Phase structure of QCD at high temperature and high density by numerical simulations of lattice QCD

Shinji Ejiri

Niigata University

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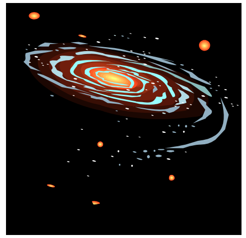
Introduction

- First principle calculation of QCD (quantum chromodynamics) is very important to understand the universe.

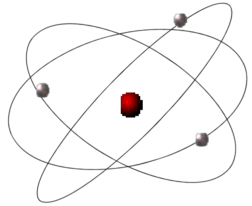
- Numerical simulations of Lattice QCD is powerful method to study QCD.

History of the Universe： Expanding Universe

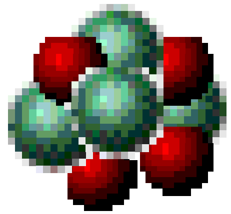
- Present (13.7 G years, 3K)
  - Gravity



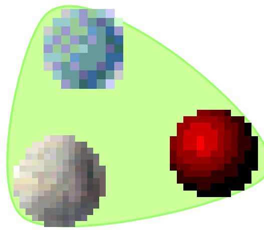
- Genesis of Atoms（380 ,000 years, 3000K）
  - Transparent to radiation
  - Electromagnetic interaction



- Genesis of Nuclei （3min., 1 G K)
  - Strong interaction



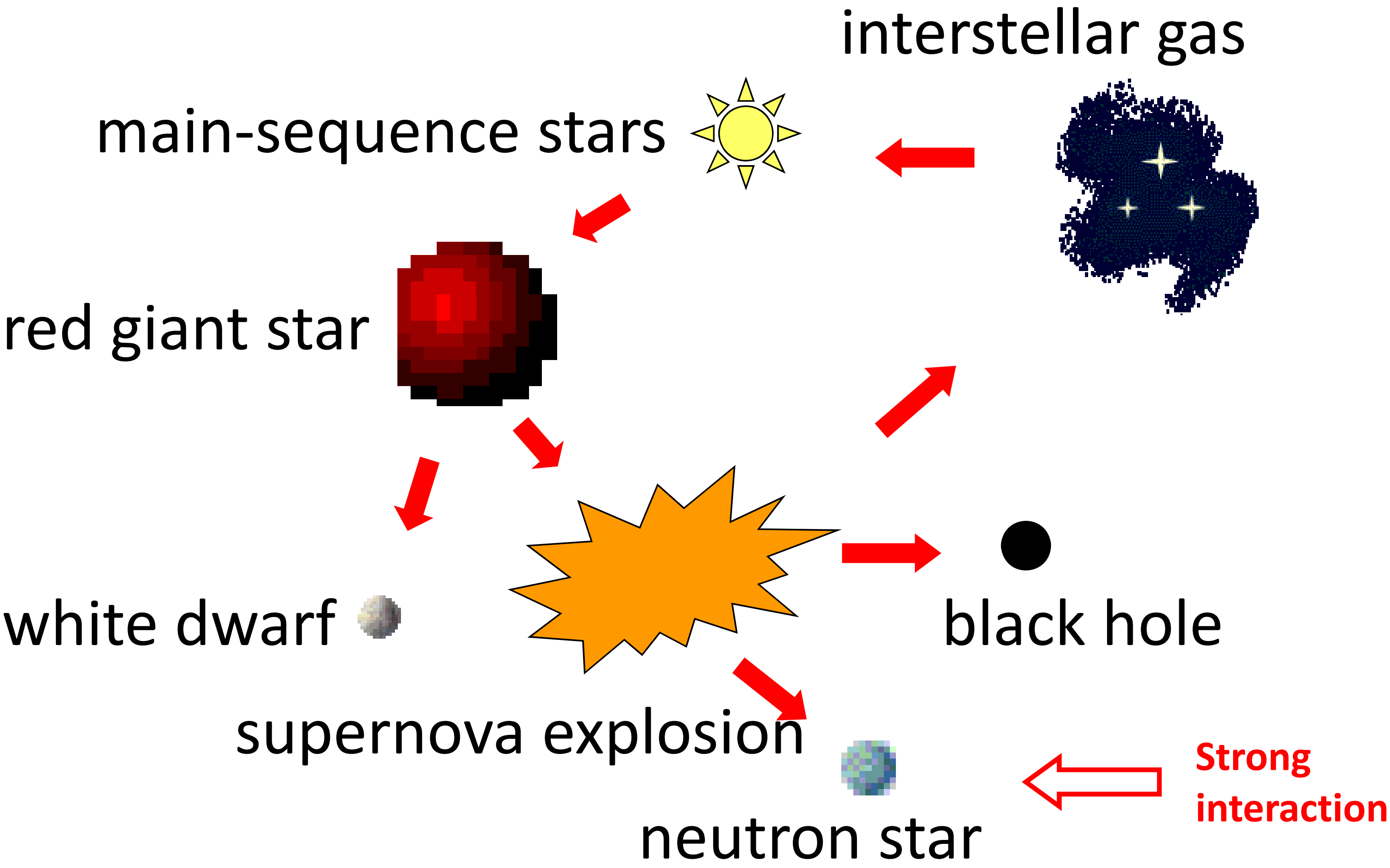
- <u>Genesis of Hadron (0.00001 sec., 1 T K)</u> ⇐ <u>**This talk**</u>
  - <u>QCD phase transition</u>
  - Strong interaction



…….

- Big Bang (Barth of the Universe)

Life of stars



Phase structure of QCD at high temperature and high density by numerical simulations of lattice QCD

Contents of this talk

- Strong interaction
- QCD phase transition at high temperature
- Basic formulation of Lattice QCD
- Transition temperature and Equation of state
- Problems of the finite density QCD

Properties of the strong interaction at high temperature and density

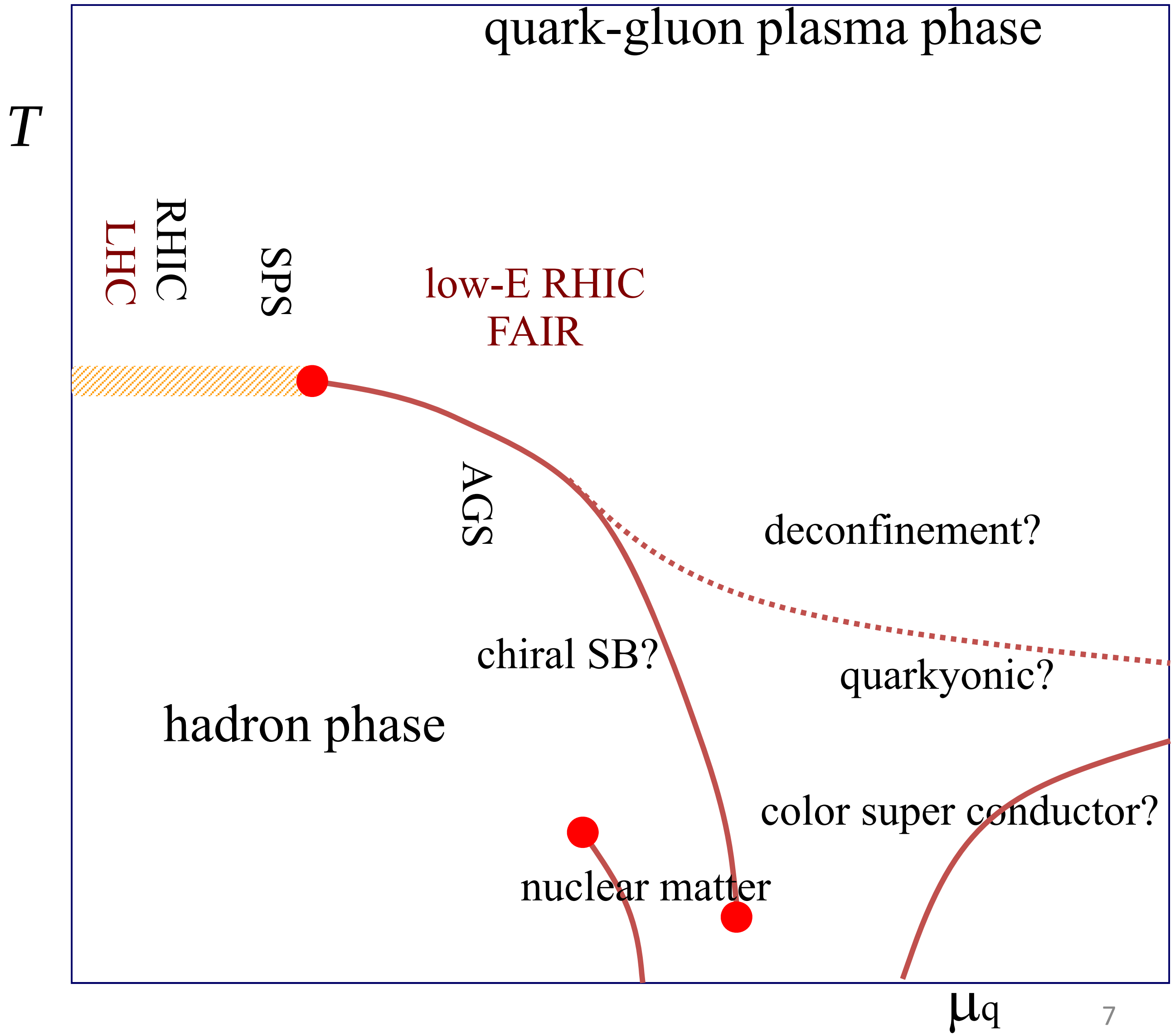
- Deconfining phase transition
  - Hadron $\Leftrightarrow$ Quark-Gluon Plasma
- Chiral symmetry restoration
- Axial U(1) anomaly restoration
- Color superconductivity
- ....

- First principle calculations are important.
- Lattice QCD using super computer.

