String Theory as an Attempt of PolyMathematics

様々な数学の交差点としての弦理論

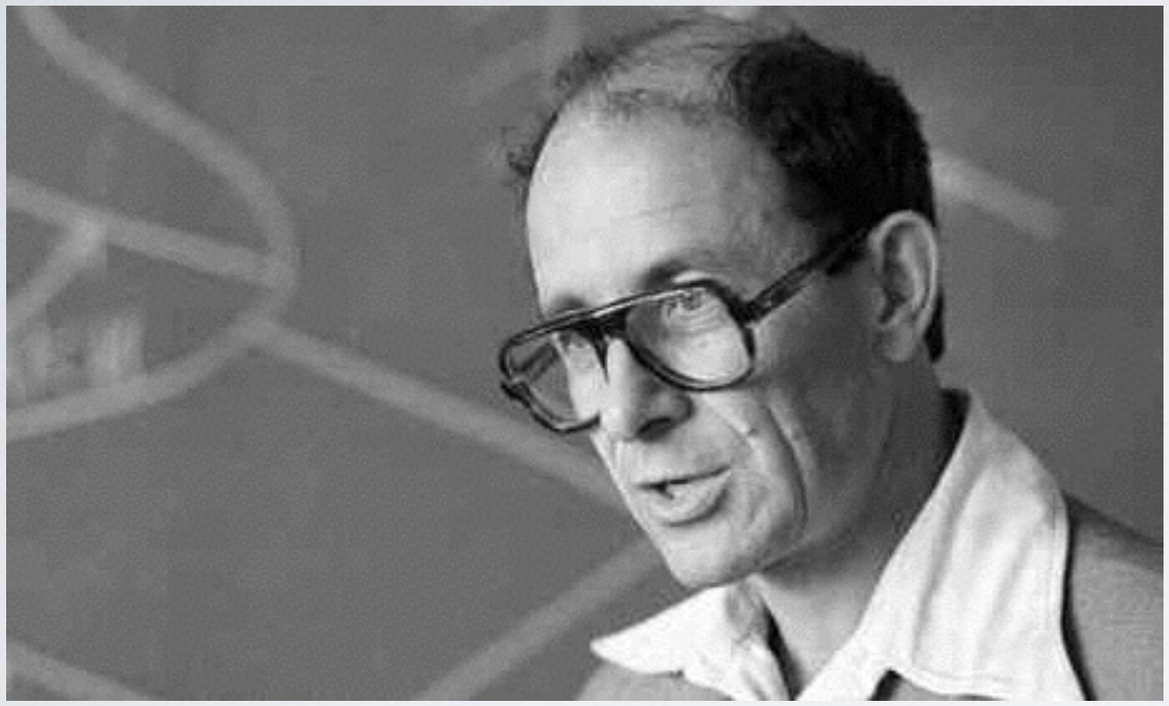
Masato Taki RIKEN, iTHES physics

2016.4/28

iTHES-AIMR-IIS

joint symposium

Polymathematics



Vladimir Igorevich Arnold

(1937 - 2010)

“Polymathematics: is mathematics a single science or a set of arts?”

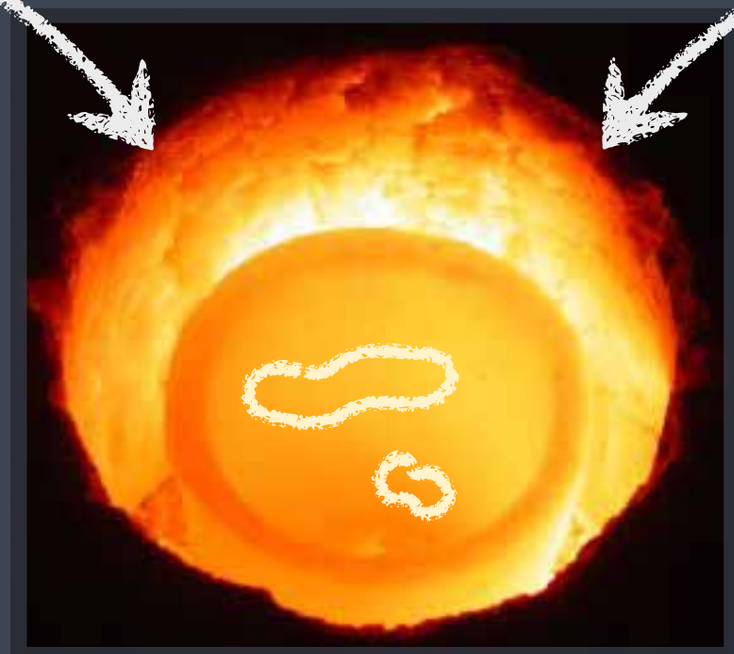
. . . The dream of the polymathematics is to transfer statements from each of these theories to the others, guessing this way new results which might be later checked or modified to become theorems. . . .

string theory:
a melting pot of mathematics !?

.....

Virasoro
symmetry

Calabi-Yau



1980's

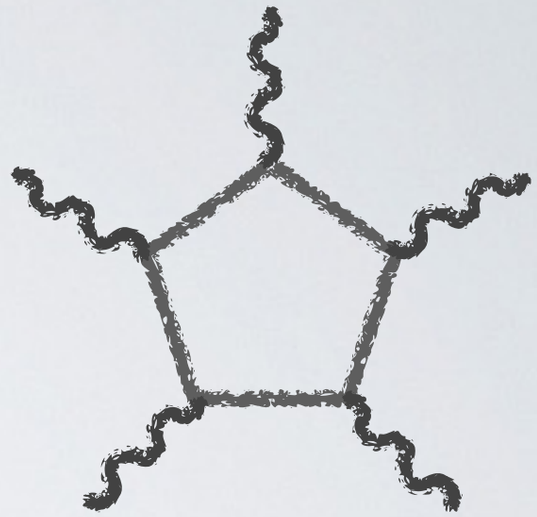
`the good old days'

Math topics in string theory (1980's)

Not so many.

