Kaon-Nucleon systems and their interaction in the Skyrme model

RCNP, Osaka University
Takashi Ezoe
Atsushi Hosaka

T. Ezoe and A. Hosaka
arXiv:1605.01203
Contents

1. Introduction

2. The Skyrme model

3. Method

4. Results and Discussions

5. Summary
1. Introduction
Introduction

Kaon nucleon systems have been receiving a lot of attention

- Strong attraction between the anti-kaon and the nucleon
- The anti-kaon nucleon ($\bar{K}N$) bound state = $\Lambda(1405)$

- Few body nuclear systems with $\bar{K}$

The properties of these systems are under debate
Introduction • Purpose

the $\bar{K}N$ interaction is important
to investigate the few body systems with $\bar{K}$

Theoretical studies of $\bar{K}N$ interaction

• Phenomenological approach

• Chiral theory: based on a 4-point local interaction
Introduction · Purpose

the $\bar{K}N$ interaction is important to investigate the few body systems with $\bar{K}$

Theoretical studies of $\bar{K}N$ interaction

- Phenomenological approach
- Chiral theory: based on a 4-point local interaction

Investigate the KN system in the Skyrme model where the nucleon is described as a solitron.
2. The Skyrme model
The Skyrme model


- Describe the interaction between mesons and baryons by mesons
- Baryon emerges as a soliton of meson fields.

\[
\phi = \frac{1}{\sqrt{2}} \lambda_a \phi_a = \begin{pmatrix}
\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & \pi^+ & K^+
\pi^- & -\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & K^0
K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}} \eta
\end{pmatrix}
\]

\[U = \exp \left[ i \frac{2}{F_\pi} \lambda_a \phi_a \right] \]

\(\lambda_a\): Gell-Mann matrices \((a = 1, 2, \ldots, 8)\)
The Skyrme model 1


- Describe the interaction between mesons and baryons by mesons
- Baryons emerge as a soliton of meson fields.

\[ \phi = \frac{1}{\sqrt{2}} \lambda_a \phi_a = \begin{pmatrix} \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}} \eta \end{pmatrix} \]

\[ U = \exp \left[ i \frac{2}{F_\pi} \lambda_a \phi_a \right] \quad \lambda_a: \text{Gell-Mann matrices (} a = 1, 2, \ldots, 8) \]

- For SU(2)

\[ L = \frac{F_\pi^2}{16} \text{tr} (\partial_\mu U \partial^\mu U^+) + \frac{1}{32e^2} \text{tr} [(\partial_\mu U) U^+, (\partial_\nu U) U^+]^2 \]

The kinetic term \( F_\pi, e: \text{parameters} \)

The interaction term (the Skyrme term)
The Skyrme model 2

- **Hedgehog ansatz**
  - $\pi$ has three degrees of freedom ($\pi^0$, $\pi^+$, $\pi^-$)
  - two of these: the angles of the radial vector, $\theta$, $\varphi$
  - the rest: a function depending on $r$
  - a special configuration called the hedgehog ansatz

  **Hedgehog ansatz:** $U_H = \exp \left[ i\tau \cdot \hat{r} F(r) \right]$

- **Quantization**
  - The hedgehog ansatz is classical
  - without spin or isospin
  - become a physical state by quantization

  $$U_H (x) \rightarrow U_H (t, x) = A(t) \exp \left[ i\tau_a R_{ab} (t) \hat{r}_b F(r) \right] A^\dagger (t)$$

  The baryon with $I=J$ from the symmetry of the hedgehog soliton
3. Method
Method

SU(3) symmetry is broken $\rightarrow m_u = m_d = 0, m_s \neq 0$

**Callan-Klebanov approach (CK approach)**

- Introduce the kaon as fluctuations *around the hedgehog soliton*
- Form a bound state of the kaon and the hedgehog soliton
- rotate the system to generate hyperons
- Follow the $1/N_c$ counting rule


**Our approach**

- Rotate the hedgehog soliton to generate the nucleon
- Introduce the kaon as fluctuations *around the nucleon*
- describe kaon-nucleon systems
- Violate the $1/N_c$ counting rule
Method

SU(3) symmetry is broken $\rightarrow m_u = m_d = 0, m_s \neq 0$

Callan-Klebanov approach (CK approach)

- Introduce the kaon as fluctuations around the hedgehog soliton
- Form a bound state of the kaon and the hedgehog soliton
- rotate the system to generate hyperons
- Follow the $1/N_c$ counting rule
- Projection after variation, The strong coupling