Phase structure of QCD at high temperature and density

- Phase structure
- Equation of State

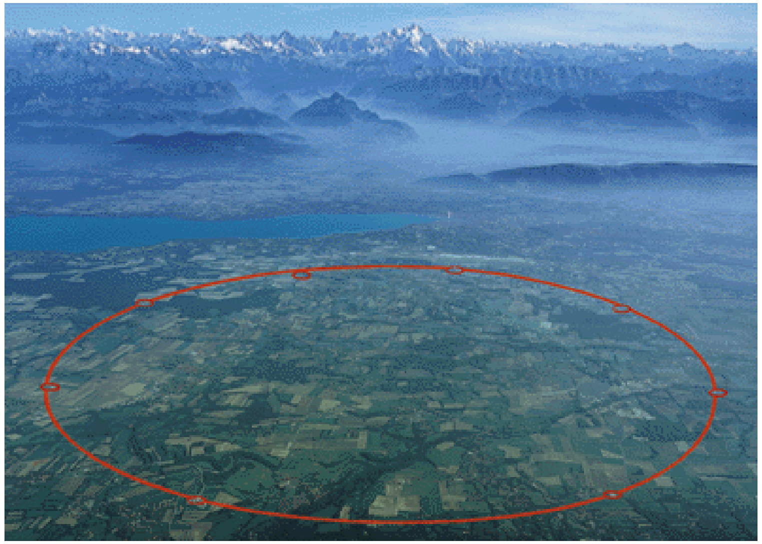
Numerical study of QCD



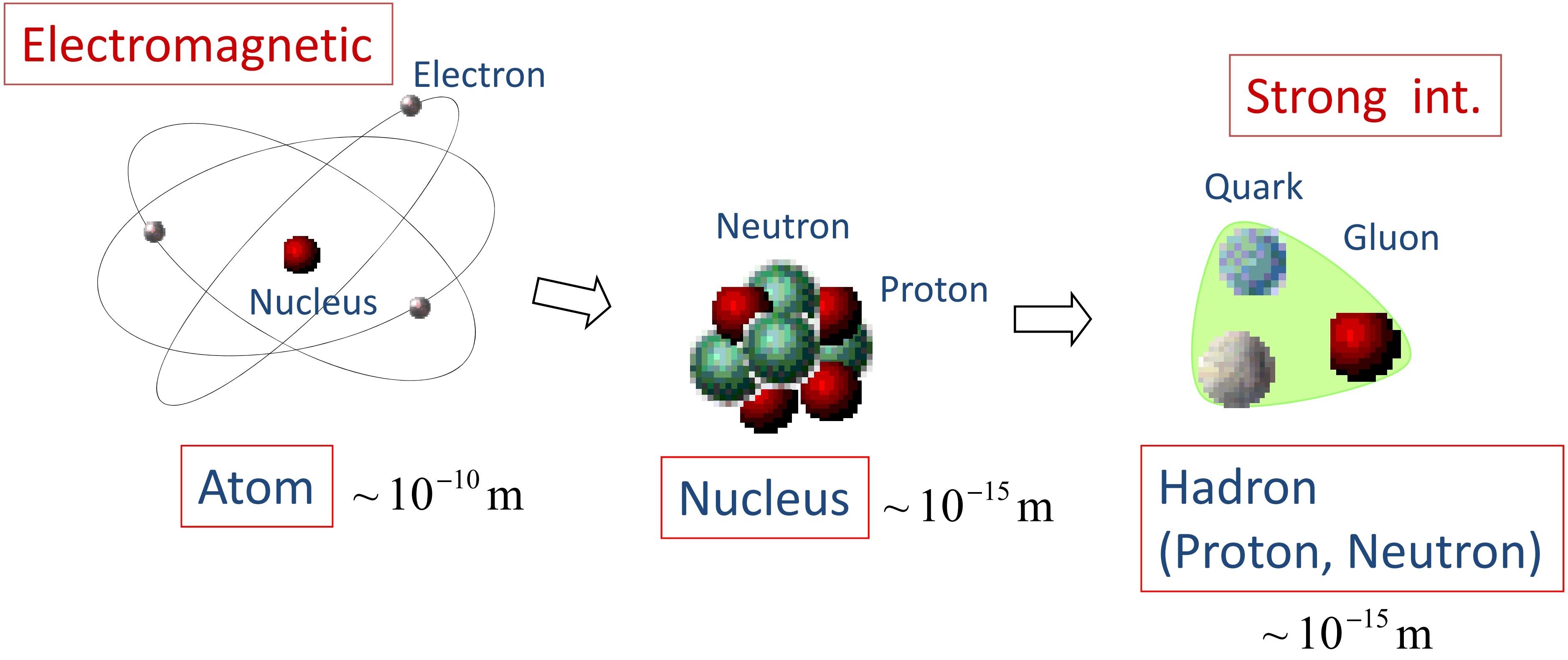
Heavy ion collision experiment RHIC, LHC

- Extremely high temperature, colliding heavy-ions.
- Brookhaven National Laboratory (BNL), RHIC (2001)
- European Organization for Nuclear Research (CERN), LHC (2010)

BNL

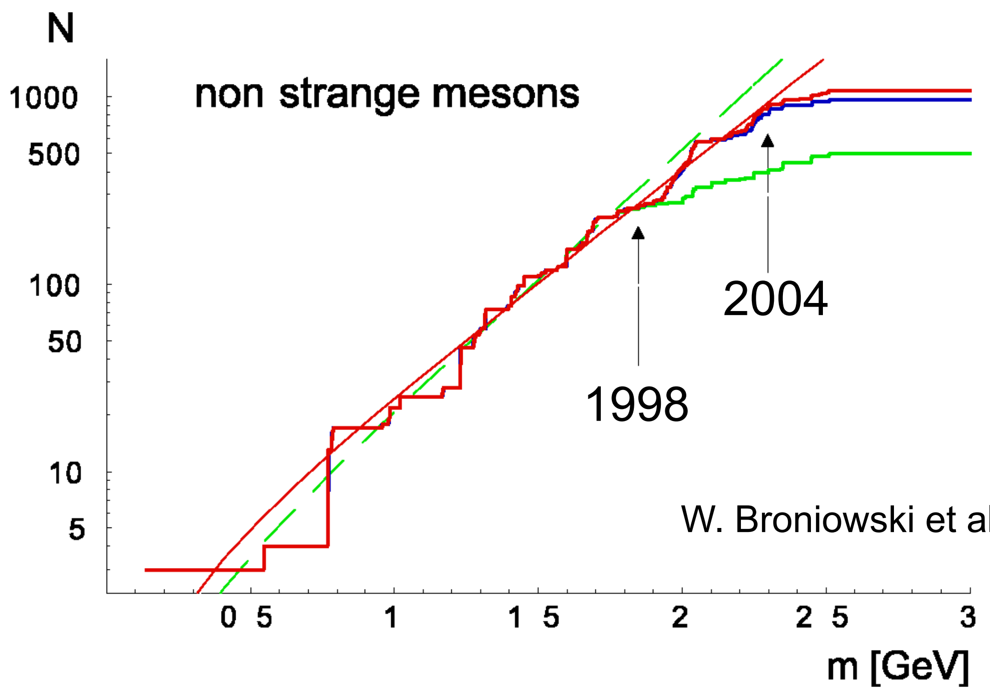

CERN

Strong interaction and quark confinement



However, quarks have never observed.

Number of hadrons (experimental data)



W. Broniowski et al., Phys. Rev. D70, 117503 (2004)

Sum of the number of mesons consist of up, down quarks

$$N = \int_0^m \rho(m')dm'$$

- Many new hadrons were discovered after the discovary of pions.
- Hagedorn's prediction (1965) ➡ $\underline{\rho(m) \approx C m^{-a} e^{m/T_H}}$

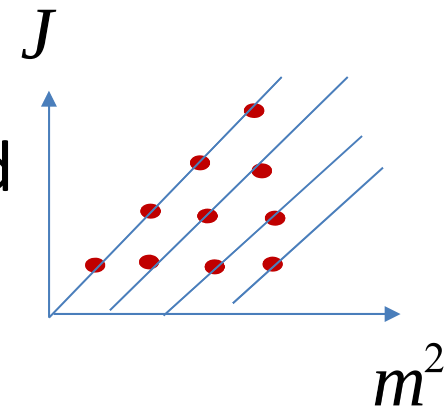
  (Statistical bootstrap model)
- Hadrons are not elementary particles.

Quark confinement

- Why are quarks never observed?

- Huge energy is required if one try to separate single quark.

➡ Linear potential between quarks



- Relation between angular momentum (J) and mass of hadrons (m)   (Regge trajectory)

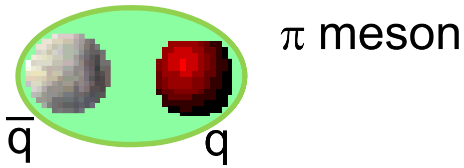
➡ Quarks are connected by a “string”.

String model of hadrons

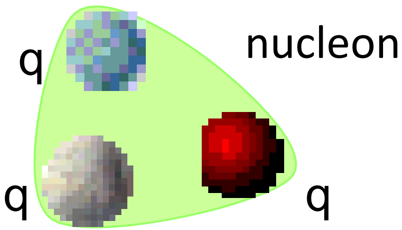
This also explains the exponential increase of the state density ρ(m).

Chiral symmetry breaking

- Why is π meson mass not 2/3 of nucleon mass?



$$m_\pi = 140\mathrm{MeV}$$



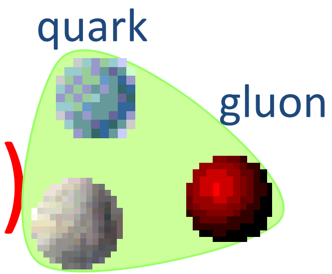
$$m_N = 939\mathrm{MeV}$$

- Chiral symmetry breaking (Y. Nambu, 1960)
  - Nucleon mass 100 times larger than the original mass of 3 quarks.
  - 99% of Nucleon mass is given by the chiral symmetry breaking.

Strong interaction

Quantum chromo dynamics(QCD)

(Y. Nambu, 1964)



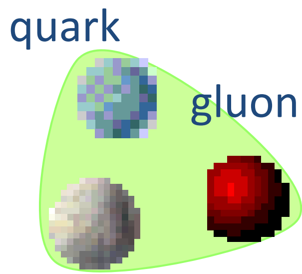
Nucleon (proton, neutron)

- Chiral condensation (Y. Nambu, 1960)
  - Mass of the nucleon is 100 times heavier than the original mass of 3 quarks.
  - 99% of mass is given by the chiral symmetry breaking.

- Quark confinement
  - Why does single quark cannot be observed?
  - Linear potential between quarks

Strong interaction
Quantum chromo dynamics(QCD)

(Y. Nambu, 1964)



Nucleon (proton, neutron)

- Force which bounds 3 quarks strongly.
  - Impossible if the charge is + or -.

- Color charge (corresponding +- in Electro.)
  - Overlapping 3 colors, color becomes white.
  - SU(3) gauge theory

- Asymptotic free: coupling constant becomes smaller as the distance becomes shorter.

➡ Strong interaction at large distance.