(family's) index theorem

(gravity) anomaly cancelation



Math topics in string theory (1980's)

(family's) index theorem

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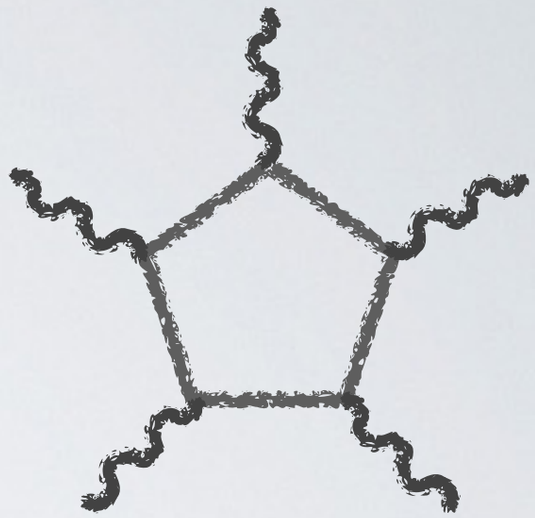


Virasoro, affine, Kac-Mody alg.

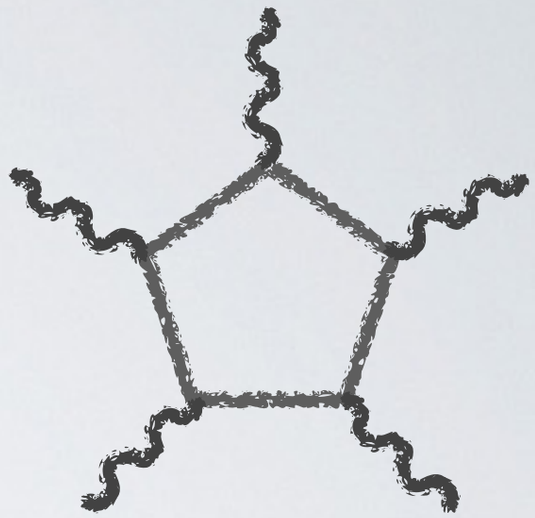
Teichmüller theory

string perturbation

Math topics in string theory (1980's)

(family's) index theorem

(gravity) anomaly cancelation



http://www.redbubble.com/people/mathartspd/

Virasoro, affine, Kac-Mody alg.

Teichmüller theory

string perturbation

moduli theory

orbifold(self-dual lattice), K3, Calabi-Yau, ...

string compactification

1990's

`a whirlwind of change'

Math topics in string theory (1990's Part 1)

knot invariants

Chern-Simons thy, branes

Gromov-Witten inv.

mirror symmetry,

topological strings

Donaldson-Witten inv./Seiberg-Witten inv.

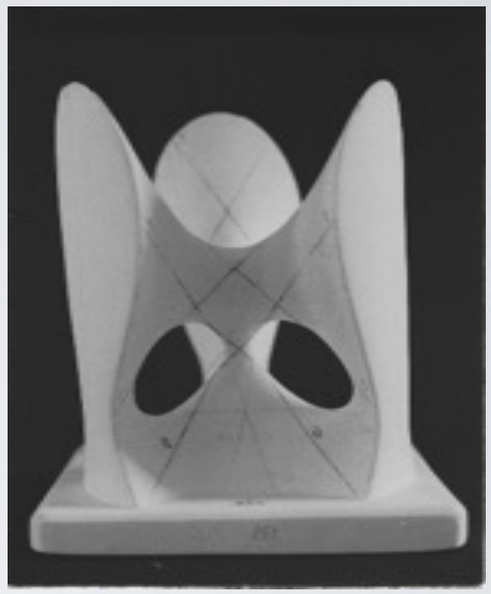
susy theory on 4-mfd

moduli space of stable curves/Gopakumar-Vafa inv.

Hilbert space of D-branes

Math topics in string theory (1990's Part 2)

ADE singularity, toric manifold, ...
string compactification



ADHM, quiver variety,
moduli space of stable sheaves,...
instanton

hyper-Kahler, Atiyah-Hitchin, ...
branes, monopoles

integrable system
susy gauge theory

elliptic fibration & singularity
F-theory

2000-

Today

Math topics in string theory (2000-)

G2 holonomy
string phenomenology

non-commutative geometry
`Magnetic` B-field

K-theory
brane/anti-brane annihilation

categolification of knot theory
topological string

twister space
amplitudes of gauge theory

symmetric polynomial and integrable system
susy QFTs, topological string

Math topics in string theory (2000-)

Langlands program
Hitchin system
electro-magnetic S-duality

Mock theta functions
vertex operator algebra
Monster groups
B.H., string compactification

(motivic) Donaldson-Thomas inv.
Kontsevich-Soibelman theory
wall crossing and B.H.

. . . . . .

Just miscellaneous collection?

Just miscellaneous collection?

No

Just miscellaneous collection?

No

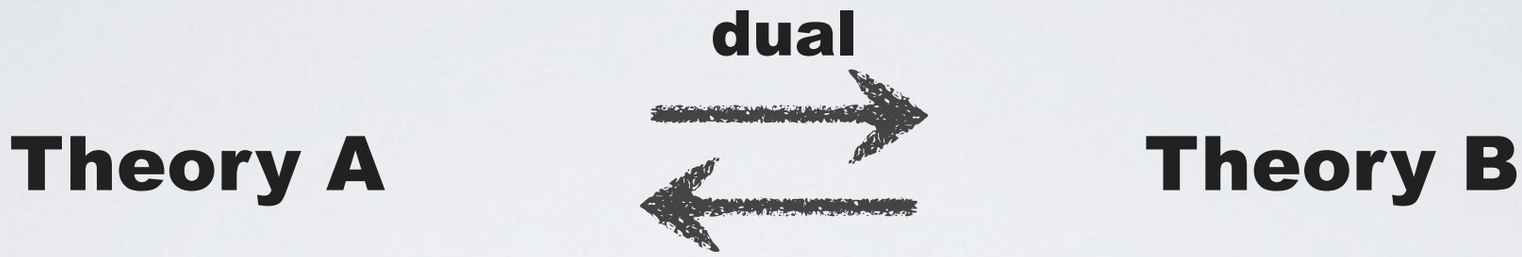
string theory connects various mathematical objects