Our approach

- Rotate the hedgehog soliton to generate the nucleon
- Introduce the kaon as fluctuations around the nucleon
- describe kaon-nucleon systems
- Violate the $1/N_c$ counting rule
- Variation after projection, The weak coupling
Lagrangian and ansatz

• Expand to the SU(3) Skyrme model

\[
L = \frac{F^2}{16} \text{tr} \left( \partial_\mu U \partial^\mu U^\dagger \right) + \frac{1}{32e^2} \text{tr} \left[ (\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger \right]^2 \\
+ L_{SB} + L_{WZ}
\]

• Ansatz

\[
U = \begin{cases} 
A(t) \sqrt{U_\pi} U_K \sqrt{U_\pi} A^\dagger(t) : \text{Callan-Klebanov ansatz} \\
A(t) \sqrt{U_\pi} A^\dagger(t) U_K A(t) \sqrt{U_\pi} A^\dagger(t) : \text{Our ansatz}
\end{cases}
\]

\[
U_\pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
Hedgehog ansatz
(2×2 matrix)

\[
U_K = \exp \left[ i \frac{2}{F_\pi} \lambda_a K_a \right], \quad a = 3, 4, 5, 6
\]

\[
\lambda_a K_a = \sqrt{2} \begin{pmatrix} 0_{2 \times 2} & K \\ K^\dagger & 0 \end{pmatrix} \quad K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \quad K^\dagger = \begin{pmatrix} K^0 \\ K^- \end{pmatrix}
\]
Derivation 1

- Substitute our ansatz for the Lagrangian

Ansatz

\[ U = A(t) \sqrt{U_\pi A^\dagger(t)U_K A(t)} \sqrt{U_\pi A^\dagger(t)} \]

\[ U_K = \exp \left[ i \frac{2}{F_\pi} \lambda_a K_a \right], \quad a = 3, 4, 5, 6 \]

\[ \lambda_a K_a = \sqrt{2} \begin{pmatrix} 0_{2 \times 2} & K \\ K^\dagger & 0 \end{pmatrix} \quad K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \quad K^\dagger = \begin{pmatrix} K^0 \\ K^- \end{pmatrix} \]

Lagrangian

\[ L = \frac{F_\pi^2}{16} \text{tr} \left( \partial_\mu U \partial^\mu U^\dagger \right) + \frac{1}{32e^2} \text{tr} \left[ (\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger \right]^2 \]

\[ + L_{SB} + L_{WZ} \]
Obtaining Lagrangian

\[ L = L_{SU(2)} + L_{KN} \]

\[ L_{SU(2)} = \frac{1}{16} F_\pi^2 \text{tr} \left[ \partial_\mu \tilde{U}^\dagger \partial^\mu \tilde{U} \right] + \frac{1}{32 e^2} \text{tr} \left[ \partial_\mu \tilde{U} \tilde{U}^\dagger, \partial_\nu \tilde{U} \tilde{U}^\dagger \right]^2 \]

\[ L_{KN} = (D_\mu K)^\dagger D^\mu K - K^\dagger a_\mu^\dagger a^\mu K - m_K^2 K^\dagger K \]

\[ + \frac{1}{(eF_\pi)^2} \left\{ -K^\dagger K \text{tr} \left[ \partial_\mu U_H U_H^\dagger, \partial_\nu U_H U_H^\dagger \right] \right\}^2 - 2 (D_\mu K)^\dagger D_\nu K \text{tr} (a^\mu a^{\nu}) \]

\[ - \frac{1}{2} (D_\mu K)^\dagger D^\mu K \text{tr} \left( \partial_\nu U_H^\dagger \partial^{\nu} U_H \right) + 6 (D_\nu K)^\dagger [a^{\nu}, a^\mu] D_\mu K \}

\[ + \frac{3i}{F_\pi^2} B^\mu \left[ (D_\mu K)^\dagger K - K^\dagger (D_\mu K) \right] \]

\[ \tilde{U} = A(t) U_H A^\dagger(t), \quad \tilde{\xi} = A(t) \sqrt{U_H} A^\dagger(t) \quad D_\mu K = \partial_\mu K + v_\mu K \]

\[ v_\mu = \frac{1}{2} \left( \tilde{\xi}^\dagger \partial_\mu \tilde{\xi} + \tilde{\xi} \partial_\mu \tilde{\xi}^\dagger \right) \quad a_\mu = \frac{1}{2} \left( \tilde{\xi}^\dagger \partial_\mu \tilde{\xi} - \tilde{\xi} \partial_\mu \tilde{\xi}^\dagger \right) \]

\[ B^\mu = - \frac{\varepsilon^{\mu\nu\alpha\beta}}{24\pi^2} \text{tr} \left[ \left( U_H^\dagger \partial_\nu U_H \right) \left( U_H^\dagger \partial_\alpha U_H \right) \left( U_H^\dagger \partial_\beta U_H \right) \right] \]