(Gross, Politzer, Wilczek, 1973)

opposite to Electromagnetic force

Analytic calculation of QCD: difficult

- Quantum Chromodynamics
  - Interaction: very strong
  - Analytic calculation of QCD: difficult

⇨ Computer simulations

Hadrons at high temperature

- Relativistic ideal gas
- Hagedorn temperature

Relativistic ideal gas (No interaction)

Energy of a particle with the momentum *k*

$$E(k)=\sqrt{k^2+m^2}$$

N(*k*): occupation number

- Grand partition function of

$$Z_{GC}=\prod_k \sum_{N(k)} e^{-\frac{N(k)[E(k)-\mu]}{T}}$$

- Bose-Einstein system $N(k)=\{0,1,2,\cdots,\infty\}$

$$Z_{GC}^B=\prod_k \sum_{N(k)=0}^{\infty} e^{-\frac{N(k)[E(k)-\mu]}{T}}=\prod_k \frac{1}{1-e^{-(E(k)-\mu)/T}}$$

$$\langle N\rangle_B=\frac{\partial \ln Z_{GC}^B}{\partial(\mu/T)}=\sum_k \frac{1}{e^{(E(k)-\mu)/T}-1}$$

- Fermi-Dirac system $N(k)=\{0,1\}$

$$Z_{GC}^F=\prod_k\left(1+e^{-(E(k)-\mu)/T}\right)$$

$$\langle N\rangle_F=\frac{\partial \ln Z_{GC}^F}{\partial(\mu/T)}=\sum_k \frac{1}{e^{(E(k)-\mu)/T}+1}$$

- Thermodynamic limit ($V\to\infty$) $\left(\Sigma_k \to \int dk\right)$

Boson: $$\ln Z_{GC}^B=-V\int\frac{d^3k}{(2\pi)^3}\ln\left(1-e^{-(E(k)-\mu)/T}\right)=-\frac{V}{2\pi^2}\int dk\, k^2 \ln\left(1-e^{-(E(k)-\mu)/T}\right)$$

Fermion: $$\ln Z_{GC}^F=V\int\frac{d^3k}{(2\pi)^3}\ln\left(1+e^{-(E(k)-\mu)/T}\right)=\frac{V}{2\pi^2}\int dk\, k^2 \ln\left(1+e^{-(E(k)-\mu)/T}\right)$$

Relativistic ideal gas

- For the case that many kinds of particles exist,

$$Z_{GC} = Z_1 Z_2 Z_3 \cdots Z_N, \quad \ln Z_{GC} = \ln Z_1 + \ln Z_2 + \ln Z_3 + \cdots + \ln Z_N$$

- Massless particle

(1) Boson (degeneracy factor: d_B, μ=0)

$$p = \frac{T}{V}\ln Z_{GC}^{B} = -\frac{d_B T}{2\pi^2}\int dk\, k^2 \ln\left(1-e^{-k/T}\right)$$

$$= -\frac{d_B T^4}{2\pi^2}\int_0^\infty dx\, x^2 \ln\left(1-e^{-x}\right) = \frac{d_B T^4}{6\pi^2}\int_0^\infty \frac{dx\, x^3}{\left(1-e^{-x}\right)}$$

$$= \frac{d_B T^4}{6\pi^2}\Gamma(4)\zeta(4) = \underline{\frac{d_B \pi^2}{90}}T^4$$

(2) Fermion (degeneracy factor: d_F)

$$p = \frac{T}{V}\ln Z_{GC}^{F} = \frac{d_F T}{2\pi^2}\int dk\, k^2 \ln\left(1+e^{-(k-\mu)/T}\right)$$

$$= \frac{d_F T^4}{2\pi^2}\int_0^\infty dx\, x^2 \ln\left(1+e^{-x+\sigma}\right) = -\frac{d_F T^4}{6\pi^2}\int_0^\infty \frac{dx\, x^3}{\left(e^{x-\sigma}+1\right)}$$

$$= \frac{d_F T^4}{6\pi^2}\Gamma(4)\phi(4,-e^{\sigma}) = \underline{d_F T^4\left(\frac{7\pi^2}{720} + \frac{1}{24}\left(\frac{\mu}{T}\right)^2 + \frac{1}{48\pi^2}\left(\frac{\mu}{T}\right)^4\right)}$$

$$(x = k/T, \quad \sigma = \mu/T)$$

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s \quad (\zeta\text{-function})$$

$$\zeta(4) = \pi^4/90, \quad \Gamma(4) = 3!$$

$$\phi(s, f) = \sum_{n=1}^{\infty} f^n/n^s$$

(modified ζ-function)

1) Massless pion gas

d_B=3 (spin: 1, isospin: 3)

$$p = \frac{\pi^2}{30} T^4 \qquad \varepsilon = \frac{T^2}{V}\frac{\partial \ln Z}{\partial T} = \frac{T^2}{V}\frac{\partial}{\partial T}\left(\frac{\pi^2 V T^3}{30}\right) = \frac{\pi^2}{10} T^4$$

2) Quark-gluon gas (N_f, μ=0)

quark: $d_F = 4 \times 3 \times N_f$

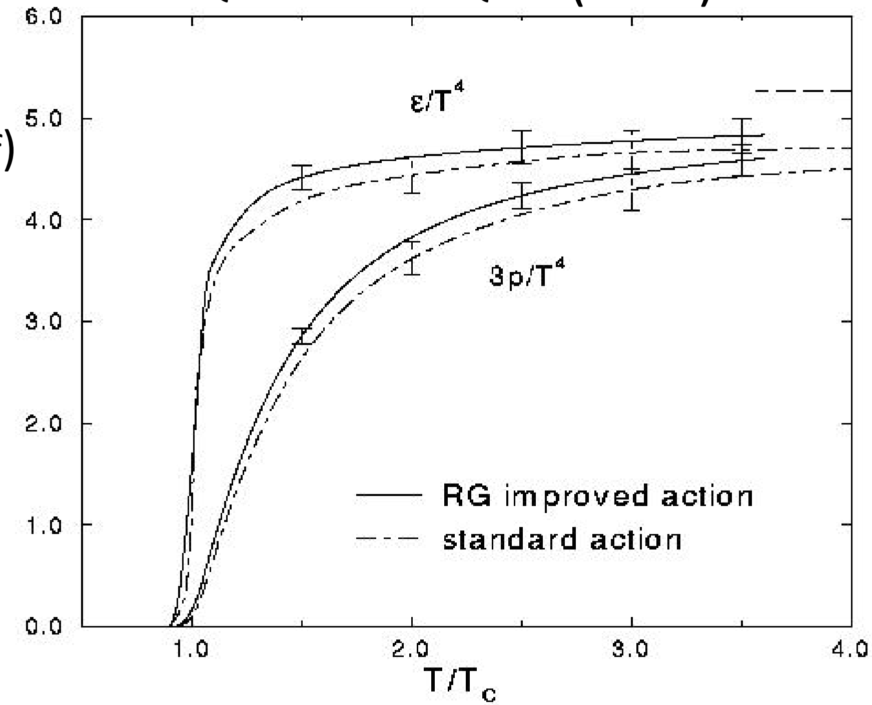
(spin: 2, antiparticle: 2, color: 3, flavor: N_f)

gluon: $d_B = 2 \times 8$ (spin: 2, color: 8)

$$p = \frac{7\pi^2 N_f}{60} T^4 + \frac{16\pi^2}{90} T^4$$

$$\varepsilon = \frac{T^2}{V}\frac{\partial \ln Z}{\partial T} = \frac{7\pi^2 N_f}{20} T^4 + \frac{16\pi^2}{30} T^4$$

Quenched QCD (N_f=0)



G. Boyd et al., Nucl. Phys. B467, 419 (1996)
CP-PACS (Okamoto), Phys. Rev. D60, 094510 (1999)

(3) Heavy particles
Hadron resonance gas model (Hagedorn model)

- For $m_i >> T$, both boson and fermion $T \approx 200\text{MeV}$

$$\frac{p_{mi}}{T^4} \approx \frac{d_i}{2\sqrt{2}\pi^{3/2}} \left(\frac{m_i}{T}\right)^{3/2} \exp\left(\frac{-m_i + 3B\mu_q + 2I_3\mu_I}{T}\right)$$

(B: baryon number, I_3: isospin number, d_i: degeneracy factor)

- Summing over all resonance states

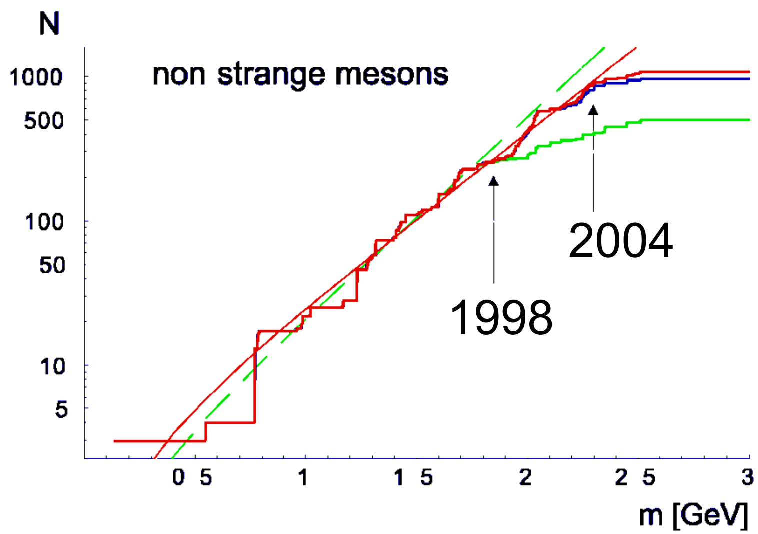
$$\frac{p}{T^4} = \sum_{i:\text{Meson}} \frac{p_{mi}}{T^4} + \sum_{i:\text{Baryon}} \frac{p_{mi}}{T^4},$$

- State density: $\rho(m)$ $\quad e.g.\quad \rho(m) = Cm^a e^{m/T_H}, \quad a \sim -2, \quad T_H \approx 200\text{MeV}$

(T_H: Hagedorn temperature)

$$\frac{p}{T^4} = \int dm \frac{\rho(m)}{2\sqrt{2}\pi^{3/2}} \left(\frac{m}{T}\right)^{3/2} \exp\left(\frac{-m + 3B\mu_q + 2I_3\mu_I}{T}\right)$$

Break down of hadron picture : QCD phase transition



$$N = \int_0^m \rho(m')dm'$$

W. Broniowski et al., Phys. Rev. D70, 117503 (2004)

$$\rho(m) \approx C m^{-a} e^{m/T_H}$$

(T_H: Hagedorn temperature)

- Number of states increases exponentially
- Pressure from multi-particle $\frac{p}{T^4} = \sum_{i:\mathrm{Meson}} \frac{p_{mi}}{T^4} + \sum_{i:\mathrm{Baryon}} \frac{p_{mi}}{T^4},$ (ignoring the interaction)

$$T_H \approx 200\mathrm{MeV}$$

- Pressure by heavy particle $\frac{p_{mi}}{T^4} \propto \left(\frac{m_i}{T}\right)^{\frac{3}{2}} \exp\left(-\frac{m_i}{T}\right)$

$$\frac{p_i}{T^4} = \frac{1}{VT^3} \ln Z$$

$$\ln Z \approx V\left(\frac{T}{2\pi}\right)^{3/2} \int_0^\infty \rho(m) m^{3/2} e^{-m/T} dm \Rightarrow V\left(\frac{T}{2\pi}\right)^{3/2} \int_0^\infty \underline{C m^{3/2-a} e^{m\left(\frac{1}{T_H}-\frac{1}{T}\right)} dm}$$

Integral does not converge for $T>T_H$

This model breaks down at $T>T_H$ ➡ quark-hadron transition: suggested

Lattice QCD

History of Lattice QCD   Monte Carlo simulations

’74, Wilson, Formulation of the lattice gauge theory
  strong coupling expansion ⇨ linear potential (confinement)

’80, Creutz, Monte Carlo simulation ⇨ linear potential

’81, McLerran & Svetitsky, Finite T phase transition

’81, Hamber & Parisi, Hadron spectral (quenched approx.)