Duality

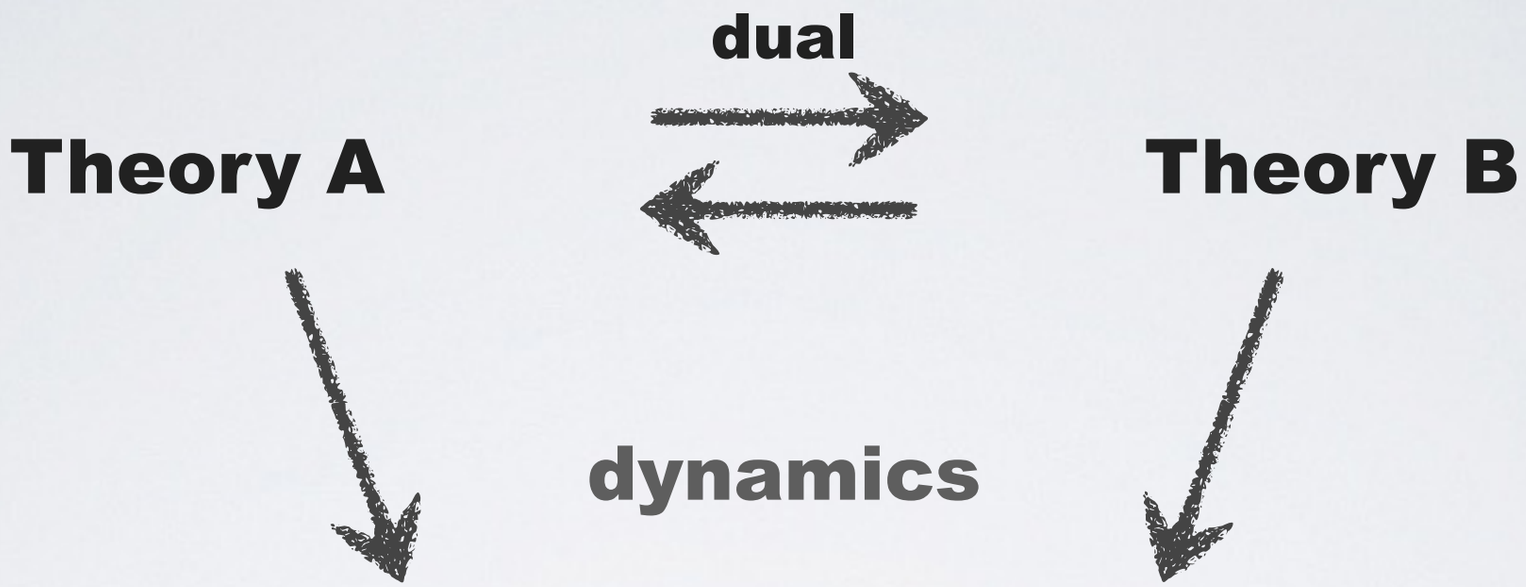
Theory A

Theory B

Duality

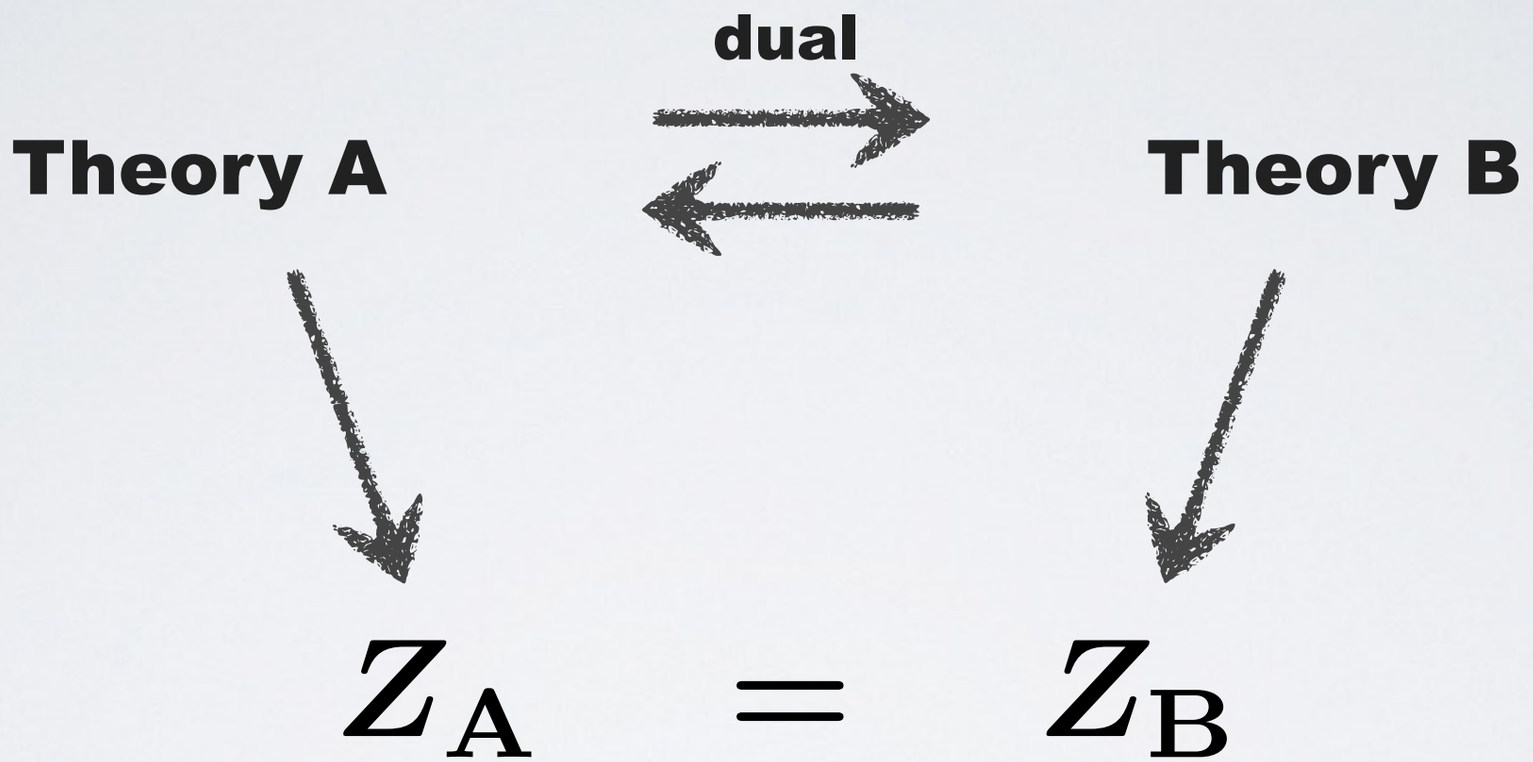


Duality



same physics

