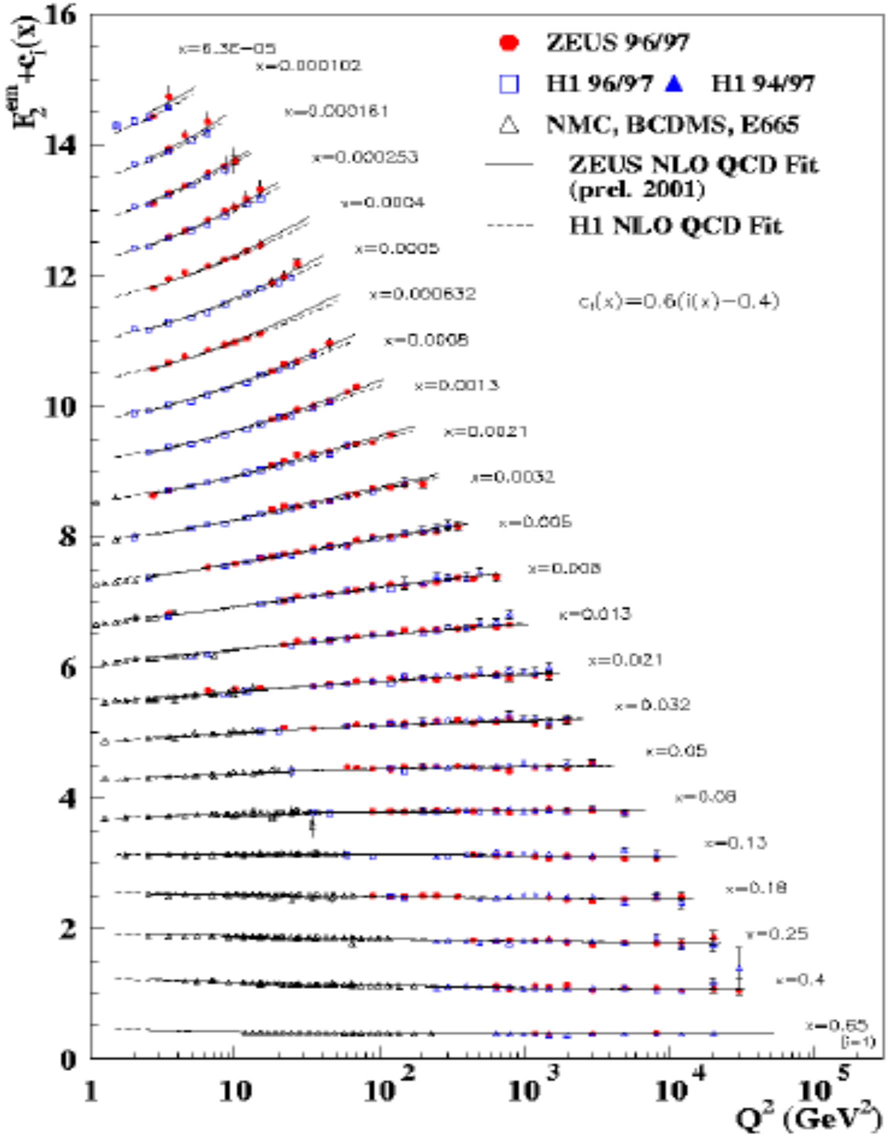
Collins, Qiu, 1989

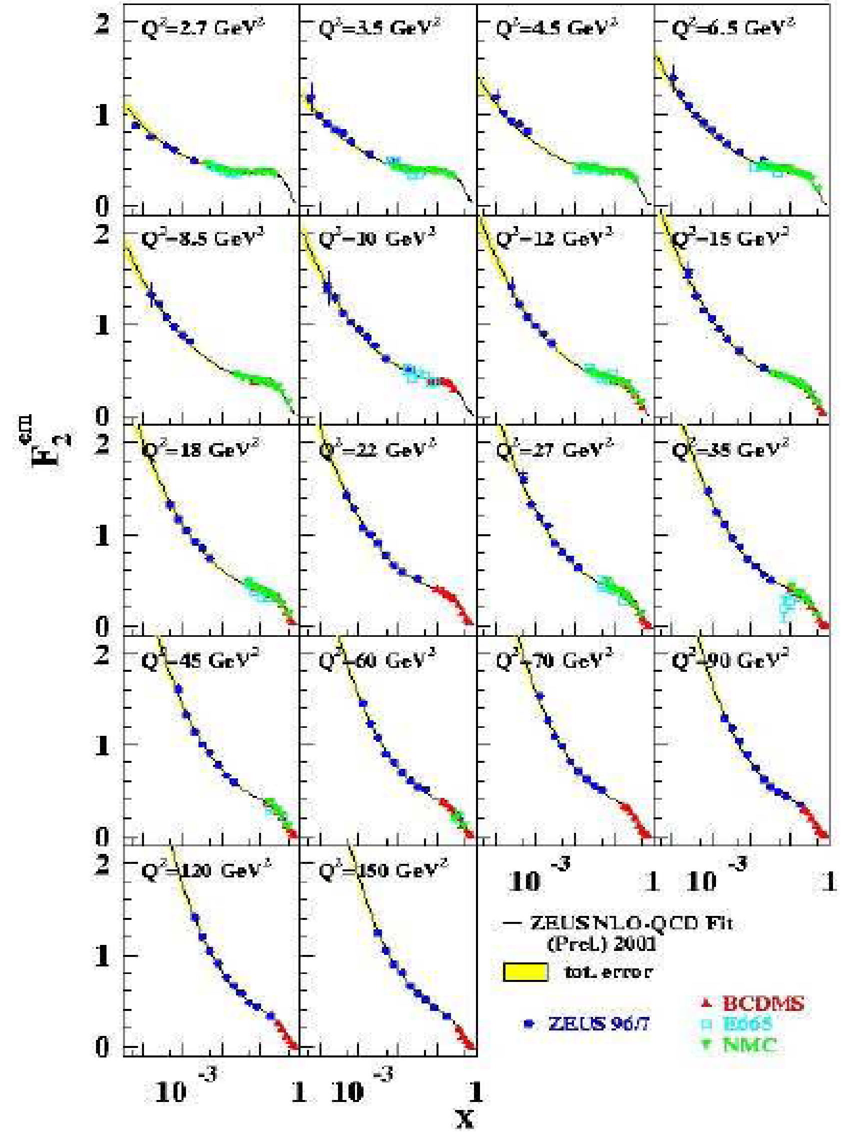
$$Q^2\frac{d}{dQ^2}q_i(x,Q^2)=\frac{\alpha_s}{2\pi}\int_x^1\frac{dx_1}{x_1}q_i(x_1,Q^2)\gamma_{qq}\left(\frac{x}{x_1}\right)-\frac{\alpha_s}{2\pi}q_i(x,Q^2)\int_0^1 dz\,\gamma_{qq}(z)$$

Change "Gain" "Loss"

  - ✧ Same is true for gluon evolution, and mixing flavor terms
- One can also extract the kernels from the CO divergence of partonic cross sections

Scaling and scaling violation





Q²-dependence is a prediction of pQCD calculation

How to calculate the perturbative parts?