’82, Karsch et al., Equation of State (p, e, S)

’83, Hasenfratz & Karsch, Finite density QCD (formulation)

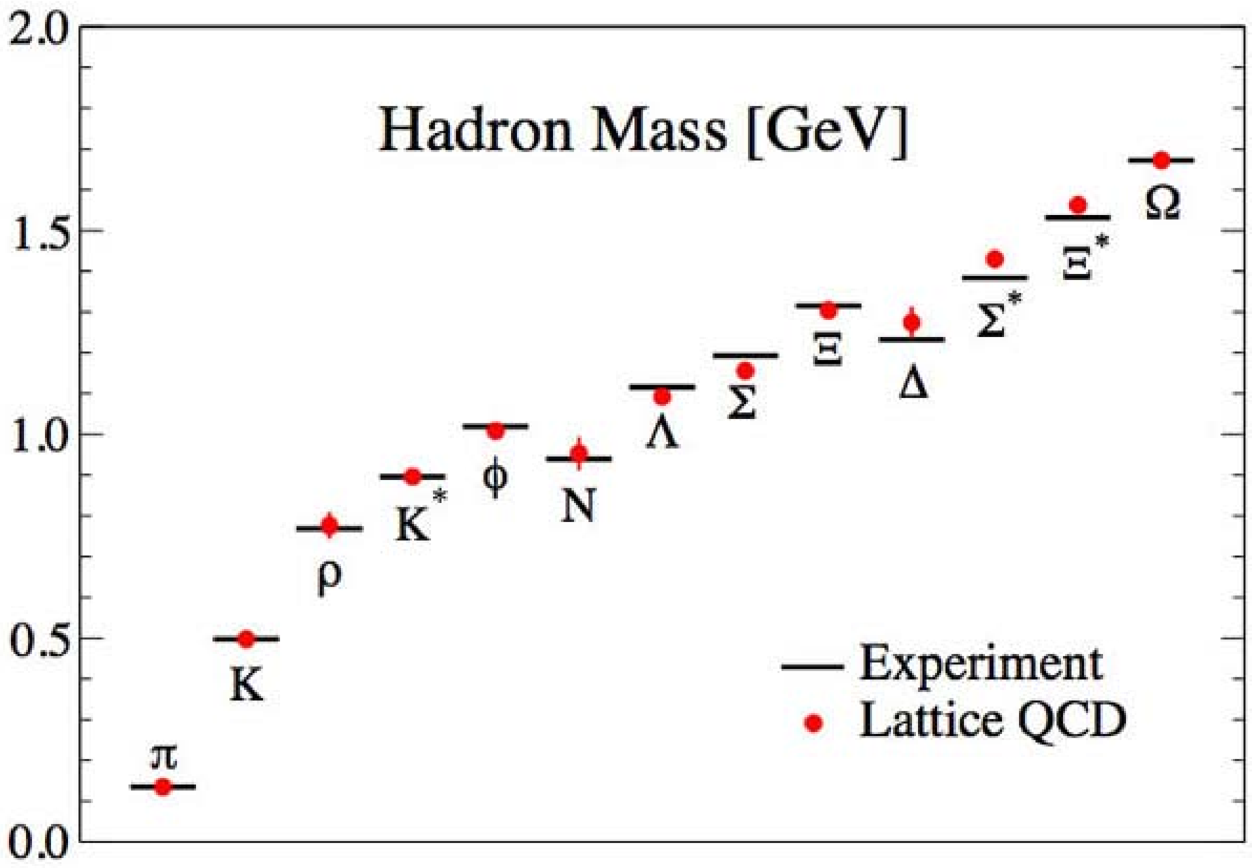
⋮

Many results are obtained.

Numerical study of Lattice QCD

Calculation of hadron mass

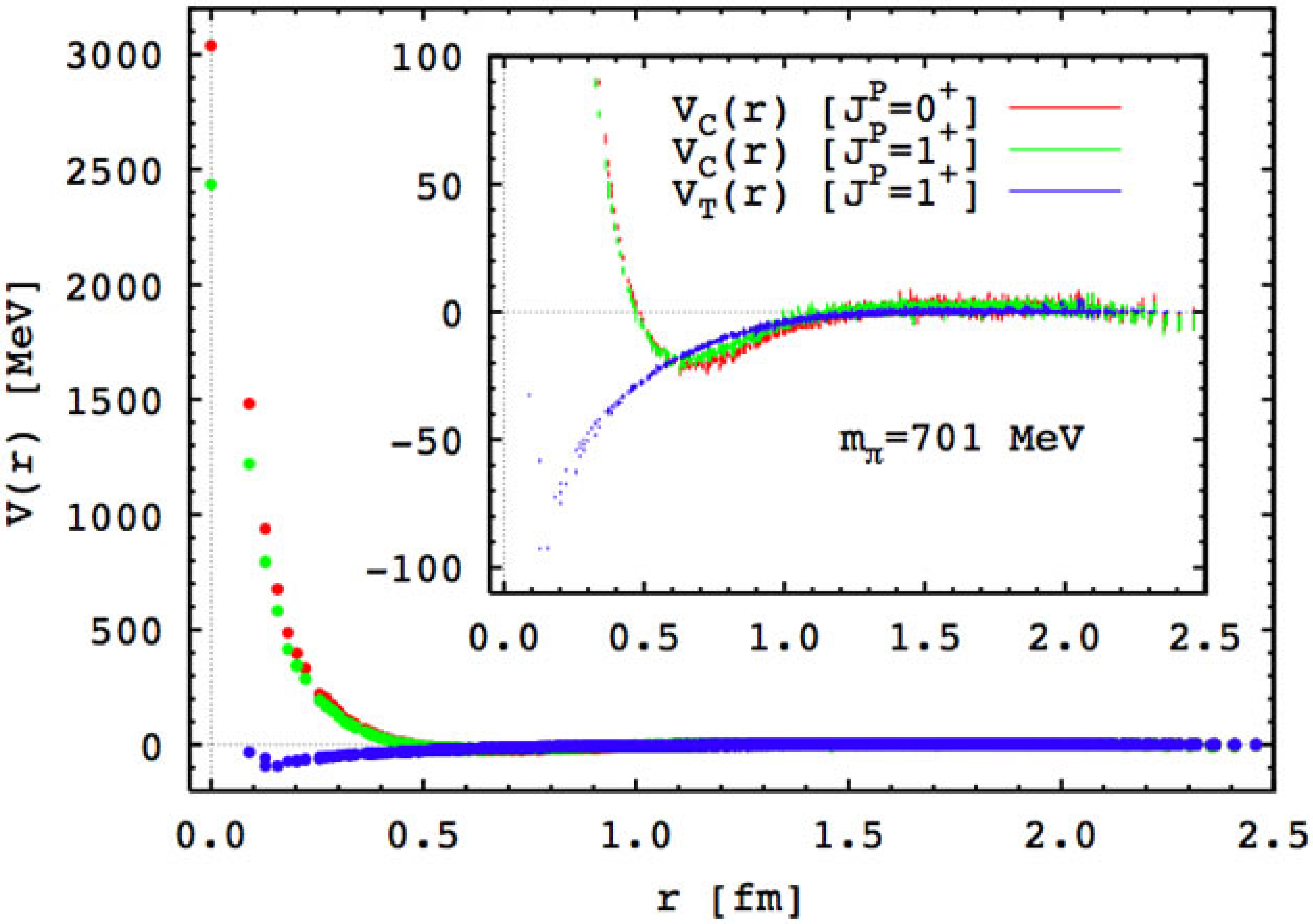
Tsukuba group (CP-PACS(97) – PACS-CS(11))



Experimental values are reproduced.

Numerical study of Lattice QCD

Nuclear force (Aoki, Hatsuda, Ishii, 2006)



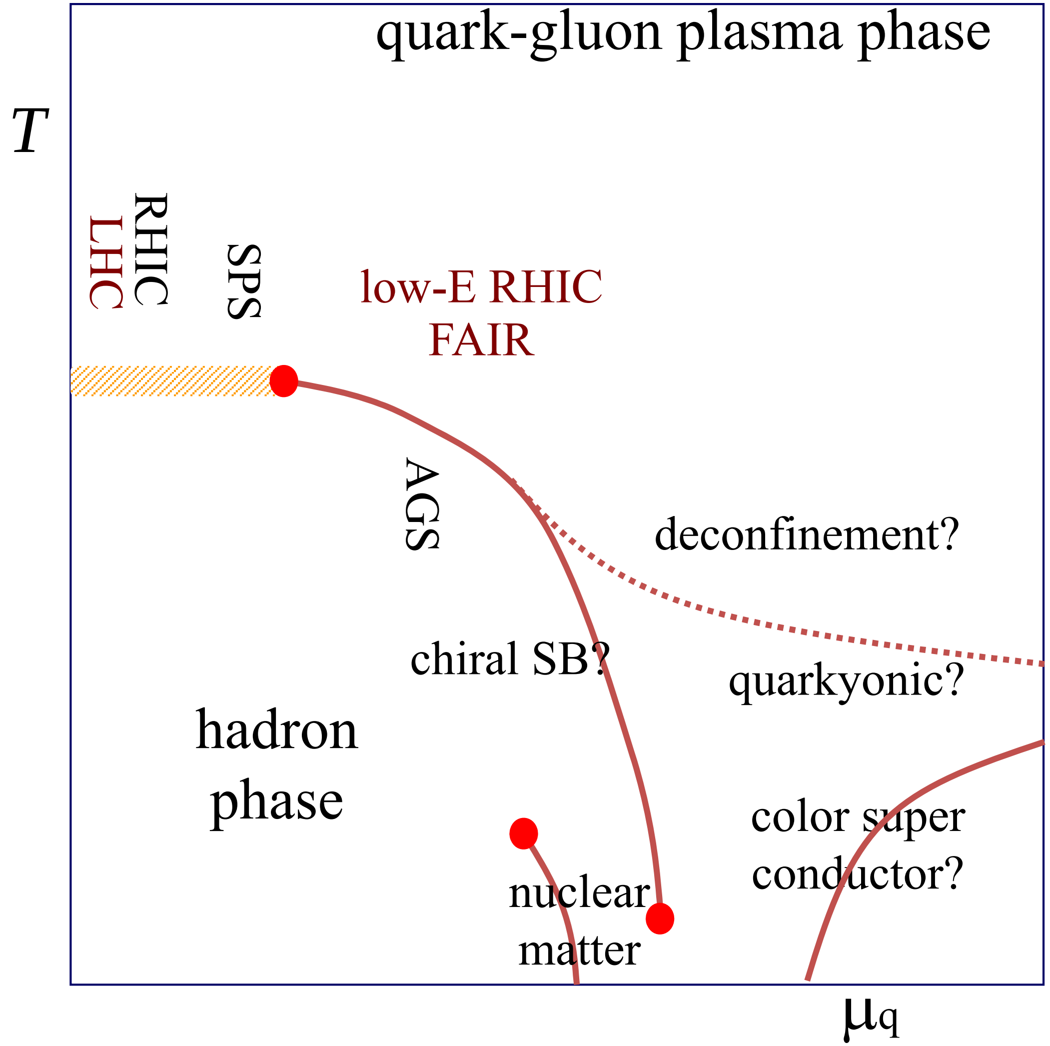
- Nuclear force potential can be computed from QCD

Phase structure of QCD at high temperature and density

- Critical point at finite μ?
- New phase?