Duality



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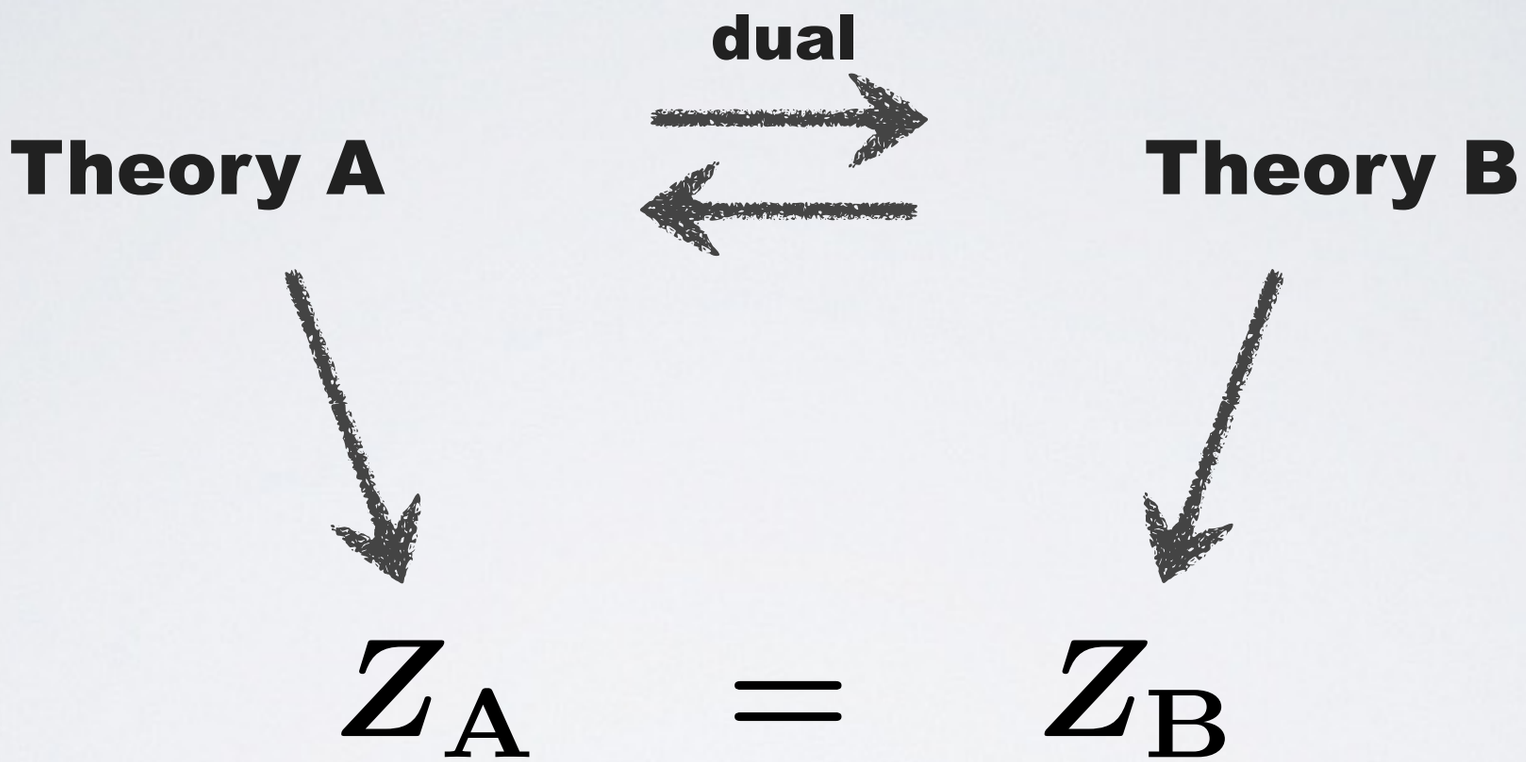
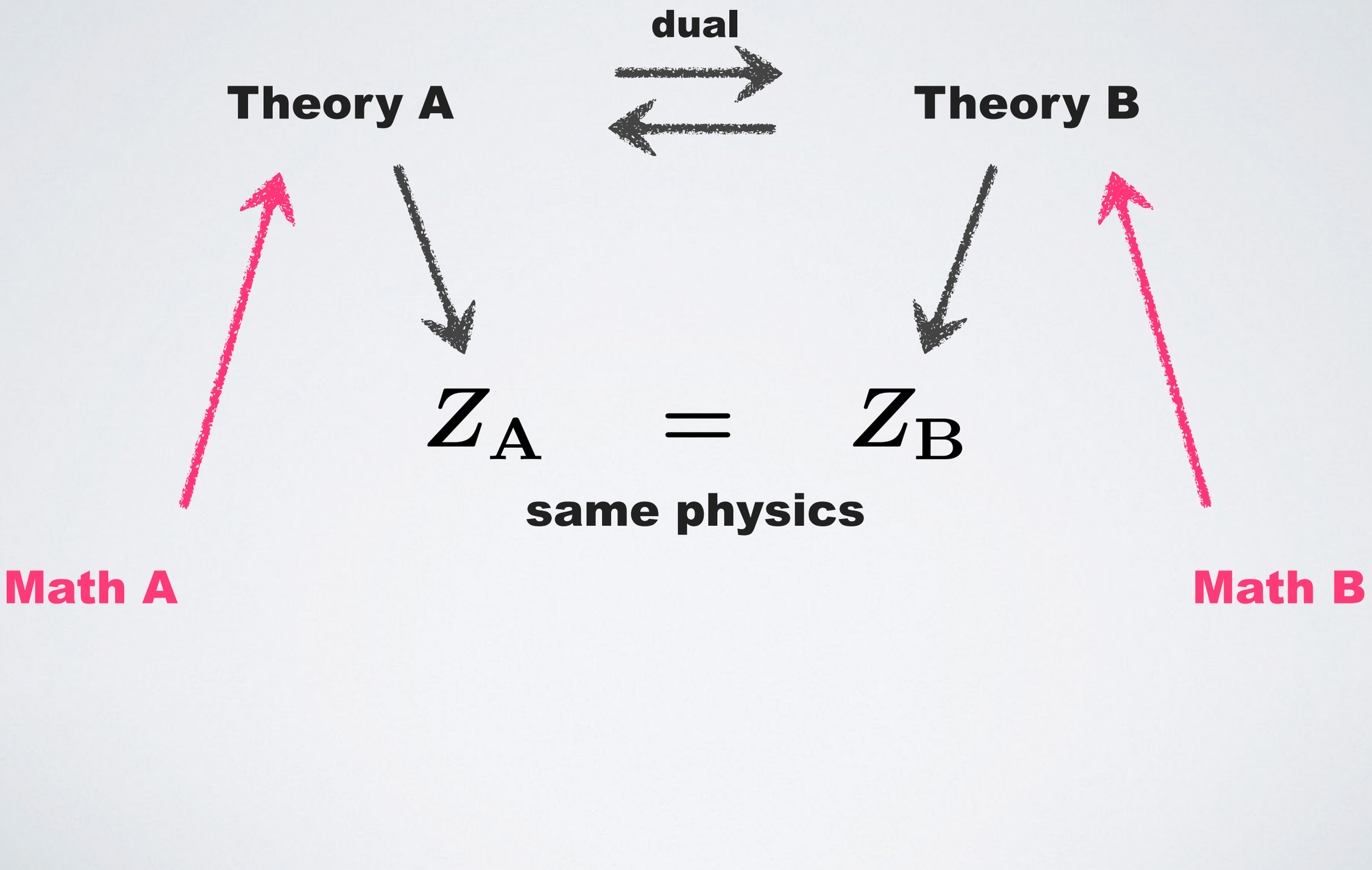
Duality