Derivation 2

- Decompose the kaon filed

\[
\begin{pmatrix} K^+ \\ K^0 \end{pmatrix} = \psi_I K (t, \mathbf{r}) \rightarrow \psi_I K (\mathbf{r}) e^{-iEt} \]

Spatial wave function

- Expand the $K(\mathbf{r})$ by the spherical harmonics

\[
K (\mathbf{r}) = \sum_{l, m} C_{l m \alpha} Y_{l m} (\theta, \phi) k_{l \alpha} (r)
\]

$Y_{l m}(\theta, \phi)$: Spherical harmonics

- Take a variation with respect to the kaon radial function

⇒ Obtain the equation of motion for the kaon around the nucleon
4. Results and Discussions
Results (Equation of motion)

- **Equation of motion (E.o.M)**

\[- \frac{1}{r^2} \frac{d}{dr} \left(r^2 h(r) \frac{dk^\alpha_i (r)}{dr} \right) - E^2 f(r) k^\alpha_i (r) + (m_K^2 + V(r)) k^\alpha_i (r) = 0\]

$h(r), f(r)$: functions depending on $r$

$m_K$ : the mass of the kaon, $E$ : the kaon energy

$V(r)$ : the kaon nucleon interaction term

---

Investigate KN systems by solving this E.o.M

---

Concentrate on the $\bar{K}N(l=0)$ bound state
**$\bar{K}N$ Bound state**

- $\bar{K}N$ bound states with $I_{tot} = 0$, $l = 0$ (Binding Energy: B.E.)

<table>
<thead>
<tr>
<th>Parameter Set</th>
<th>$F_\pi$ [MeV]</th>
<th>$e$</th>
<th>B.E. [MeV]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter set 1</td>
<td>129</td>
<td>5.45</td>
<td>82.9</td>
</tr>
<tr>
<td>Parameter set 2</td>
<td>186</td>
<td>5.45</td>
<td>27.2</td>
</tr>
<tr>
<td>Parameter set 3</td>
<td>186</td>
<td>4.82</td>
<td>32.9</td>
</tr>
</tbody>
</table>

$m_K = 495$ MeV

when $F_\pi$ is the exp. value, of order ten MeV
Bound state properties

- Root mean square radii for N and K

\[ \langle r_N^2 \rangle = \int_0^\infty dr \ r^2 \rho_B(r), \ \rho_B(r) = -\frac{2}{\pi} \sin^2 FF' \]


\[ \langle r_K^2 \rangle = \int dV \ r^2 \ [Y_{00}(\hat{r}) k_0^0(r)]^2 = \int_0^\infty dr \ r^4 k^2(r) \]

\[ Y_{00} = \frac{1}{\sqrt{4\pi}} \]

- properties of the \( \bar{K}N (I = 0) \) bound states

<table>
<thead>
<tr>
<th></th>
<th>( F_\pi ) [MeV]</th>
<th>( e )</th>
<th>B.E. [MeV]</th>
<th>( \langle r_N^2 \rangle^{1/2} ) [fm]</th>
<th>( \langle r_K^2 \rangle^{1/2} ) [fm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>parameter set 1</td>
<td>129</td>
<td>5.45</td>
<td>82.9</td>
<td>0.59</td>
<td>0.99</td>
</tr>
<tr>
<td>parameter set 2</td>
<td>186</td>
<td>5.45</td>
<td>27.2</td>
<td>0.41</td>
<td>1.19</td>
</tr>
<tr>
<td>parameter set 3</td>
<td>186</td>
<td>4.82</td>
<td>32.9</td>
<td>0.46</td>
<td>1.18</td>
</tr>
</tbody>
</table>

The anti-kaon is weakly binding to the nucleon
Potential

\[- \frac{1}{r^2} \frac{d}{dr} \left( r^2 h(r) \frac{dk_l^\alpha (r)}{dr} \right) - \frac{1}{m_K + E} \frac{d}{dr} \left( r^2 \frac{dk_l^\alpha (r)}{dr} \right) + U(r) k_l^\alpha (r) = \varepsilon k_l^\alpha (r) \]

\[ U(r) = - \frac{1}{m_K + E} \left[ h(r) - 1 \right] \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{dh(r)}{dr} \frac{d}{dr} - \frac{(f(r) - 1) E^2}{m_K + E} \]