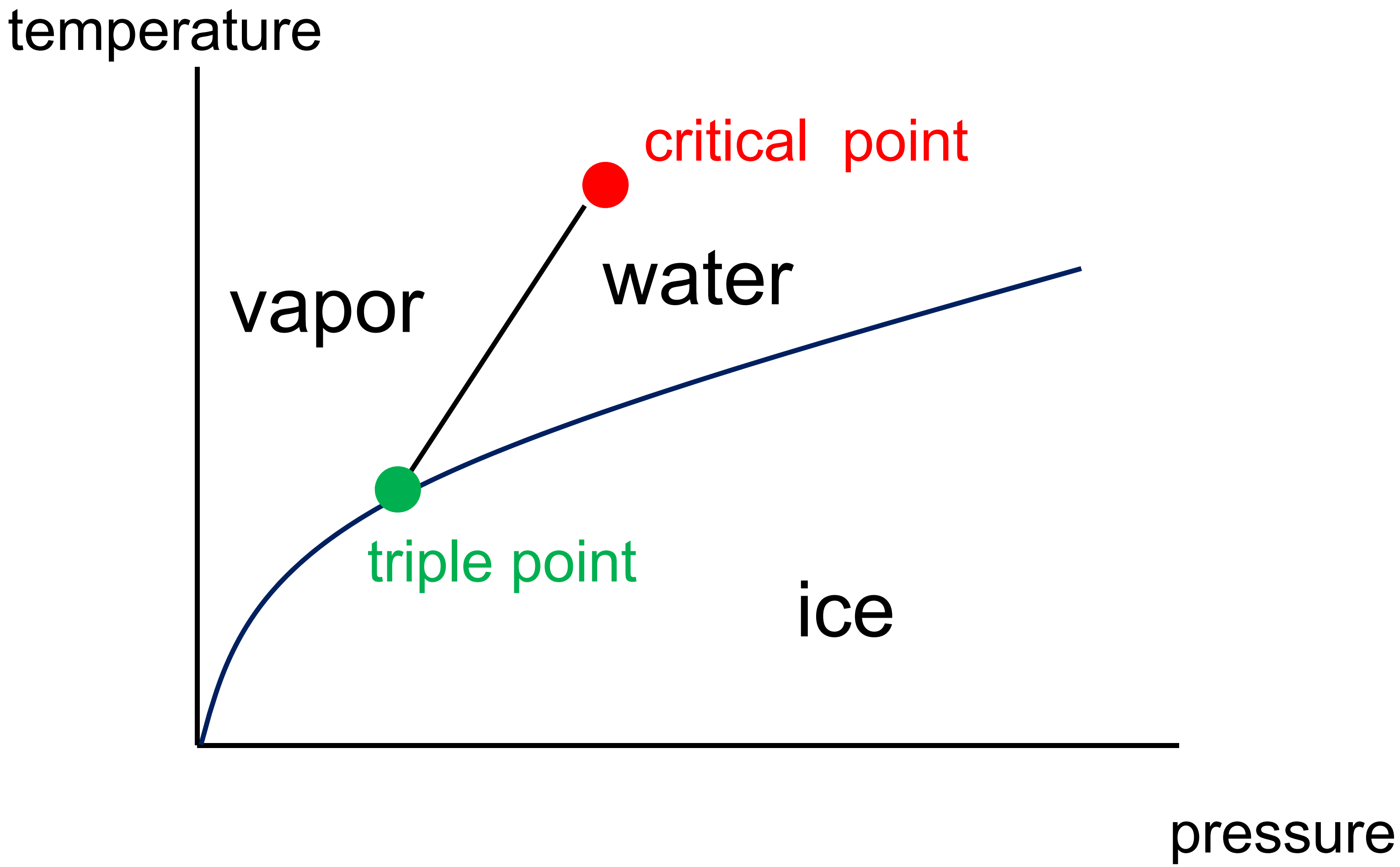
Numerical study of QCD

- Low density region: Important to support the heavy-ion collision experiments.
- High density region: the calculation becomes difficult as the density increases.



Phase structure of water

- Very similar to QCD



Equation of State

- Thermodynamic quantities

Calculation of Thermodynamic quantities

- Pressure $$p = \frac{T}{V}\ln Z$$ or $$p = T\frac{\partial \ln Z}{\partial V}$$
- Derivatives of lnZ:

  (almost all thermodynamic quantities are given by the derivatives)

Quark number density: $$n_q = \frac{T}{V}\frac{\partial \ln Z}{\partial \mu_q}$$

Energy density: $$\varepsilon = -\frac{1}{V}\left(\frac{\partial \ln Z}{\partial T^{-1}}\right)_{\mu/T}$$

Chiral condensate: $$\langle \bar{\psi}\psi \rangle = \frac{T}{V}\frac{\partial \ln Z}{\partial m}$$

Finite-temperature QCD

Finite-temperature field theory (Matsubara formalism)

- Canonical partition function

$$Z = \sum_{\alpha} \left\langle \alpha \middle| e^{-H/T} \middle| \alpha \right\rangle = \sum_{\alpha} e^{-E_\alpha/T}$$

- It is the same as the calculation of transition amplitude $\left\langle \beta \middle| e^{-itH} \middle| \alpha \right\rangle$

$$\left\langle \beta \middle| e^{-itH} \middle| \alpha \right\rangle = \int \prod_{x,\mu} dA_\mu(x) \prod_{x} d\psi(x) d\overline{\psi}(x) \exp\left(iS_{\mathrm{QCD}}[A, \psi, \overline{\psi}] \right)$$

- replacing $t = -i/T$, Minkowski $\rightarrow$ Euclidean

$$Z = \sum_{\alpha} \left\langle \alpha \middle| e^{-H/T} \middle| \alpha \right\rangle = \int \prod_{x,\mu} dA_\mu(x) \prod_{x} d\psi(x) d\overline{\psi}(x) \exp\left(-S^{\mathrm{E}}_{\mathrm{QCD}}[A, \psi, \overline{\psi}] \right)$$

($S^{\mathrm{E}}_{\mathrm{QCD}}$ is the Euclidian action.)

- (Anti-) Periodic boundary condition $\left(\because \ \mathrm{Tr}\, A = \int \left\langle \psi \middle| A \middle| -\psi \right\rangle d\psi \right)$

$$A_\mu(x_4 = 1/T, \vec{x}) = A_\mu(x_4 = 0, \vec{x})$$

$$\psi(x_4 = 1/T, \vec{x}) = -\psi(x_4 = 0, \vec{x}) \qquad \overline{\psi}(x_4 = 1/T, \vec{x}) = -\overline{\psi}(x_4 = 0, \vec{x})$$

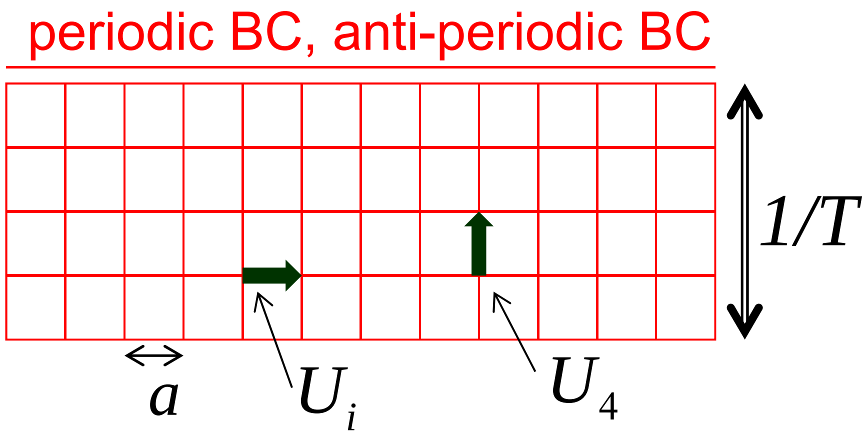
- Expectation value of *O*

$$\langle O \rangle = \frac{1}{Z} \int \prod_{x,\mu} dA_\mu(x) \prod_x d\psi(x) d\overline{\psi}(x) O[A, \psi, \overline{\psi}] \exp\left(- S_{\rm QCD}^{\rm E}[A, \psi, \overline{\psi}]\right)$$

- On the lattice,

$$Z = \int \prod_{x,\mu} dU_\mu(x) \prod_x d\psi(x) d\overline{\psi}(x) \exp\left(- S_{\rm QCD}^{\rm Lattice}[U, \psi, \overline{\psi}]\right)$$

$$\langle O \rangle = \frac{1}{Z} \int \prod_{x,\mu} dU_\mu(x) \prod_x d\psi(x) d\hat{\psi}(x) \exp\left(- S_{\rm QCD}^{\rm Lattice}[U, \psi, \overline{\psi}]\right)$$



$$U_\mu \approx e^{igaA_\mu} \in SU(N_C)$$

Temperature: $T = 1/(N_t a)$

Temperature can determined after the calculation of the lattice sapacing

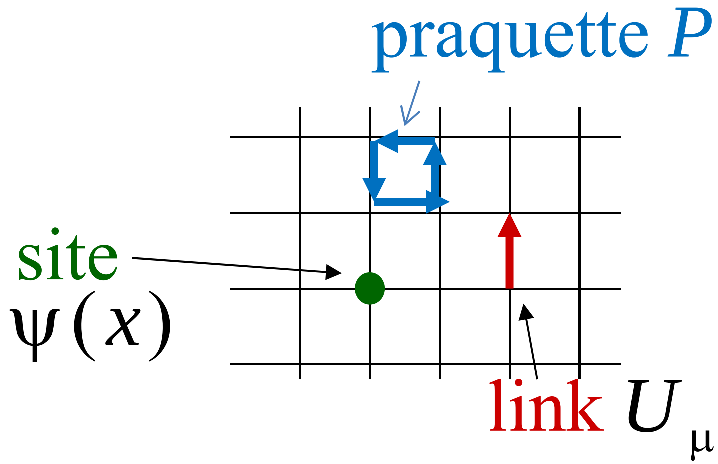
Partition function of QCD on a lattice

Dynamical variables

- Gauge field: $U_\mu \in SU(3)$ on a link $\quad U_\mu(x) = e^{igA_\mu(x)}$
- Quark field: $\psi, \bar{\psi}$, Grassmann number on a site