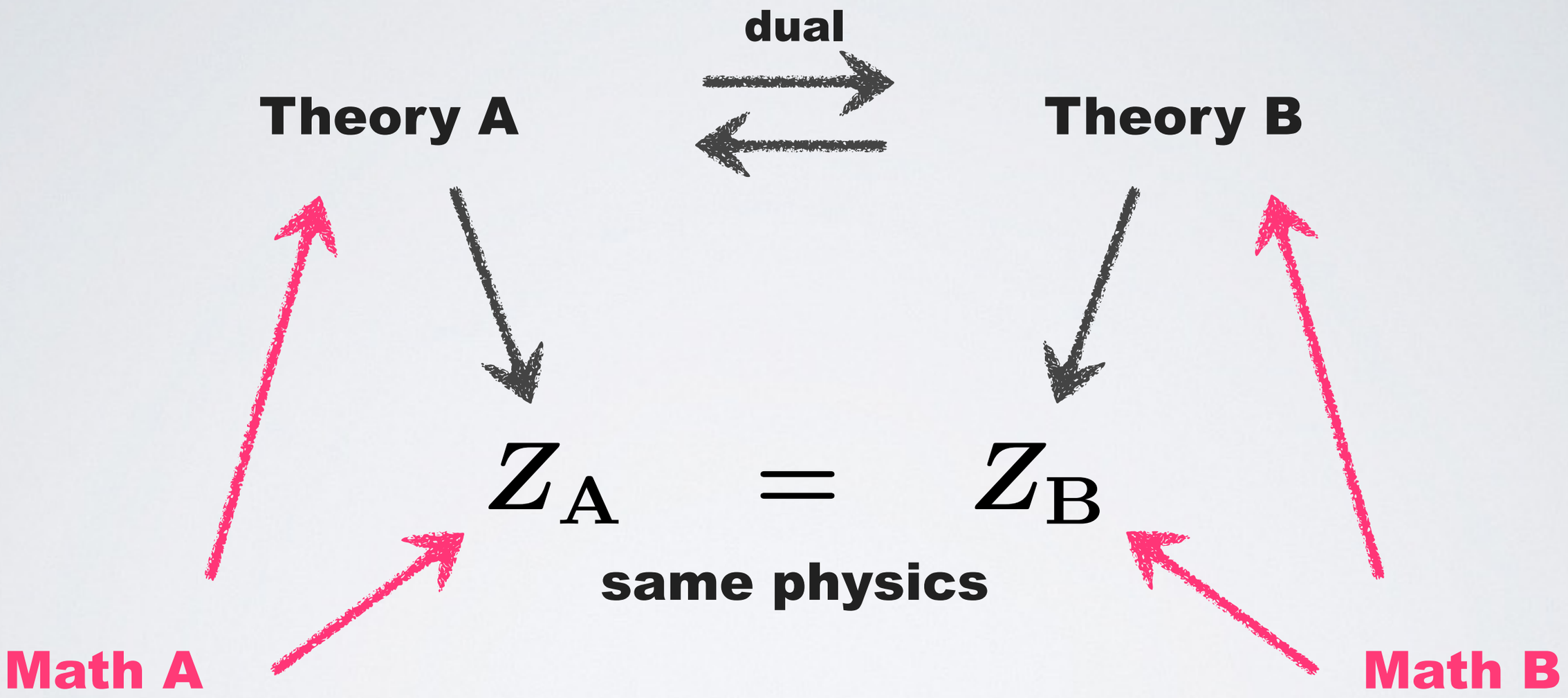
same physics

So what?

Duality



Duality



Duality

handy example :

AGT correspondence (2009)

[Alday-Gaiotto-Tachikawa] [Wyllard]

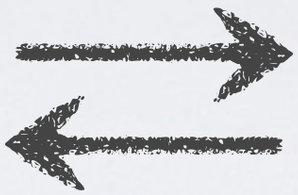
AGT correspondence

2-dim
representation of 2d
Virasoro/W algebra

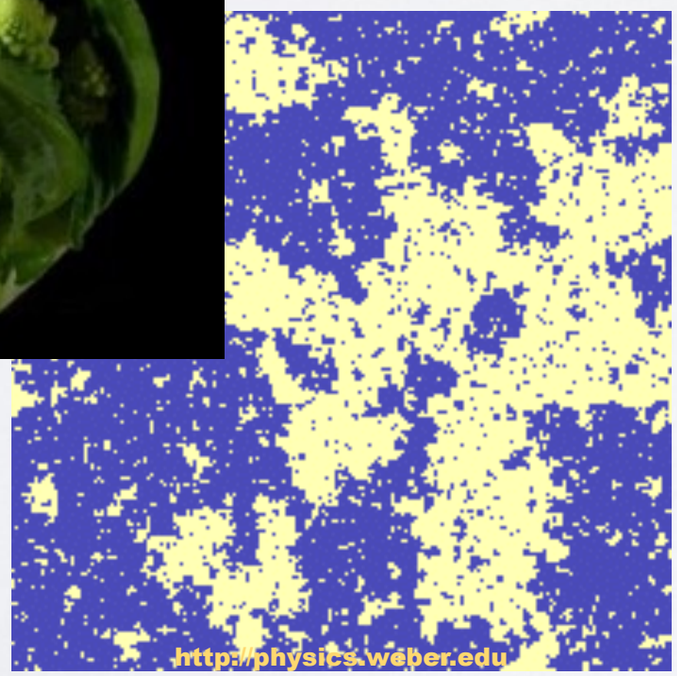
4-dim
instanton moduli space
(torsion free sheaves)