**Properties of resulting potential \( U \)**

1. Nonlocal and depend on the kaon energy
2. Contain isospin dependent and independent central forces and the similar spin-orbit (LS) forces
3. In the short range, behave as a repulsive force proportional to \( 1/r^2 \)
\( \bar{K}N \ (L = 0, \ I_{tot} = 0) \) potential

\[ U(r) = \frac{U(k)}{k} \text{[MeV]} \]

\[ m_K = 495 \text{ MeV} \]

 Equivalent local potential

\[ \tilde{U}(r) = \frac{U(r)k(r)}{k(r)} \]

<table>
<thead>
<tr>
<th>parameter set</th>
<th>( F_\pi ) [MeV]</th>
<th>( e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>parameter set 1</td>
<td>129</td>
<td>5.45</td>
</tr>
<tr>
<td>parameter set 2</td>
<td>186</td>
<td>5.45</td>
</tr>
<tr>
<td>parameter set 3</td>
<td>186</td>
<td>4.82</td>
</tr>
</tbody>
</table>

Repulsion proportional to \( 1/r^2 \)
5. Summary
Summaries

Construct the new method to investigate the kaon-nucleon systems in the Skyrme model and apply to a channel

**Results**
1. $\bar{K}N (I = 0)$ bound states exist with B.E. of order ten MeV
2. The obtaining potential is nonlocal and depends on the kaon energy
3. Contain central and LS terms with and without isospin dependence
4. repulsion proportional to $1/r^2$ for small $r$

**Future works**
1. Scattering states of the KN system (ongoing)
2. The $\pi \Sigma$ system
3. The properties of $\Lambda(1405)$
4. few body nuclear system with kaon
Thank you for your attention
back-up
The Skyrme model 2

- **Hedgehog ansatz**
  - $\pi$ has three degrees of freedom ($\pi^0$, $\pi^+$, $\pi^-$)
  - two of these: the angles of the radial vector, $\theta$, $\varphi$
  - the rest: a function depending on $r$
  - a special configuration called the hedgehog ansatz

Hedgehog ansatz: $U_H = \exp \left[ i \mathbf{r} \cdot \hat{r} F(r) \right]$

- minimize the mass of the soliton with B.C. for $F(r)$: $F(\infty) = 0, F(0) = \pi$

The Skyrme model 3

• Quantization
The hedgehog ansatz is classical
→ without spin or isospin
→ become a physical state by quantization
\[
U_H (\mathbf{x}) \rightarrow U_H (t, \mathbf{x}) = A(t) \exp \left[ i \tau_a R_{ab} (t) \hat{r}_b F(r) \right] A^\dagger (t)
\]
\( A(t) \): 2×2 isospin rotation matrix
\( R_{ab}(t) \): 3×3 spatial rotation matrix

The baryon with \( I=J \) from the symmetry
which the hedgehog ansatz has

• Quantized Hamiltonian
\[
H = M_{\text{sol}} + \frac{J (J+1)}{2\Lambda}
\]
\( M_{\text{sol}} \): soliton mass
\( J \): spin or isospin value
\( \Lambda \): moment of inertia

the rotation energy
1/$N_c$ expansion

**Callan-Klebanov approach**

- Hedgehog soliton
  - $O(N_c)$
  - $O(N_c^0)$
  - $O(1/N_c)$

- Soliton + Kaon
  - $O(N_c^0)$

- Hyperon di-quark + S-quark
  - $O(1/N_c)$

**Our approach**

- Hedgehog soliton
  - $O(N_c)$

- Nucleon
  - $O(1/N_c)$

- Nucleon + Kaon
  - $O(1/N_c)$

$O(N_c^0)$ is missing

2016/2/9 修論発表
Lagrangian and ansatz

- Expand to the SU(3) Skyrme model

\[ L = \frac{F^2}{16} \text{tr} \left( \partial_\mu U \partial^\mu U^\dagger \right) + \frac{1}{32e^2} \text{tr} \left( (\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger \right)^2 + L_{SB} + L_{WZ} \]