Boltzmann weight: $\quad e^{-S_g - S_f}$

$$\beta = 6/g^2$$



gauge field $\quad S_g = -\beta \sum_{n,\mu\neq\nu} \frac{1}{3} \mathrm{tr}\left[U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n)\right] \equiv -6 N_{\mathrm{site}} \beta P$

quark field $\quad S_q = \sum_{i=1}^{N_f} \bar{\psi}_i M \psi_i \qquad$ (M: quark matrix)

Continuum limit $\quad L = \bar{\psi}\left[\gamma_\mu\left(\partial_\mu + igA_\mu\right) + m\right]\psi + \frac{1}{2} Tr F_{\mu\nu}^2$

Lattice QCD simulations

- Partition function:

$$Z = \int \prod_{x,\mu} dU_\mu(x) \prod_x d\psi(x) d\overline{\psi}(x) \exp\left(-S_g[U] - S_q\left[U, \psi, \overline{\psi}\right]\right)$$

- Integral of Grassmann variables (One cannot deal with Grassmann number on a computer.)

$$Z = \int \prod_{x,\mu} dU_\mu(x) \left(\det M\right)^{N_f} e^{-S_g}, \quad S_q = \sum_{i=1}^{N_f} \overline{\psi}_i M \psi_i$$

- Generate gauge field configurations by Monte-Carlo method and perform the path integral.
- Calculation of a physical quantity *O*

$$\langle O \rangle_{(\beta,m)} = \frac{1}{Z} \int DU_\mu O\left[U_\mu\right] \underline{\left(\det M\right)^{N_f} e^{-S_g(\beta)}}$$

Generate configurations with this weight

Critical temperature

- Finding the critical parameter (β) performing simulations at finite T
  - Order parameter
    - Chiral condensate
    - Polyakov loop (an order parameter of confinement)
- Determination of the lattice spacing
  - Performing T=0 simulation.
  - The lattice spacing is independent of temperature.
  - Measurements of physical quantities, hadron masses, decay constants etc.
  - Comparison with the experimental values.

Results of transition temperature

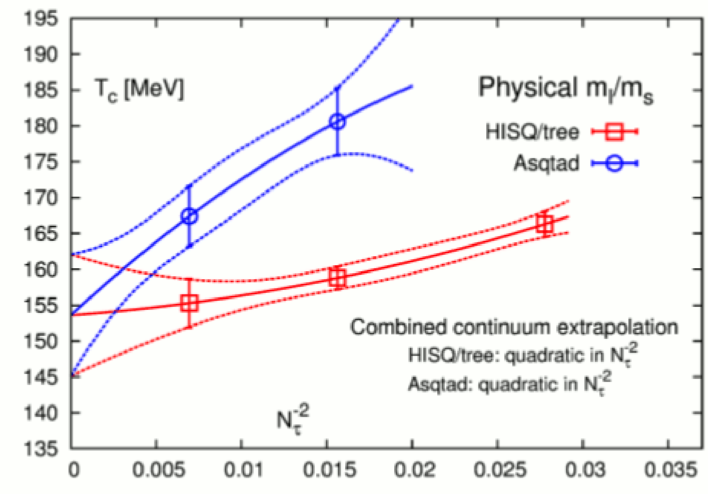
Staggered-type quarks at physical quark mass

- HotQCD Collaboration (2012)

$$T_C = 154(9)\ \text{MeV}$$

- Wuppertal-Budapest group(2010)

$$T_C = 147(2)(3)\ \text{MeV}$$

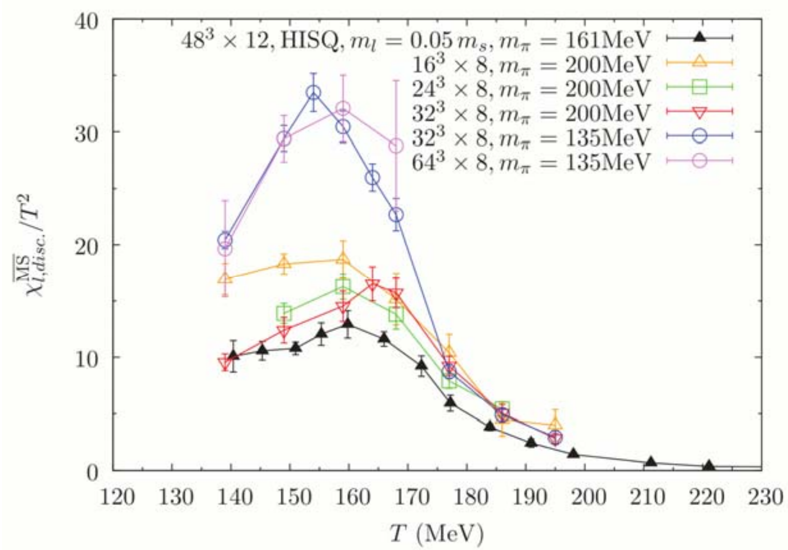


HotQCD: Phys. Rev. D85, 054503 (2012)

Wilson-type quark

- Domein-wall fermion
- HotQCD Collab. (2014)

$$T_C = 155(9)\ \text{MeV}$$

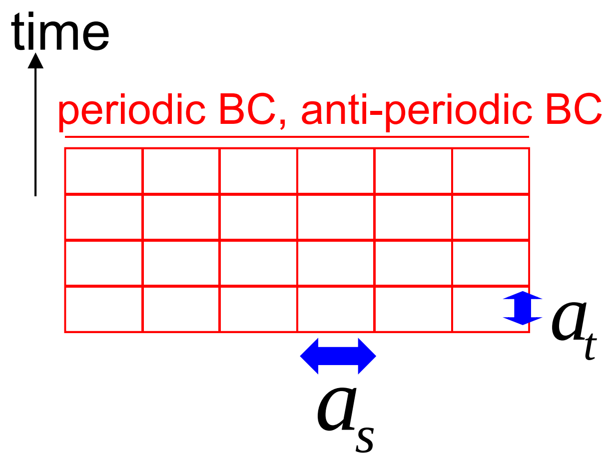


Equation of state

- Energy density and pressure

$$\frac{\varepsilon}{T^4} = -\frac{1}{VT^4}\left(\frac{\partial \ln Z}{\partial T^{-1}}\right)_V \qquad \frac{p}{T^4} = -\frac{1}{T^3}\left(\frac{\partial \ln Z}{\partial V}\right)_T$$

$$T = (N_t a_t)^{-1} \qquad V = (N_s a_s)^3$$



- Equation of state on an anisotropic lattice

$$Z = \int DU\, e^{-S} \qquad \langle O \rangle = \frac{1}{Z}\int DU\, O\, e^{-S}$$

$$\frac{\varepsilon}{T^4} = -\frac{N_t^4 a_t^4}{N_s^3 N_t a_s^3}\left(\frac{\partial \ln Z}{\partial a_t}\right)_{a_s} = -\frac{N_t^4}{N_s^3 N_t}\left(\frac{a_t}{a_s}\right)^3 \left\langle a_t \frac{\partial S}{\partial a_t}\bigg|_{a_s} \right\rangle$$

$$\frac{p}{T^4} = \frac{N_t^4 a_t^4}{3 N_s^3 N_t a_s^2 a_t}\left(\frac{\partial \ln Z}{\partial a_s}\right)_{a_t} = -\frac{N_t^4}{3 N_s^3 N_t}\left(\frac{a_t}{a_s}\right)^3 \left\langle a_s \frac{\partial S}{\partial a_s}\bigg|_{a_t} \right\rangle$$

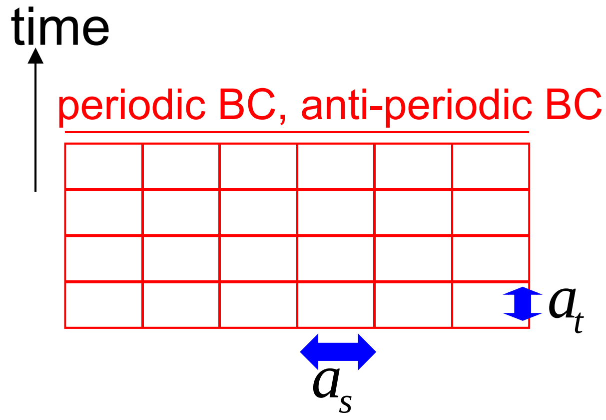
Equation of state

Differential method: pure gauge theory (no dynamical quarks)

- Energy density and pressure

$$\frac{\varepsilon}{T^4} = -\frac{1}{VT^4}\left(\frac{\partial \ln Z}{\partial T^{-1}}\right)_V \qquad \frac{p}{T^4} = -\frac{1}{T^3}\left(\frac{\partial \ln Z}{\partial V}\right)_T$$

$$T = (N_t a_t)^{-1} \qquad V = (N_s a_s)^3$$



- Pure gauge theory on an anisotropic lattice

$$Z = \int DU\, e^{-S_g} \qquad S_g = -\beta_s \sum_{n,i<j} \mathrm{Re} P_{ij}(n) - \beta_t \sum_{n,i} \mathrm{Re} P_{i4}(n)$$

$$\frac{\varepsilon}{T^4} = -\frac{N_t^4 a_t^4}{N_s^3 N_t a_s^3}\left(\frac{\partial \ln Z}{\partial a_t}\right)_{a_s} = -3N_t^4\left(\frac{a_t}{a_s}\right)^3\left[a_t \left.\frac{\partial \beta_s}{\partial a_t}\right|_{a_s}\left(\langle \overline{P}_s\rangle - \langle \overline{P}_s\rangle_0\right) + a_\tau \left.\frac{\partial \beta_t}{\partial a_t}\right|_{a_s}\left(\langle \overline{P}_t\rangle - \langle \overline{P}_t\rangle_0\right)\right]$$

$$\frac{p}{T^4} = \frac{N_t^4 a_t^4}{3N_s^3 N_t a_s^2 a_t}\left(\frac{\partial \ln Z}{\partial a_s}\right)_{a_t} = N_t^4\left(\frac{a_t}{a_s}\right)^3\left[a_s \left.\frac{\partial \beta_s}{\partial a_s}\right|_{a_t}\left(\langle \overline{P}_s\rangle - \langle \overline{P}_s\rangle_0\right) + a_s \left.\frac{\partial \beta_t}{\partial a_s}\right|_{a_s}\left(\langle \overline{P}_t\rangle - \langle \overline{P}_t\rangle_0\right)\right]$$

where $\overline{P}_{s(t)} = \left(3N_s^3 N_t\right)^{-1} \sum_n P_{s(t)}(n)$, $\langle \overline{p}_{s(t)} \rangle_0$ is the expectation value at T=0.