AGT correspondence

2-dim
representation of 2d
Virasoro/W algebra



4-dim
instanton moduli space
(torsion free sheaves)

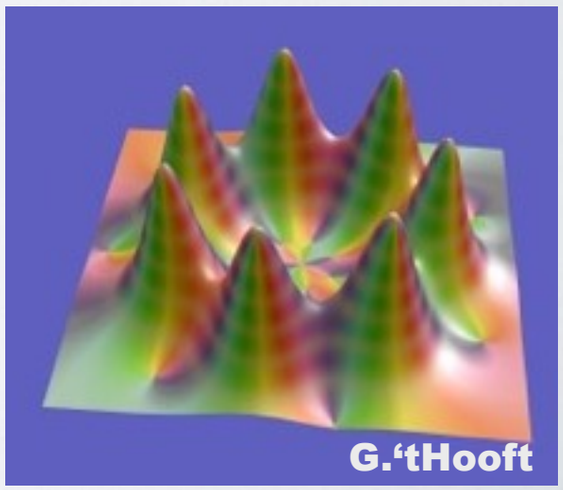


2-dim critical phenomena

curvature of principal
U(N) bundle

$$F = dA + A \wedge A$$

$$F = *F$$



AGT correspondence

2-dim representation of 2d Virasoro/W algebra ⟷ **4-dim instanton moduli space (torsion free sheaves)**

$$B(C_{g,n};\mathcal{W}_N) = \sum_{k=0}^{\infty} \Lambda^{2Nk} \int_{\mathcal{M}(N,k)} 1$$

conformal block **instanton partition function**

AGT correspondence

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conformal block — **instanton partition function**

CFT correlation function

partition function including contribution from k-instantons (evaluating path integral)

AGT correspondence

2-dim representation of 2d Virasoro/W algebra ⟷ **4-dim instanton moduli space (torsion free sheaves)**

$$B(C_{g,n};\mathcal{W}_N) = \sum_{k=0}^{\infty} \Lambda^{2Nk} \int_{\mathcal{M}(N,k)} 1$$

fixed point th'm of equivariant cohomology

$$= \sum_{p \in \mathcal{M}^T(N,k)} \frac{1}{e(T_p\mathcal{M}(N,k))}$$

AGT correspondence: simplified version

[Gaiotto, 09] [MT, 09] ... [Maulik-Okounkov, 2013]

2d side : Virasoro algebra

$$[L_m, L_n] = L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

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Highest weight rep. (Verma module)

$$L_{-1,-2,\cdots} \quad \longleftrightarrow \quad J_+ \qquad a^\dagger$$

$$L_0 \quad \longleftrightarrow \quad J_3$$

$$L_{1,2,\cdots} \quad \longleftrightarrow \quad J_- \qquad a$$

‘Chevalley basis’

AGT correspondence: simplified version

[Gaiotto, 09] [MT, 09] ... [Maulik-Okounkov, 2013]

highest weight vector (ground state)

$$L_0 \vec{v}_0 = \lambda \vec{v}_0$$

$$L_{n \geq 1} \vec{v}_0 = 0$$

$L_{-1,-2,\cdots}$	$\longleftrightarrow$	J_+	$a^\dagger$
L_0	$\longleftrightarrow$	J_3	
$L_{1,2,\cdots}$	$\longleftrightarrow$	J_-	a

'Chevalley basis'

AGT correspondence: simplified version

[Gaiotto, 09] [MT, 09] ... [Maulik-Okounkov, 2013]

Whittaker vector (coherent state)

$$\vec{v} = \vec{v}_0 + C_{[1]} L_{-1}\vec{v}_0 + C_{[11]} (L_{-1})^2 \vec{v}_0 + C_{[2]} L_{-2}\vec{v}_0 + \cdots$$

$L_{-1,-2,\cdots}$	$\longleftrightarrow$	J_+	$a^\dagger$
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$$L_1 \vec{v} = \Lambda^2 \vec{v}, \quad L_{n\geq 2}\vec{v} = 0$$

$L_{-1,-2,\cdots}$	$\longleftrightarrow$	J_+	$a^\dagger$
L_0	$\longleftrightarrow$	J_3	
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$$L_1 \vec{v} = \Lambda^2 \vec{v}, \quad L_{n\geq 2}\vec{v} = 0$$

$$\langle \vec{v}, \vec{v} \rangle = Z^{inst.}_{SU(2)}$$

Why does AGT phenomenon happen?