- Use DIS structure function F_2 as an example:

$$F_{2h}(x_B,Q^2)=\sum_{q,f}C_{q/f}\left(\frac{x_B}{x},\frac{Q^2}{\mu^2},\alpha_s\right)\otimes\varphi_{f/h}\left(x,\mu^2\right)+O\left(\frac{\Lambda^2_{\rm QCD}}{Q^2}\right)$$

✧ Apply the factorized formula to parton states: $h\to q$

Feynman diagrams → $$F_{2q}(x_B,Q^2)=\sum_{q,f}C_{q/f}\left(\frac{x_B}{x},\frac{Q^2}{\mu^2},\alpha_s\right)\otimes\varphi_{f/q}\left(x,\mu^2\right)$$ ← Feynman diagrams

✧ Express both SFs and PDFs in terms of powers of α_s:

0th order: $$F^{(0)}_{2q}(x_B,Q^2)=C^{(0)}_q(x_B/x,Q^2/\mu^2)\otimes\varphi^{(0)}_{q/q}\left(x,\mu^2\right)$$

⇒ $$C^{(0)}_q(x)=F^{(0)}_{2q}(x) \qquad \varphi^{(0)}_{q/q}\left(x\right)=\delta_{qq}\delta\left(1-x\right)$$

1th order: $$F^{(1)}_{2q}(x_B,Q^2)=C^{(1)}_q(x_B/x,Q^2/\mu^2)\otimes\varphi^{(0)}_{q/q}\left(x,\mu^2\right)+C^{(0)}_q(x_B/x,Q^2/\mu^2)\otimes\varphi^{(1)}_{q/q}\left(x,\mu^2\right)$$

⇒ $$C^{(1)}_q(x,Q^2/\mu^2)=F^{(1)}_{2q}(x,Q^2)-F^{(0)}_{2q}(x,Q^2)\otimes\varphi^{(1)}_{q/q}\left(x,\mu^2\right)$$

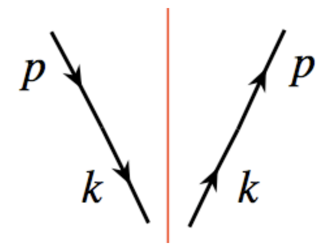
PDFs of a parton

- **Change the state without changing the operator:**

$$\phi_{q/h}(x,\mu^2) = \int \frac{dy^-}{2\pi}\, e^{ixp^+y^-} \langle h(p)|\overline{\psi}_q(0)\frac{\gamma^+}{2}\, U^n_{[0,y^-]}\, \psi_2(y^-)|h(p)\rangle$$

$|h(p)\rangle \;\Rightarrow |\text{parton}(p)\rangle$ $\quad\phi_{f/q}(x,\mu^2)$ **– given by Feynman diagrams**

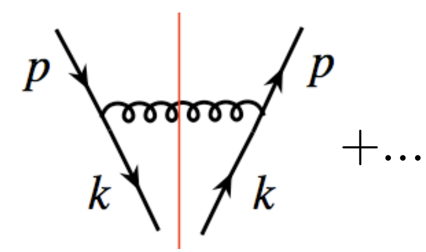
- **Lowest order quark distribution:**



  - ✧ **From the operator definition:**

$$\begin{aligned}\phi^{(0)}_{q'/q}(x) &= \delta_{qq'} \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\left(\frac{1}{2}\gamma\cdot p\right)\left(\frac{\gamma^+}{2p^+}\right)\right] \delta\left(x - \frac{k^+}{p^+}\right) (2\pi)^4\delta^4(p-k)\\ &= \delta_{qq'}\; \delta(1-x)\end{aligned}$$

- **Leading order in α_s quark distribution:**



  - ✧ **Expand to $(g_s)^2$ – logarithmic divergent:**

$$\phi^{(1)}_{q/q}(x) = C_F \frac{\alpha_s}{2\pi} \int \frac{dk_T^2}{k_T^2} \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\,\delta(1-x)\right] + \text{UVCT}$$

UV and CO divergence

Partonic cross sections

❑ Projection operators for SFs:

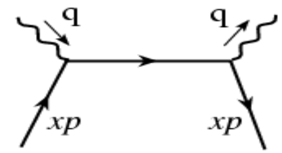
$$W_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_1\left(x, Q^2\right) + \frac{1}{p\cdot q}\left(p_\mu - q_\mu \frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu \frac{p\cdot q}{q^2}\right) F_2\left(x, Q^2\right)$$

$$F_1(x, Q^2) = \frac{1}{2}\left(-g^{\mu\nu} + \frac{4x^2}{Q^2} p^\mu p^\nu\right) W_{\mu\nu}(x, Q^2)$$

$$F_2(x, Q^2) = x\left(-g^{\mu\nu} + \frac{12x^2}{Q^2} p^\mu p^\nu\right) W_{\mu\nu}(x, Q^2)$$

❑ 0th order:

$$F_{2q}^{(0)}(x) = x g^{\mu\nu} W_{\mu\nu,q}^{(0)} = x g^{\mu\nu} \left[\frac{1}{4\pi}\ \cdots\ \right]$$



$$= \left(x g^{\mu\nu}\right) \frac{e_q^2}{4\pi} \mathrm{Tr}\left[\frac{1}{2}\gamma\cdot p \gamma_\mu \gamma\cdot\left(p+q\right)\gamma_\nu\right] 2\pi\delta\left((p+q)^2\right)$$

$$= e_q^2 x \delta(1-x)$$

$$C_q^{(0)}(x) = e_q^2 x \delta(1-x)$$

NLO coefficient function – complete example

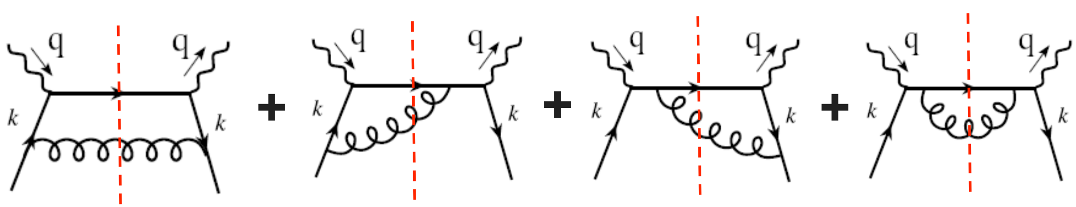
$$C_q^{(1)}(x, Q^2/\mu^2) = F_{2q}^{(1)}(x, Q^2) - F_{2q}^{(0)}(x, Q^2) \otimes \varphi_{q/q}^{(1)}\left(x, \mu^2\right)$$

- **Projection operators in n-dimension:** $g_{\mu\nu} g^{\mu\nu} = n \equiv 4 - 2\varepsilon$

$$\left(1-\varepsilon\right) F_2 = x\left(-g^{\mu\nu} + (3-2\varepsilon)\frac{4x^2}{Q^2} p^\mu p^\nu\right) W_{\mu\nu}$$

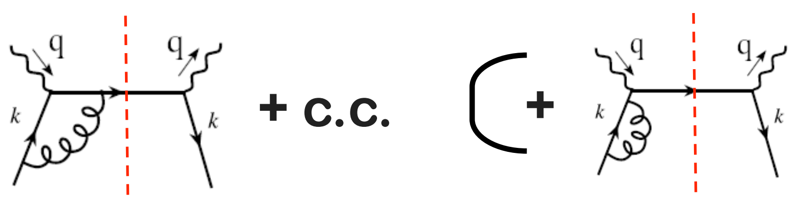
- **Feynman diagrams:**

$W_{\mu\nu,q}^{(1)}$



} Real

Virtual {+



+ c.c. (+ … + c.c. + UV CT)

- **Calculation:** $-g^{\mu\nu} W_{\mu\nu,q}^{(1)}$ and $p^\mu p^\nu W_{\mu\nu,q}^{(1)}$