- Ansatz

\[ U = \begin{cases} 
  A(t) \sqrt{U_\pi} U_K \sqrt{U_\pi} A^\dagger(t) : \text{Callan-Klebanov ansatz} \\
  A(t) \sqrt{U_\pi} A^\dagger(t) U_K A(t) \sqrt{U_\pi} A^\dagger(t) : \text{Our ansatz} 
\end{cases} \]

the kaon around the rotating hedgehog soliton
Interaction term

\[ V(r) = V_{nor}(r) + V_{WZ}(r) \]

\[
V_{nor}(r) = -\frac{1}{4} \left( \frac{2 \sin^2 F}{r^2} + F'^2 \right) + \frac{2 s^4}{r^2} - \frac{1}{(eF_\pi)^2} \left[ \frac{2 \sin^2 F}{r^2} \left( \frac{\sin^2 F}{r^2} + 2 F'^2 \right) - 2 \frac{s^4}{r^2} \left( F'^2 + \frac{\sin^2 F}{r^2} \right) \right]
+ \frac{1}{(eF_\pi)^2} \left[ \frac{6 s^4 \sin^2 F}{r^2} + \frac{d}{dr} \{ s^2 \sin FF' \} \right]
+ \frac{2E}{\Lambda} \frac{s^2}{r^2} \left[ 1 + \frac{1}{(eF_\pi)^2} \left( F'^2 + \frac{5}{r^2} \sin^2 F \right) \right] + \frac{8E}{3\Lambda} s^2 I_{KN} + \frac{1}{(eF_\pi)^2} \frac{8Es^2}{3\Lambda} \left[ F'^2 + \frac{4}{r^2} \sin^2 F \right] I_{KN}
+ \frac{1}{r^2} \frac{d}{dr} \left[ r^2 \left( \frac{4}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} I_{KN} + \frac{3}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} \right) \right]
+ \left[ 1 + \frac{1}{(eF_\pi)^2} \left( \frac{\sin^2 F}{r^2} + F'^2 \right) \right] \frac{l (l+1)}{r^2} - \left[ 1 + \frac{1}{(eF_\pi)^2} \left( \frac{4 \sin^2 F}{r^2} + F'^2 \right) \right] \frac{16 s^2}{3r^2} J_{KN} I_{KN}
+ \frac{1}{(eF_\pi)^2} \frac{2E \sin^2 F}{\Lambda r^2} J_{KN} - \frac{1}{(eF_\pi)^2} \frac{8}{r^2} \frac{d}{dr} (\sin FF') J_{KN} I_{KN}

\[
V_{WZ}(r) = \frac{3E}{(\pi F_\pi)^2} \frac{\sin^2 F}{r^2} F' - \frac{3}{(\pi F_\pi)^2} \frac{\sin^2 F s^2}{\Lambda r^2} F' + \frac{3}{(\pi F_\pi)^2} \frac{\sin^2 F}{\Lambda r^2} F' J_{KN}
\]

\[ s = \sin(F/2) \]

\[ I_{KN} = I^K \cdot I^N, \quad J_{KN} = L^K \cdot J^N \]
Results 1 E.o.M with l = 0

\[- \frac{1}{r^2} \frac{d}{dr} \left( r^2 h(r) \frac{dk_l^\alpha(r)}{dr} \right) - E^2 f(r) k_l^\alpha(r) + (m_K^2 + V(r)) k_l^\alpha(r) = 0 \]

\[- \frac{1}{m_K + E} r^2 \frac{d}{dr} \left( \frac{r^2 dk}{dr} \right) + \frac{2}{m_K + E} \frac{1}{r^2} k \simeq 0, \quad (r \sim 0) \]

\[
h(r) = 1 + \frac{1}{(e F \pi)^2} \frac{2}{r^2} \sin^2 F \quad f(r) = 1 + \frac{1}{(e F \pi)^2} \left( \frac{2}{r^2} \sin^2 F + F'^2 \right) \quad V(r) = V_{eff}(r) \pm V_{WZ}(r) \]

\[
V_{nor}(r) = - \frac{1}{4} \left( \frac{2 \sin^2 F}{r^2} + F'^2 \right) + 2 \frac{s^4}{r^2} - \frac{1}{(e F \pi)^2} \left[ \frac{2 \sin^2 F}{r^2} \left( \frac{\sin^2 F}{r^2} + 2F'^2 \right) - 2 \frac{s^4}{r^2} \left( F'^2 + \sin^2 F \right) \right] + \frac{1}{(e F \pi)^2} \left[ \frac{6 s^4 \sin^2 F}{r^2} + \frac{d}{dr} \left( s^2 \sin FF' \right) \right] + \frac{2E}{\Lambda} \left( \frac{s^4 \sin^2 F}{r^2} + \frac{d}{dr} \left( s^2 \sin FF' \right) \right) + \frac{8E}{3\Lambda} \left[ \frac{s^2 I_{KN}}{r^2} + \frac{8Es^2}{3\Lambda} \left[ \frac{F'^2}{r^2} + \frac{4}{r^2} \sin^2 F \right] I_{KN} \right] + \frac{1}{r^2} \frac{d}{dr} \left[ \frac{4}{(e F \pi)^2} \Lambda I_{KN} + \frac{3}{(e F \pi)^2} \Lambda I_{KN} \right] \]