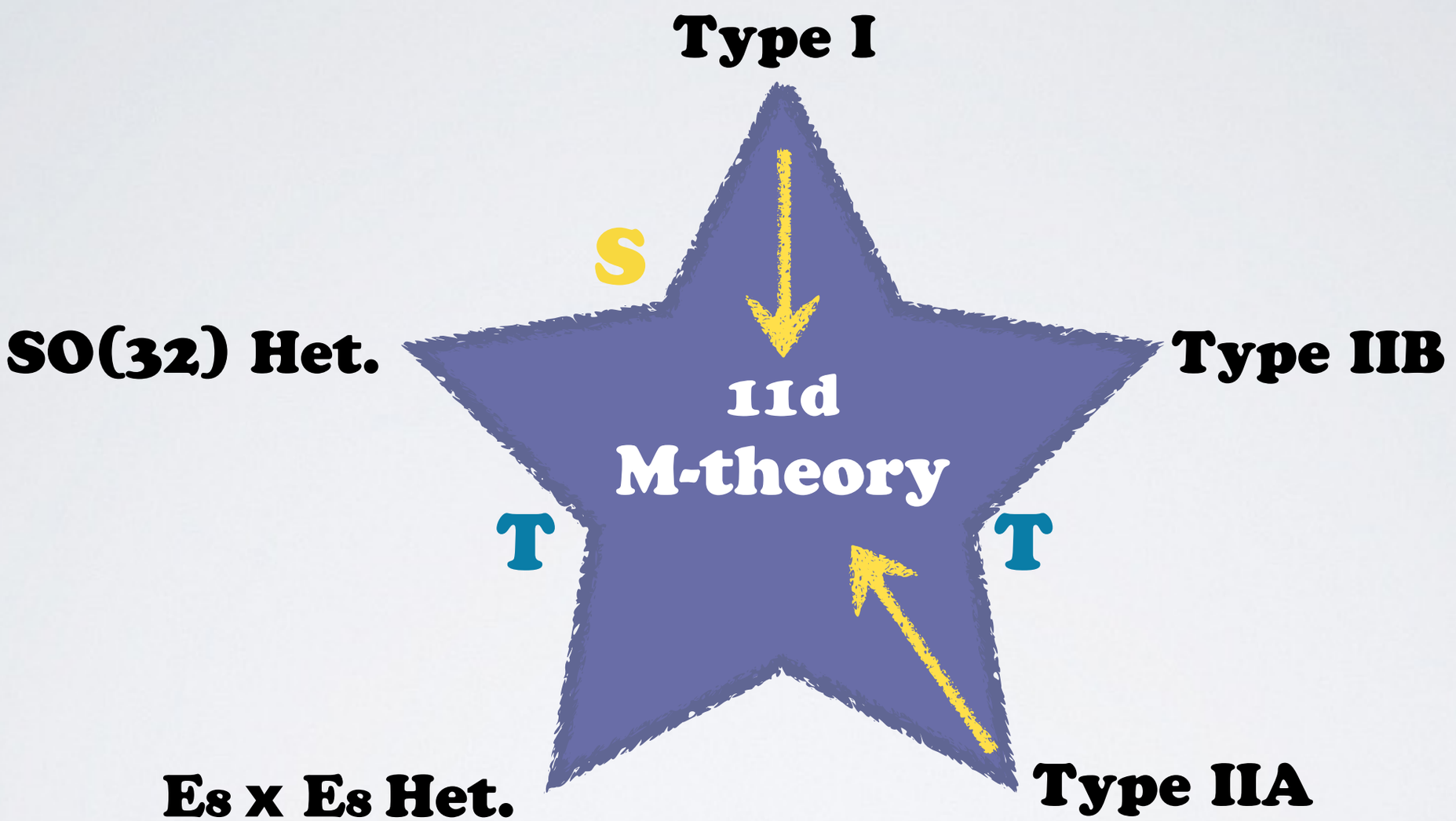
$$\langle \vec{v}, \vec{v} \rangle = Z^{inst.}_{SU(2)}$$

- mathematical proof (geometric representation theory)

- ‘clear’ physical origin

The key is M-theory

Unification of string theories: M-theory

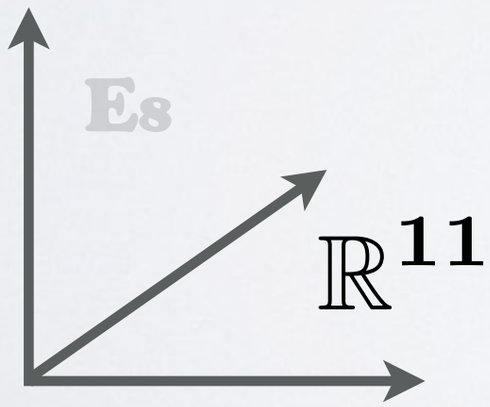


Unification of string theories: M-theory

Fundamental object: M-branes

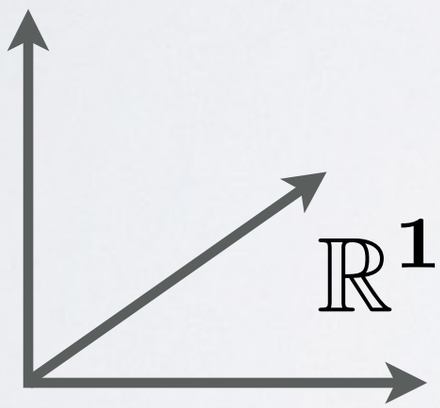
Unification of string theories: M-theory

Fundamental object: M-branes



6=2+4

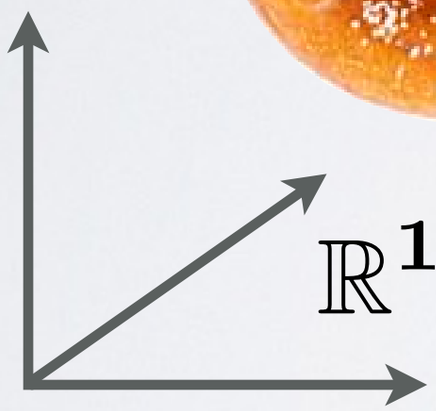
$$C_{g,n} \quad \times \quad S^4$$



$$\mathbb{R}^{11} \to \mathbb{R}^3 \times T^* C_{g,n} \times S^4$$

6=2+4

$$C_{g,n} \times S^4$$



$$\mathbb{R}^{11} \to \mathbb{R}^3 \times T^*C_{g,n} \times S^4$$

6=2+4

$$C_{g,n} \times S^4$$

Assumption: M5 physics does not depend their relative size

6=2+4

S^4

4-dim gauge theory (instanton)

6=2+4

$$C_{g,n} \times S^4$$

×

6=2+4

$$C_{g,n}$$

×

2-dim CFT
(Virasoro algebra)

6=2+4

$C_{g,n}$

=

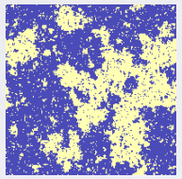
S^4

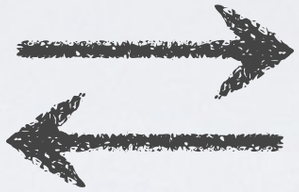
2-dim CFT
(Virasoro algebra)

4-dim gauge theory
(instanton)