Contribution from the trace of $W_{\mu\nu}$

- **Lowest order in n-dimension:**

$$-g^{\mu\nu}W^{(0)}_{\mu\nu,q} = e_q^2(1-\varepsilon)\delta(1-x)$$

- **NLO virtual contribution:**

$$-g^{\mu\nu}W^{(1)V}_{\mu\nu,q} = e_q^2(1-\varepsilon)\delta(1-x)$$

$$*\left(-\frac{\alpha_s}{\pi}\right)C_F\left[\frac{4\pi\mu^2}{Q^2}\right]^{\varepsilon}\frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left[\frac{1}{\varepsilon^2}+\frac{3}{2}\frac{1}{\varepsilon}+4\right]$$

- **NLO real contribution:**

$$-g^{\mu\nu}W^{(1)R}_{\mu\nu,q} = e_q^2(1-\varepsilon)C_F\left(-\frac{\alpha_s}{2\pi}\right)\left[\frac{4\pi\mu^2}{Q^2}\right]^{\varepsilon}\frac{\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$*\left\{-\frac{1-\varepsilon}{\varepsilon}\left[1-x+\left(\frac{2x}{1-x}\right)\left(\frac{1}{1-2\varepsilon}\right)\right]+\frac{1-\varepsilon}{2(1-2\varepsilon)(1-x)}+\frac{2\varepsilon}{1-2\varepsilon}\right\}$$

❑ The "+" distribution:

$$\left(\frac{1}{1-x}\right)^{1+\varepsilon} = -\frac{1}{\varepsilon}\delta(1-x) + \frac{1}{(1-x)_+} + \varepsilon\left(\frac{\ln(1-x)}{1-x}\right)_+ + O\left(\varepsilon^2\right)$$

$$\int_z^1 dx \frac{f(x)}{(1-x)_+} \equiv \int_z^1 dx \frac{f(x)-f(1)}{1-x} + \ln(1-z) f(1)$$

❑ One loop contribution to the trace of $W_{\mu\nu}$:

$$-g^{\mu\nu} W^{(1)}_{\mu\nu,q} = e_q^2 (1-\varepsilon)\left(\frac{\alpha_s}{2\pi}\right)\left\{-\frac{1}{\varepsilon} P_{qq}(x) + P_{qq}(x)\ln\left(\frac{Q^2}{\mu^2(4\pi e^{-\gamma_E})}\right)\right.$$
$$+C_F\left[\left(1+x^2\right)\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{3}{2}\left(\frac{1}{1-x}\right)_+ - \frac{1+x^2}{1-x}\ln(x)\right.$$
$$\left.\left. +3-x-\left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-x)\right]\right\}$$

❑ Splitting function:

$$P_{qq}(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]$$

- One loop contribution to $p^\mu p^\nu W_{\mu\nu}$:

$$p^\mu p^\nu W^{(1)V}_{\mu\nu,q} = 0 \qquad\qquad p^\mu p^\nu W^{(1)R}_{\mu\nu,q} = e_q^2 C_F \frac{\alpha_s}{2\pi}\frac{Q^2}{4x}$$

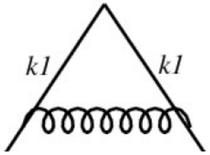
- One loop contribution to F_2 of a quark:

$$F^{(1)}_{2q}(x,Q^2) = e_q^2 x \frac{\alpha_s}{2\pi}\left\{\left(-\frac{1}{\varepsilon}\right)_{\rm CO} P_{qq}(x)\left(1+\varepsilon\ln(4\pi e^{-\gamma_E})\right) + P_{qq}(x)\ln\left(\frac{Q^2}{\mu^2}\right)\right.$$
$$\left. + C_F\left[(1+x^2)\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{3}{2}\left(\frac{1}{1-x}\right)_+ - \frac{1+x^2}{1-x}\ln(x) + 3 + 2x - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-x)\right]\right\}$$

$$\Rightarrow \quad \infty \quad \text{as} \quad \varepsilon \to 0$$

- One loop contribution to quark PDF of a quark:

$$\varphi^{(1)}_{q/q}(x,\mu^2) = \left(\frac{\alpha_s}{2\pi}\right) P_{qq}(x)\left\{\left(\frac{1}{\varepsilon}\right)_{\rm UV} + \left(-\frac{1}{\varepsilon}\right)_{\rm CO}\right\} + \text{UV-CT}$$



– in the dimensional regularization

Different UV-CT = different factorization scheme!