\[
V_{WZ}(r) = - \frac{3E}{(\pi F \pi)^2} \frac{\sin^2 F}{r^2} F' - \frac{3}{(\pi F \pi)^2} \frac{\sin^2 F s^2}{\Lambda r^2} F' \quad I_{KN} = I^K . I^N \]

Blue: dominant in \( r \sim 0 \)
Red: \( O(1/Nc) \) contributions

2016/7/29 MENU @Kyoto University
Results 1 potential for I = 0

\[
U(r) = -\frac{1}{m_K + E} \left[ \frac{h(r) - 1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{dh(r)}{dr} \frac{d}{dr} \right] - \frac{(f(r) - 1) E^2}{m_K + E} \\
+ \frac{V(r)}{m_K + E}
\]

\[
h(r) = 1 + \frac{1}{(eF_\pi)^2} \frac{2}{r^2} \sin^2 F \\
\frac{1}{(eF_\pi)^2} \frac{d}{dr} \left( \frac{2}{r^2} \sin^2 F + F'^2 \right) \\
f(r) = 1 + \frac{1}{(eF_\pi)^2} \left( \frac{2}{r^2} \sin^2 F + F'^2 \right) \\
s = \sin(F/2)
\]

\[
V_{nor}(r) = -\frac{1}{4} \left( \frac{2 \sin^2 F}{r^2} + F'^2 \right) + \frac{2 s^4}{r^2} - \frac{1}{(eF_\pi)^2} \left[ \frac{2 \sin^2 F}{r^2} \left( \frac{\sin^2 F}{r^2} + 2 F'^2 \right) - \frac{2 s^4}{r^2} \left( F'^2 + \frac{\sin^2 F}{r^2} \right) \right] \\
+ \frac{1}{(eF_\pi)^2} \frac{6}{r^2} \left[ \frac{s^4 \sin^2 F}{r^2} + \frac{d}{dr} \left\{ s^2 \sin FF' \right\} \right] \\
+ \frac{2E}{\Lambda} s^2 \left[ 1 + \frac{1}{(eF_\pi)^2} \left( F'^2 + \frac{5}{r^2} \sin^2 F \right) \right] + \frac{8E}{3\Lambda} s^2 I_{KN} + \frac{1}{(eF_\pi)^2} \frac{8E s^2}{3\Lambda} \left[ F'^2 + \frac{4}{r^2} \sin^2 F \right] I_{KN} \\
+ \frac{d}{r^2} \left[ \frac{4}{(eF_\pi)^2} \frac{E F' \sin F}{\Lambda} \frac{I_{KN}}{\Lambda} + \frac{3}{(eF_\pi)^2} \frac{E F' \sin F}{\Lambda} \right] \\
V_{WZ} = \frac{3E}{(\pi F_\pi)^2} \frac{\sin^2 F}{r^2} F' - \frac{3}{(\pi F_\pi)^2} \frac{\sin^2 F s^2}{\Lambda r^2} F'
\]