Landscape of dualities

2-dim
representation of
Virasoro/W algebra





4-dim

instanton moduli space

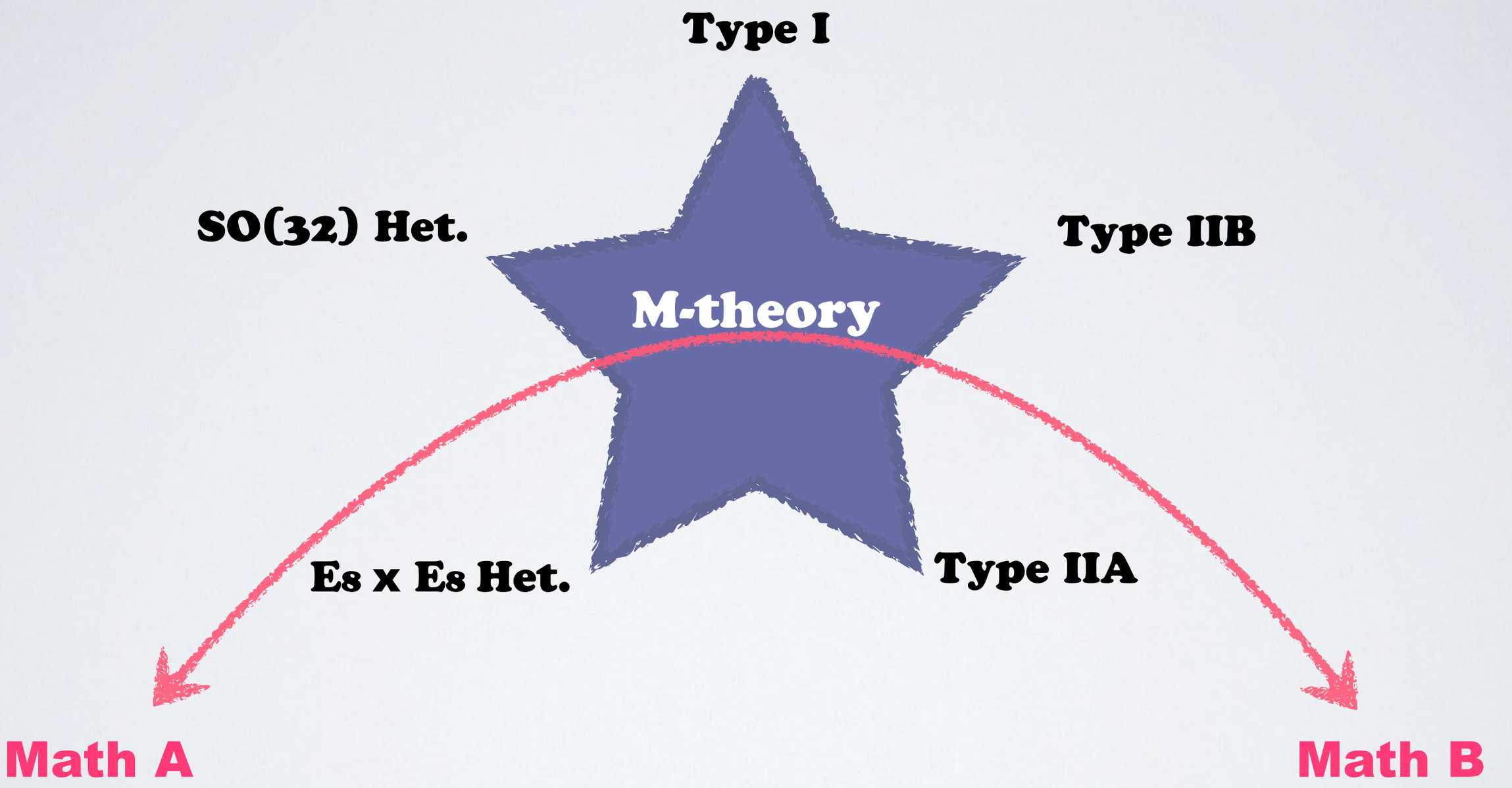
mirror symmetry

random matrix model **Gromov-Witten inv of CY**

. . . . . . **knot (CS) theory**

Summary

string: toy model of polymathemastics



FIN

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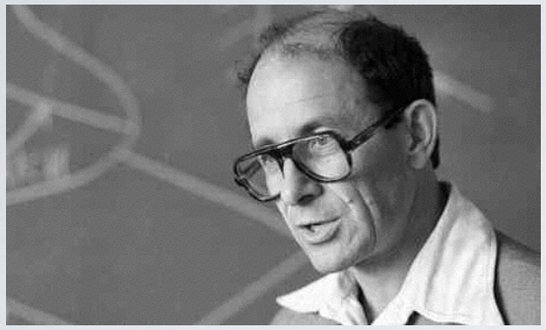

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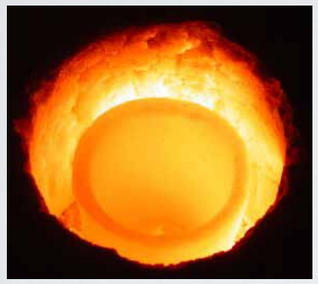

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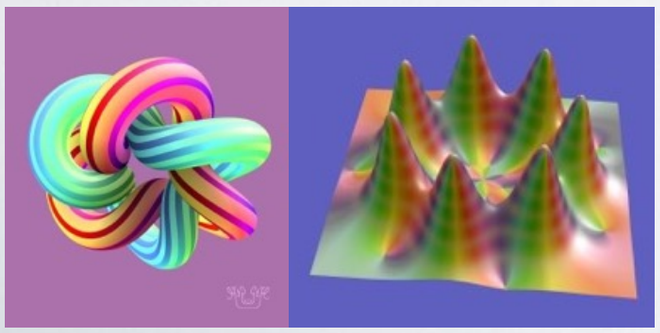

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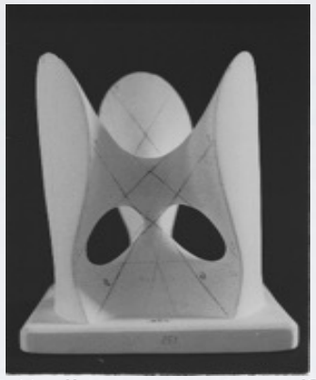

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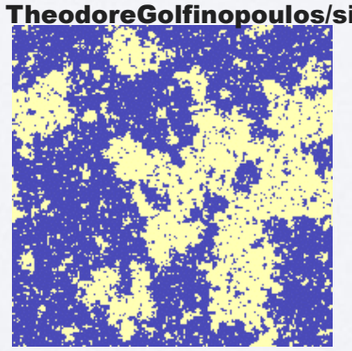

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