❑ Common UV-CT terms:

✧ **MS scheme:** $\text{UV-CT}\big|_{\text{MS}} = -\frac{\alpha_s}{2\pi} P_{qq}(x)\left(\frac{1}{\varepsilon}\right)_{\text{UV}}$

✧ **$\overline{\text{MS}}$ scheme:** $\text{UV-CT}\big|_{\overline{\text{MS}}} = -\frac{\alpha_s}{2\pi} P_{qq}(x)\left(\frac{1}{\varepsilon}\right)_{\text{UV}} \left(1+\varepsilon \ln(4\pi e^{-\gamma_E})\right)$

✧ **DIS scheme: choose a UV-CT, such that** $C_q^{(1)}(x, Q^2/\mu^2)\big|_{\text{DIS}} = 0$

❑ One loop coefficient function:

$$C_q^{(1)}(x, Q^2/\mu^2) = F_{2q}^{(1)}(x, Q^2) - F_{2q}^{(0)}(x, Q^2) \otimes \varphi_{q/q}^{(1)}\left(x, \mu^2\right)$$

$$C_q^{(1)}(x, Q^2/\mu^2) = e_q^2 x \frac{\alpha_s}{2\pi} \left\{ P_{qq}(x) \ln\left(\frac{Q^2}{\mu_{\overline{\text{MS}}}^2}\right) + C_F \left[(1+x^2)\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{3}{2}\left(\frac{1}{1-x}\right)_+ - \frac{1+x^2}{1-x}\ln(x) + 3 + 2x - \left(\frac{9}{2} + \frac{\pi^2}{3}\right)\delta(1-x) \right] \right\}$$

Summary of lecture one

- QCD is a SU(3) color non-Abelian gauge theory of quark and gluon fields
- QCD perturbation theory works at high energy because of the Asymptotic Freedom
- Perturbative QCD calculations make sense only for infrared safe (IRS) quantities – e^+e^- total cross section
- Jets in high energy collisions provide us the "trace" of energetic quarks and gluons
- Factorization is necessary for pQCD to treat observables (cross sections) with "identified hadrons"
- Predictive power of QCD factorization relies on the universality of PDFs (or TMDs, GPDs, …), the calculations of perturbative coefficient functions – hard parts

Backup slides

Fundamental forces

❑ **Known "fundamental" interactions:**

AdS/CFT

- "Strong" – QCD
- Weak – Fermi's theory, …
- Electromagnetic – QED
- …
- Gravity – GR, SuperGravity, String, …

Electro-Weak (Weak + Electromagnetic)

Standard "Model" ("Strong" + Electro-Weak)

❑ **Status:**

✧ QED is the best tested, but, can only be an effective theory

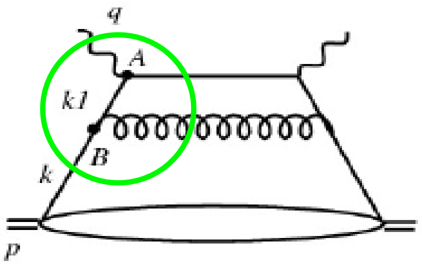
✧ Electro-weak is only tested at low energy (as an effective theory)

✧ GR is successful, what is the theory for quantum gravity?

✧ QCD is successful at short distance, but, connection to hadrons?

Scaling violation and factorization

- NLO partonic diagram to structure functions:



$$\propto \int_0^{-Q^2} \frac{dk_1^2}{k_1^2}$$ **Dominated by** $$\begin{cases} k_1^2 \sim 0 \\ t_{AB} \rightarrow \infty \end{cases}$$

Diagram has both long- and short-distance physics

- Factorization, separation of short- from long-dis