\[I_{KN} = I^K \cdot I^N\]
\(V_{\text{nor}}(r) = -\frac{1}{4} \left(2 \frac{\sin^2 F}{r^2} + F r^2 \right) + 2 \frac{s^4}{r^2} - \frac{1}{(eF\pi)^2} \left[2 \frac{\sin^2 F}{r^2} \left(\frac{\sin^2 F}{r^2} + 2F r^2 \right) - 2 \frac{s^4}{r^2} \left(F r^2 + \frac{\sin^2 F}{r^2} \right) \right] + \frac{1}{(eF\pi)^2} \frac{6}{r^2} \left[\frac{s^4 \sin^2 F}{r^2} + \frac{d}{dr} \left\{s^2 \sin FF' \right\} \right] + \frac{2E}{\Lambda} s^2 \left[1 + \frac{1}{(eF\pi)^2} \left(F r^2 + \frac{5}{r^2} \sin^2 F \right) \right] + \frac{8E}{3\Lambda} \sin^2 I_{KN} + \frac{1}{(eF\pi)^2} \frac{8Es^2}{3\Lambda} \left[F r^2 + \frac{4}{r^2} \sin^2 F \right] I_{KN} + \frac{1}{r^2} \frac{d}{dr} \left[r^2 \left(\frac{4}{(eF\pi)^2} \frac{EF' \sin F}{\Lambda} I_{KN} + \frac{3}{(eF\pi)^2} \frac{EF' \sin F}{\Lambda} \right) \right] + \left[1 + \frac{1}{(eF\pi)^2} \left(\frac{\sin^2 F}{r^2} + F r^2 \right) \right] \frac{l(l+1)}{r^2} - \left[1 + \frac{1}{(eF\pi)^2} \left(4 \frac{\sin^2 F}{r^2} + F r^2 \right) \right] \frac{16s^2}{3r^2} J_{KN} I_{KN} + \frac{1}{(eF\pi)^2} \frac{2E \sin^2 F}{\Lambda r^2} J_{KN} - \frac{1}{(eF\pi)^2} \frac{8}{r^2} \frac{d}{dr} \left(\sin FF' \right) J_{KN} I_{KN} \]

\(V_{WZ}(r) = \frac{3E}{(\pi F\pi)^2} \frac{\sin^2 F}{r^2} F' - \frac{3}{(\pi F\pi)^2} \frac{\sin^2 Fs^2}{\Lambda r^2} F' + \frac{3}{(\pi F\pi)^2} \frac{\sin^2 F}{\Lambda r^2} F' J_{KN} \)

\(s = \sin(F/2) \quad I_{KN} = I^K \cdot I^N, \quad J_{KN} = L^K \cdot J^N \)

**Red:** New contributions for \(I \neq 0 \)
Results 2 potential for $l \neq 0$ ($r \sim 0$)

$$V_{nor}(r) = -\frac{1}{4} \left( 2 \frac{\sin^2 F}{r^2} + F'^2 \right) + \frac{2s^4}{r^2} - \frac{1}{(eF_\pi)^2} \left[ 2 \frac{\sin^2 F}{r^2} \left( \frac{\sin^2 F}{r^2} + 2F'^2 \right) - \frac{2s^4}{r^2} \left( F'^2 + \frac{\sin^2 F}{r^2} \right) \right]$$

$$+ \frac{1}{(eF_\pi)^2} \frac{6}{r^2} \left[ \frac{s^4 \sin^2 F}{r^2} + \frac{d}{dr} \{s^2 \sin FF'\} \right]$$

$$+ \frac{2E}{\Lambda} s^2 \left[ 1 + \frac{1}{(eF_\pi)^2} \left( F'^2 + \frac{5}{r^2} \sin^2 F \right) \right] + \frac{8E}{3\Lambda} s^2 I_{KN} + \frac{1}{(eF_\pi)^2} \frac{8Es^2}{3\Lambda} \left[ F'^2 + 4 \frac{r^2 \sin^2 F}{r^2} \right] I_{KN}$$

$$+ \frac{1}{r^2} \frac{d}{dr} \left[ \frac{4}{(eF_\pi)^2} \frac{E F' \sin F}{\Lambda} I_{KN} + \frac{3}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} \right]$$

$$+ \left[ 1 + \frac{1}{(eF_\pi)^2} \left( \frac{\sin^2 F}{r^2} + F'^2 \right) \right] \frac{l(l+1)}{r^2} - \left[ 1 + \frac{1}{(eF_\pi)^2} \left( 4 \frac{\sin^2 F}{r^2} + F'^2 \right) \right] \frac{16s^2}{3r^2} J_{KN} I_{KN}$$

$$+ \frac{1}{(eF_\pi)^2} \frac{2E \sin^2 F}{\Lambda r^2} J_{KN} - \frac{1}{(eF_\pi)^2} \frac{8}{r^2} \frac{d}{dr} \left( \sin FF' \right) J_{KN} I_{KN}$$

$$F(r \sim 0) \simeq \pi - ar$$

$$V(r) = \frac{2}{r^2} + \frac{a^2}{(eF_\pi)^2} \frac{4}{r^2} + \left[ 1 + \frac{2a^2}{(eF_\pi)^2} \right] \frac{l(l+1)}{r^2}$$

$$- \left[ 1 + \frac{5a^2}{(eF_\pi)^2} \right] \frac{16}{3r^2} J_{KN} I_{KN} - \frac{a^2}{(eF_\pi)^2} \frac{8}{r^2} J_{KN} I_{KN}$$

(No contribution from WZ term)