- Anisotropic coefficients (Karsch coefficients)

$$a_t \frac{\partial \beta_s}{\partial a_t}\bigg|_{a_s}, \quad a_t \frac{\partial \beta_t}{\partial a_t}\bigg|_{a_s}, \quad a_s \frac{\partial \beta_s}{\partial a_s}\bigg|_{a_t}, \quad a_s \frac{\partial \beta_t}{\partial a_s}\bigg|_{a_t}$$

  - Perturbation theory (Karsch, Nucl. Phys. B205, 285 (1982))
  - Comparison of temporal and spatial Wilson loop (G. Burger et al., Nucl. Phys. B304, 587 (1988), J. Engels et al., Nucl. Phys. B564, 303 (2002))
  - Phase transition line in the (β_s, β_t) plane (S. Ejiri et al. Phys. Rev. D58. 094505, (1998))

➡ Difficult to compute

- At ξ=1, $\left(\xi = a_s / a_t\,,\;\; a_t = a_s = a,\;\; \beta_t = \beta_s = \beta,\;\; \langle \bar{p}_t \rangle = \langle \bar{p}_s \rangle = \langle \bar{p} \rangle\right)$

  Interaction measure:

$$\frac{\varepsilon - 3p}{T^4} = -3N_t^4 \left(\frac{a_t}{a_s}\right)^3 \left[\left(a_t \frac{\partial \beta_s}{\partial a_t}\bigg|_{a_s} + a_s \frac{\partial \beta_s}{\partial a_s}\bigg|_{a_t} \right) \left(\langle \bar{P}_s \rangle - \langle \bar{P}_s \rangle_0 \right) + \left(a_t \frac{\partial \beta_t}{\partial a_t}\bigg|_{a_s} + a_s \frac{\partial \beta_t}{\partial a_s}\bigg|_{a_t} \right) \left(\langle \bar{P}_t \rangle - \langle \bar{P}_t \rangle_0 \right) \right]$$

$$= 6N_\tau^4 a \frac{d\beta}{da} \left(\langle \bar{P} \rangle - \langle \bar{P} \rangle_0 \right) \qquad \left(\because\; a \frac{d\beta_{s(t)}}{da} = a_t \frac{\partial \beta_{s(t)}}{\partial a_t}\bigg|_{a_s} + a_s \frac{\partial \beta_{s(t)}}{\partial a_s}\bigg|_{a_t} \right)$$

Equation of State

- Integral method
  - Interaction measure $\dfrac{\varepsilon - 3p}{T^4} = -\dfrac{1}{VT^3}\dfrac{\partial \ln Z}{\partial \ln a},$

    computed by plaquette (1x1 Wilson loop) $\langle P \rangle$ and the derivative of det*M*.
  - Pressure
    - Integral

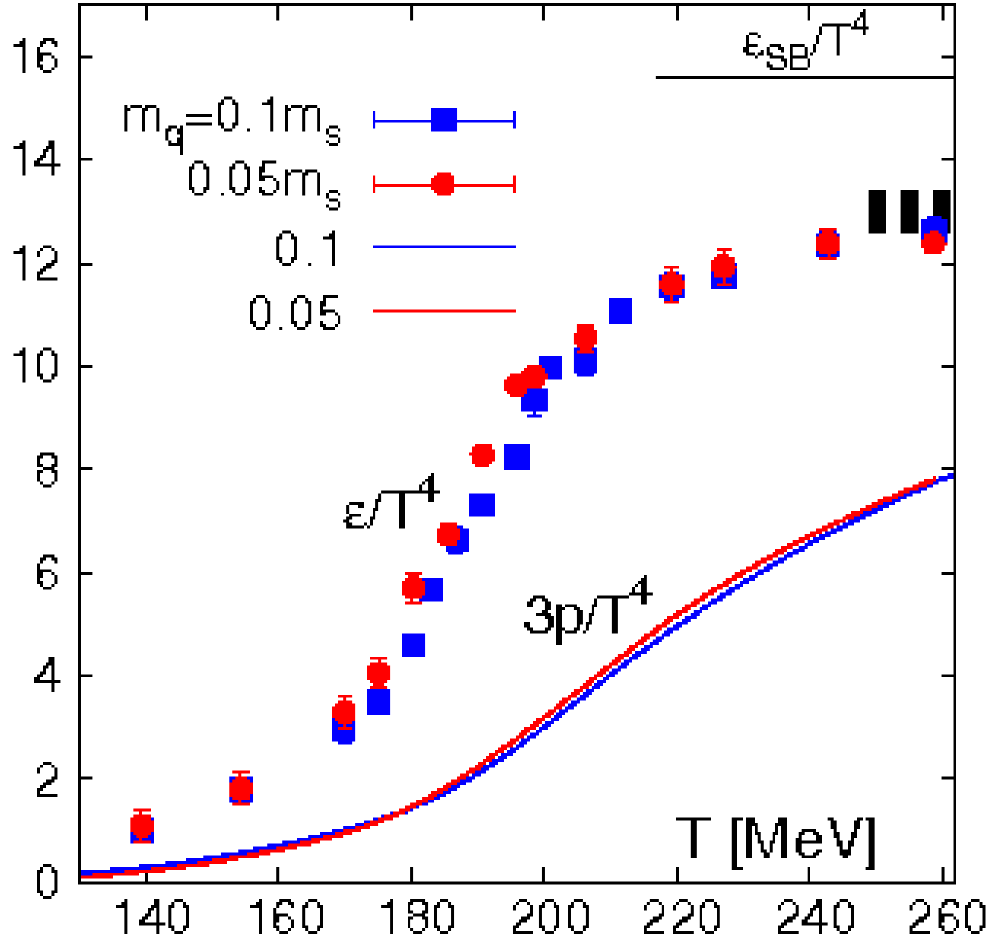
$$\frac{p}{T^4} = \frac{1}{VT^3}\ln Z$$

$$\left.\frac{p}{T^4}\right|_{a} - \left.\frac{p}{T^4}\right|_{a_0} = -\int_{a_0}^{a} \frac{\varepsilon' - 3p'}{T'^4}\, d\left(\ln a'\right)$$

↖ a0: start point p=0

Equation of state with physical mass

- Pressure and energy density by the integral method (staggered-type quarks)



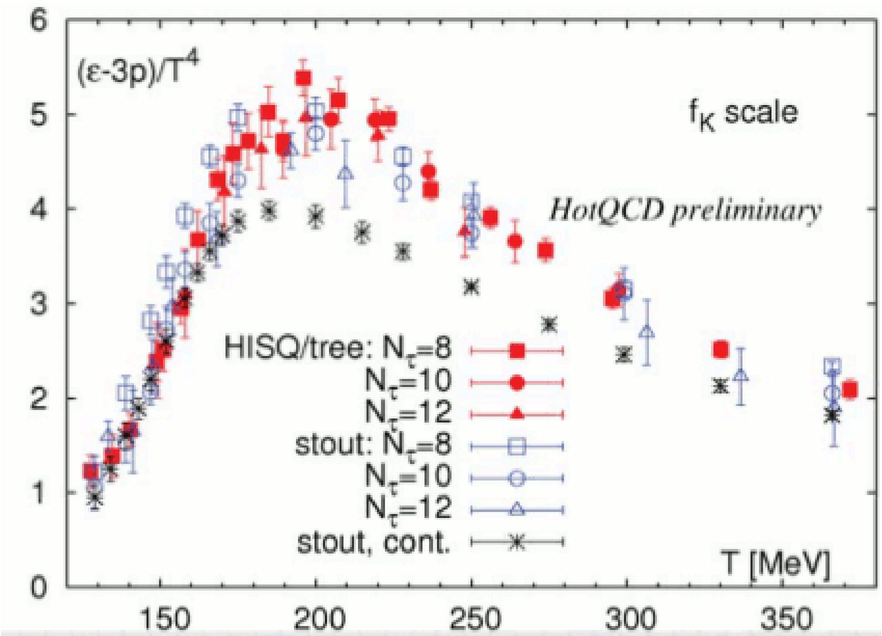
hotQCD Collab., PRD81, 054504(2010)

p4 staggered quark near the physical point

$N\tau$=8: $a\approx$0.14fm

Finite temperature QCD by Monte-Carlo simulations

- Many interesting results in QCD thermodynamics at $\mu=0$



$(\varepsilon\text{-}3p)/T^4$ vs T
at the Physical quark mass

hotQCD Collaboration (HISQ)
Budapest-Wuppertal group (stout)
Lattice 2012

- Study at $\mu\neq0$: in the stage of development.

Problem of Lattice QCD at $\mu \neq 0$

- Problem of Complex Determinant at $\mu \neq 0$

$$(M(\mu))^{+} = \gamma_5 M(-\mu)\gamma_5 \quad \Rightarrow \quad \underline{(\det M(\mu))^{*} = \det M(-\mu) \neq \det M(\mu)}$$

  - Boltzmann weight: complex at $\mu \neq 0$
    - Monte-Carlo method is not applicable.
    - Configuration cannot be generated.

- Various approaches
  - Taylor expansion in μ
  - Reweighting method: Simulations at $\mu=0$, Modify the weight
  - Histogram method: Provability distribution of a physical quantity
  - Analytic continuation from imaginary chemical potential simulations
  - Stochastic quantization for QCD at finite μ: Langevin Equation

Taylor expansion method for EoS

- Heavy-ion collisions: low density
  - In the heavy-ion collision at RHIC, the interesting regime of μ_q is around $\mu_q/T_c \approx 0.1$.

- Taylor expansion in μ at μ=0.

$$\frac{p}{T^4}(\mu) = \frac{p}{T^4}(0) + c_2\left(\frac{\mu_q}{T}\right)^2 + c_4\left(\frac{\mu_q}{T}\right)^4 + c_6\left(\frac{\mu_q}{T}\right)^6 + \cdots \qquad \frac{p}{T^4} = \frac{1}{VT^3}\ln Z$$

$$c_2 = \frac{N_t^3}{2N_s^3}\frac{\partial^2 \ln Z}{\partial(\mu_q/T)^2}, \quad c_4 = \frac{N_t^3}{4!N_s^3}\frac{\partial^4 \ln Z}{\partial(\mu_q/T)^4}, \ldots \qquad \boxed{\frac{\partial^n \ln Z}{\partial(\mu_q/T)^n} = 0 \ \text{ for } n:\text{odd}}$$

- Simulations at μ=0: Free from the complex determinant problem.
- Calculation of the derivatives: a basic technique for QCD thermodynamics.

  e.g. Energy density, Quark number, Quark number susceptibility

$$\frac{\varepsilon - 3p}{T^4} = -\frac{N_t^3}{N_s^3}\frac{\partial \ln Z}{\partial \ln a}, \qquad n_q = \frac{N_t^3}{N_s^3}\frac{\partial \ln Z}{\partial(\mu_q/T)}, \qquad \frac{\chi_q}{T^2} = 9\frac{\chi_B}{T^2} = \frac{N_t^3}{N_s^3}\frac{\partial^2 \ln Z}{\partial(\mu_q/T)^2}$$