Bule: dominant for $r \sim 0$
### Short range behavior of $V_{\text{eff}}$

<table>
<thead>
<tr>
<th>$I_{\text{tot}}$</th>
<th>$J_{\text{tot}} = l - 1/2$</th>
<th>$J_{\text{tot}} = l + 1/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{NK}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>$J_{NK} = -(l+1)/2$</td>
<td>$J_{NK} = l/2$</td>
</tr>
<tr>
<td>$-3/4$</td>
<td>$J_{NK} = 3(l+1)/8$</td>
<td>$J_{NK} = -3l/8$</td>
</tr>
<tr>
<td>$1/4$</td>
<td>$J_{NK} = -(l+1)/8$</td>
<td>$J_{NK} = l/8$</td>
</tr>
</tbody>
</table>

- Repulsive or Attractive depending on $l$
- Analytically, attraction ($1 \leq l \leq 3$) repulsion ($l=0$, $4 \leq l$)

Attractive for small $r$
Comparisons with the chiral theory

- **Weinberg-Tomozawa interaction**

\[
L_{WT} = \frac{2}{F^2_\pi} \left\{ \bar{N} I^N \gamma^\mu N \cdot (\partial_\mu K^\dagger K K^K K - K^\dagger I^K K^\dagger K) \right\} \propto \frac{1}{F^2_\pi}
\]

The strength of \(L_{WT} \propto 1/F^2_\pi\)

→For \(F_\pi = 129\) MeV and 186 MeV,

\[1/129^2 : 1/186^2 \sim 15 : 7\]

- **The interaction for \(KN(I = 0)\) bound state**

\[
W \equiv 4\pi \int r^2 dr \tilde{U}(r)
\]

<table>
<thead>
<tr>
<th>(F_\pi) [MeV]</th>
<th>(\epsilon)</th>
<th>(-W \times 10^5 [1/\text{MeV}^2])</th>
</tr>
</thead>
<tbody>
<tr>
<td>129</td>
<td>5.45</td>
<td>(1.2 \times 4\pi)</td>
</tr>
<tr>
<td>186</td>
<td>5.45</td>
<td>(0.48 \times 4\pi)</td>
</tr>
</tbody>
</table>

\[5:2\]

Comparisons with the chiral theory 2

- The interaction for KN($I = 0$) bound state

$$W \equiv 4\pi \int r^2 dr \tilde{U}(r)$$

<table>
<thead>
<tr>
<th>$F_\pi$ [MeV]</th>
<th>$e$</th>
<th>$-W \times 10^5$ [1/MeV$^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>129</td>
<td>5.45</td>
<td>$1.2 \times 4\pi$</td>
</tr>
<tr>
<td>186</td>
<td>5.45</td>
<td>$0.48 \times 4\pi$</td>
</tr>
</tbody>
</table>
Comparisons with the CK approach

- Comparisons between the CK and our approach

<table>
<thead>
<tr>
<th>$l$</th>
<th>$l_{\text{eff}}$</th>
<th>B.E. [MeV]</th>
<th>$\langle r_K^2 \rangle^{1/2}$ [fm]</th>
<th>$l$</th>
<th>B.E. [MeV]</th>
<th>$\langle r_K^2 \rangle^{1/2}$ [fm]</th>
<th>Physical state</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>61.7</td>
<td>0.93</td>
<td>0</td>
<td>32.9</td>
<td>1.18</td>
<td>$\Lambda (1405)$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>326.6</td>
<td>0.54</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>$\Lambda (1116)$</td>
</tr>
</tbody>
</table>

- Anti-kaon hedgehog potential

\[ l_{\text{eff}} (l_{\text{eff}} + 1) = l (l + 1) + 4 \mathbf{I} \cdot \mathbf{L} + 2 \]

Parameters:
\[ F_\pi = 186 \text{ MeV}, \quad e = 4.82 \]
Comparisons with the CK approach

- Comparisons between the CK and our approach

<table>
<thead>
<tr>
<th>Callan-Klebanov approach</th>
<th>Our approach</th>
<th>Physical state</th>
</tr>
</thead>
<tbody>
<tr>
<td>l</td>
<td>l_{eff}</td>
<td>B.E. [MeV]</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>61.7</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>326.6</td>
</tr>
</tbody>
</table>

- The CK approach

- The present approach
Potential

\[ U(r) = U_{\text{nor}}(r) + U_{\text{WZ}}(r) \]

parameter set 1: \((F_\pi, e) = (129\text{MeV}, 5.45)\)