- Taylor expansion method ➡ Useful for EoS study

Derivatives of grand partition function

$$c_2 = \frac{N_\tau}{2!N_\sigma^3}\frac{\partial^2 \ln Z}{\partial \mu^2} = \frac{N_\tau}{2!N_\sigma^3}\mathrm{A}_2, \quad c_4 = \frac{1}{4!N_\sigma^3 N_\tau}\frac{\partial^4 \ln Z}{\partial \mu^4} = \frac{1}{4!N_\sigma^3 N_\tau}\left(\mathrm{A}_4 - 3\mathrm{A}_2^2\right),$$

$$c_6 = \frac{1}{6!N_\sigma^3 N_\tau^3}\frac{\partial^6 \ln Z}{\partial \mu^6} = \frac{1}{6!N_\sigma^3 N_\tau^3}\left(\mathrm{A}_6 - 15\mathrm{A}_4\mathrm{A}_2 + 30\mathrm{A}_2^3\right).$$

$\left(\mu \equiv \mu_q a = \left(\mu_q / T\right)/N_\tau\right)$ Lattice size : $N_\sigma^3 \times N_\tau$

$$\mathrm{A}_2 = \left\langle \frac{N_f}{4}\frac{\partial^2 \ln\det M}{\partial\mu^2}\right\rangle + \left\langle\left(\frac{N_f}{4}\frac{\partial \ln\det M}{\partial\mu}\right)^2\right\rangle$$

$$\mathrm{A}_4 = \left\langle \frac{N_f}{4}\frac{\partial^4 \ln\det M}{\partial\mu^4}\right\rangle + 4\left\langle\left(\frac{N_f}{4}\right)^2\frac{\partial^3 \ln\det M}{\partial\mu^3}\frac{\partial \ln\det M}{\partial\mu}\right\rangle + 3\left\langle\left(\frac{N_f}{4}\frac{\partial^2 \ln\det M}{\partial\mu^2}\right)^2\right\rangle + 6\left\langle\left(\frac{N_f}{4}\right)^3\frac{\partial^2 \ln\det M}{\partial\mu^2}\left(\frac{\partial \ln\det M}{\partial\mu}\right)^2\right\rangle + \left\langle\left(\frac{N_f}{4}\frac{\partial \ln\det M}{\partial\mu}\right)^4\right\rangle$$

$$\mathrm{A}_6 = \left\langle \frac{N_f}{4}\frac{\partial^6 \ln\det M}{\partial\mu^6}\right\rangle + 6\left\langle\left(\frac{N_f}{4}\right)^2\frac{\partial^5 \ln\det M}{\partial\mu^5}\frac{\partial \ln\det M}{\partial\mu}\right\rangle + 15\left\langle\left(\frac{N_f}{4}\right)^2\frac{\partial^4 \ln\det M}{\partial\mu^4}\frac{\partial^2 \ln\det M}{\partial\mu^2}\right\rangle + 10\left\langle\left(\frac{N_f}{4}\frac{\partial^3 \ln\det M}{\partial\mu^3}\right)^2\right\rangle$$

$$+15\left\langle\left(\frac{N_f}{4}\right)^3\frac{\partial^4 \ln\det M}{\partial\mu^4}\left(\frac{\partial \ln\det M}{\partial\mu}\right)^2\right\rangle + 60\left\langle\left(\frac{N_f}{4}\right)^3\frac{\partial^3 \ln\det M}{\partial\mu^3}\frac{\partial^2 \ln\det M}{\partial\mu^2}\frac{\partial \ln\det M}{\partial\mu}\right\rangle + 15\left\langle\left(\frac{N_f}{4}\frac{\partial^2 \ln\det M}{\partial\mu^2}\right)^3\right\rangle + 20\left\langle\left(\frac{N_f}{4}\right)^4\frac{\partial^3 \ln\det M}{\partial\mu^3}\left(\frac{\partial \ln\det M}{\partial\mu}\right)^3\right\rangle$$

$$+45\left\langle\left(\frac{N_f}{4}\right)^4\left(\frac{\partial^2 \ln\det M}{\partial\mu^2}\right)^2\left(\frac{\partial \ln\det M}{\partial\mu}\right)^2\right\rangle + 15\left\langle\left(\frac{N_f}{4}\right)^5\frac{\partial^2 \ln\det M}{\partial\mu^2}\left(\frac{\partial \ln\det M}{\partial\mu}\right)^4\right\rangle + \left\langle\left(\frac{N_f}{4}\frac{\partial \ln\det M}{\partial\mu}\right)^6\right\rangle$$

Derivatives of grand partition function

$$\frac{\partial \ln\det M}{\partial \mu} = \mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial \mu} \right)$$

$$\frac{\partial^2 \ln\det M}{\partial \mu^2} = \mathrm{Tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2} \right) - \mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial \mu} M^{-1}\frac{\partial M}{\partial \mu} \right)$$

$$\frac{\partial^3 \ln\det M}{\partial \mu^3} = \mathrm{Tr}\left(M^{-1}\frac{\partial^3 M}{\partial \mu^3} \right) - 3\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial \mu} M^{-1}\frac{\partial^2 M}{\partial \mu^2} \right) + 2\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial \mu} M^{-1}\frac{\partial M}{\partial \mu} M^{-1}\frac{\partial M}{\partial \mu} \right)$$

$$\frac{\partial^4 \ln\det M}{\partial \mu^4} = \mathrm{Tr}\left(M^{-1}\frac{\partial^4 M}{\partial \mu^4} \right) - 4\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial \mu} M^{-1}\frac{\partial^3 M}{\partial \mu^3} \right) - 3\mathrm{Tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2} M^{-1}\frac{\partial^2 M}{\partial \mu^2} \right)$$
$$+12\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial \mu} M^{-1}\frac{\partial M}{\partial \mu} M^{-1}\frac{\partial^2 M}{\partial \mu^2} \right) - 6\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial \mu} M^{-1}\frac{\partial M}{\partial \mu} M^{-1}\frac{\partial M}{\partial \mu} M^{-1}\frac{\partial M}{\partial \mu} \right)$$

These are calculated by the random noise method.

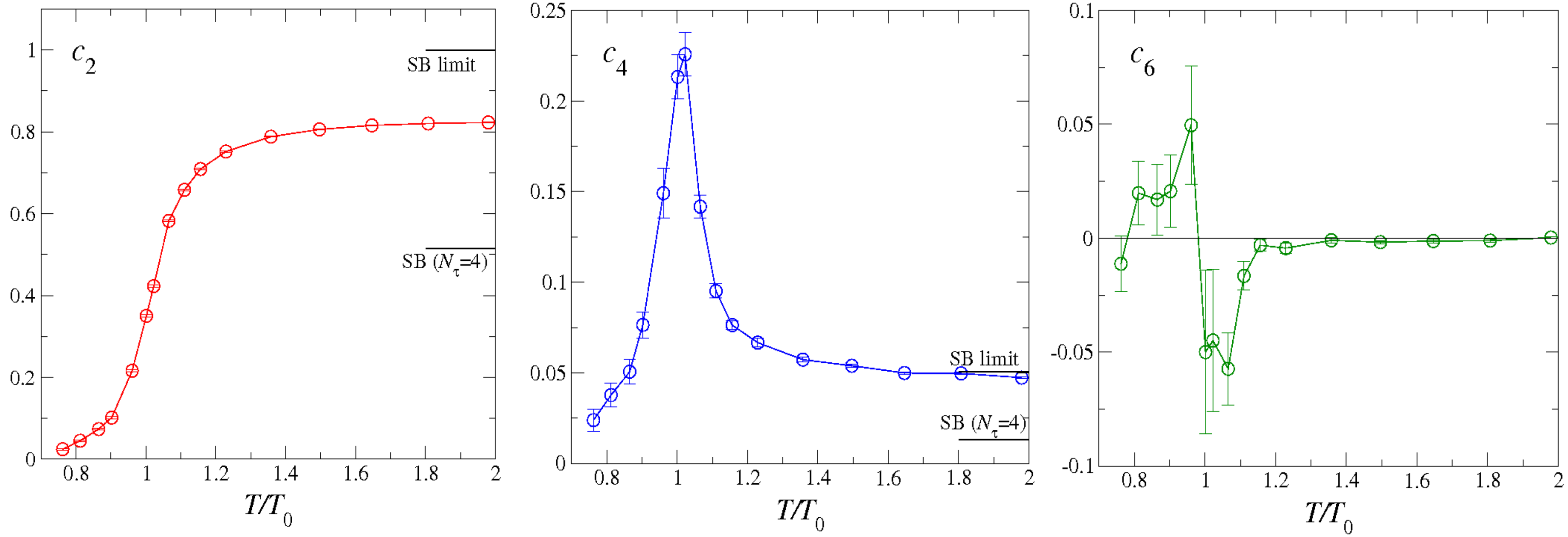
Derivatives of pressure and susceptibilities

- Taylor expansion at μ=0

$$\frac{p}{T^4} = \frac{1}{VT^3}\ln Z = c_0 + c_2\left(\frac{\mu_q}{T}\right)^2 + c_4\left(\frac{\mu_q}{T}\right)^4 + c_6\left(\frac{\mu_q}{T}\right)^6 + \cdots$$

$$c_n = \frac{N_\tau^3}{n!N_\sigma^3}\frac{\partial^n \ln Z}{\partial(\mu_q/T)^n}$$

- Bielefeld-Swansea Collab., Phys.Rev.D68(2003)014507; D71, 054508 (2005)
  - 2-flavor p4-improved staggered fermion action $m_\pi \sim 770$MeV, $m_\pi/m_\rho \sim 0.7$



This method works well in the low density region.

Reweighting method for μ≠0 and Sign problem

(Ferrenberg-Swendsen method)

- Reweighting method
  - Boltzmann weight: Complex for μ>0
    - Monte-Carlo method is not applicable directly.
  - Perform Simulation at μ=0.

partition function:

$$Z = \int DU \left(\det M(\mu)\right)^{N_f} e^{-S_g}$$

$$\det M \equiv \left|\det M\right| e^{i\theta}$$

$$\langle O \rangle_{(\beta,\mu)} = \frac{1}{Z}\int DUO\left(\det M(\mu)\right)^{N_f} e^{-S_g(\beta)} = \frac{\left\langle O e^{i\theta} \left|\det^{N_f} M(\mu)/\det^{N_f} M(0)\right| \right\rangle_{(\beta,0)}}{\left\langle e^{i\theta} \left|\det^{N_f} M(\mu)/\det^{N_f} M(0)\right| \right\rangle_{(\beta,0)}}$$

- Sign problem
  - If $e^{i\theta}$ changes its sign frequently, $\langle O e^{i\theta} \cdots \rangle_{(\beta,0)}$ and $\langle e^{i\theta} \cdots \rangle_{(\beta,0)}$ become smaller than their statistical errors.
  - Then $\langle O \rangle_{(\beta,\mu)}$ cannot be computed.

Summary

- I introduced studies of Quantum chromodynamics (QCD) by computer simulations.
- The study of QCD is very important to understand the evolution of the Universe.
- The computer simulation is very powerful method to study QCD.