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AUGMENTED LAGRANGIAN METHODS FOR CONVEX OPTIMIZATION

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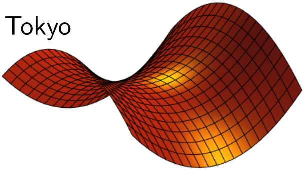


Optimization Problems: Types and Algorithms[1]

- Types of Optimization
  - Linear Programming
  - Quadratic Programming
  - Semidefinite Programming
  - MPEC
  - Nonlinear Least Squares
  - Nonlinear Equations
  - Nondifferentiable Optimization
  - Global Optimization
  - Derivative-Free Optimization
  - Network Optimization

- Algorithms
  - Line Search
  - Trust-Region
  - (Nonlinear) Conjugate Gradient
  - (Quasi-)Newton
  - Truncated Newton
  - Gauss Newton
  - Levenberg-Marquardt
  - Homotopy
  - Gradient Projection
  - Simplex Method
  - Interior Point Methods
  - Active Set Methods
  - Primal-Dual Active Set Methods
  - Augmented Lagrangian Methods
  - Reduced Gradient
  - Sequential Quadratic Programming

[1] neos Optimization Guide: *http://www.neos-guide.org*

Problem and Objective

- Problem: A class of nonsmooth convex optimization

$$\min_x f(x) + \phi(Ex) \tag{1}$$

  - X, H: real Hilbert spaces
  - $f\colon X \to \mathbb{R}$: differentiable, $\nabla^2 f(x) \succeq mI$
  - $\phi\colon H \to \mathbb{R}$: convex, nonsmooth(nondifferentiable)
  - $E\colon X \to H$: linear

⋆ Optimality system

$$0 \in \nabla_x f(x) + E^\top \partial\phi(Ex) \tag{2}$$

- Objective: To develop Newton (-like) methods
  - Applicable to a wide range of applications
  - Fast and accurate
  - Easy to implement
- Tool: **Augmented Lagrangian in the sense of Fortin.**

Motivating Examples (1/4)

- Optimal control problem

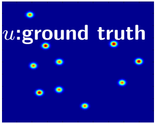
$$\begin{aligned} \min \quad & \frac{1}{2}\int_0^T x(t)^\top Q x(t) + u(t)^\top R u(t)\, dt \\ \text{s.t.} \quad & \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ & \|u(t)\| \le 1. \end{aligned}$$

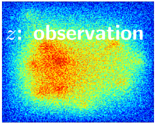
- Inverse source problem

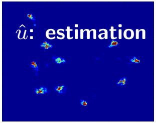
$$(\hat{y}, \hat{u}) = \arg \min_{(y,u)\in H_0^1(\Omega)\times L^2(\Omega)} \quad (1/2)\|y - z\|_{L^2}^2 + (\alpha/2)\|u\|_{L^2}^2 + \beta\|u\|_{L^1}$$

$$\text{s.t.} \quad -\Delta y = u \quad \text{on} \quad \Omega,$$

$$y = 0 \quad \text{on} \quad \partial\Omega$$







Motivating Examples (2)

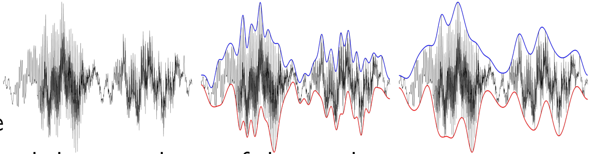
- lasso (least absolute shrinkage and selection operator) [R.Tibshinari, 96]

$$\min_{\beta \in \mathbb{R}^p} \|X\beta - y\|_2^2 \text{ subject to } \|\beta\|_1 \leq t. \quad \left(\min_{\beta \in \mathbb{R}^p} \|X\beta - y\|_2^2 + \lambda \|\beta\|_1 \right)$$

  - $X = [\boldsymbol{x}^1, \ldots, \boldsymbol{x}^N]^\top$
  - $\boldsymbol{x}^i = (x_{i1}, \ldots, x_{ip})^\top$: the predictor variables, $i = 1, 2, \ldots, N$
  - y_i: the responses, $i = 1, 2, \ldots, N$

- envelope construction

$$\color{blue}{\hat{x}} \color{black} = \arg\min_{x \in \mathbb{R}^n} \frac{1}{2}\|x - s\|_2^2 + \alpha \|(D^\top D)^2 x\|_1 \text{ subject to } s \leq x.$$



  - s: noisy signal
  - D: finite difference
  - $\|(D^\top D)^2 x\|_1$: control the smoothness of the envelope x.

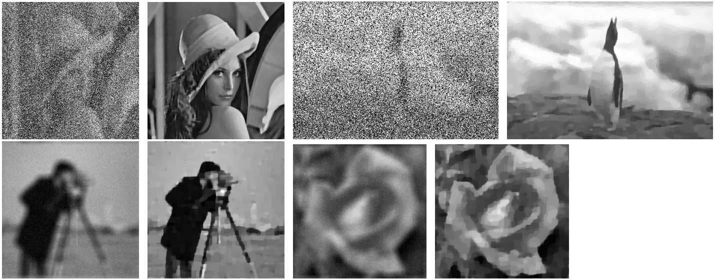
Motivating Examples (3/4)

- total variation image reconstruction

$$\min \frac{1}{p}\|Kx-y\|_{L^p} + \alpha|x|_{TV} + \beta/2\|x\|_{H^1}^2$$

for $p = 1$ or $p = 2$. K: blurring kernel.

$$|x|_{TV} = \sum_{i,j}\sqrt{([D_1 x]_{i,j})^2 + ([D_2 x]_{i,j})^2}$$



Motivating Examples (4/4)

- Quadratic programming

$$\min_x \frac{1}{2}(x, Ax) - (a, x) \quad \text{s.t.} \quad Ex = b, \quad Gx \geq g.$$

- Group LASSO

$$\min_\beta \frac{1}{2}\|X\beta - y\|^2 + \alpha\|\beta\|_G,$$

e.g. $G = \{\{1,2\},\{3,4,5\}\}$, $\|\beta\|_G = \sqrt{\beta_1^2 + \beta_2^2} + \sqrt{\beta_3^2 + \beta_4^2 + \beta_5^2}$.

- SVM
- $\cdots$

outline

1. Augmented Lagrangian

2. Newton Methods

3. Numerical Test

Augmented Lagrangian

Method of multipliers for equality constraint

- equality constrained optimization problem

$$(P) \qquad \min_x \; f(x) \text{ subject to } g(x) = 0$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}^m$ are differentiable.

- augmented Lagrangian (Hestenes, Powell 69), $c > 0$

$$\begin{aligned} L_c(x,\lambda) &= f(x) + \lambda^\top g(x) + (c/2)\|g(x)\|_{\mathbb{R}^m}^2 \\ &= f(x) + (c/2)\|g(x) + \lambda/c\|_{\mathbb{R}^m}^2 - 1/(2c)\|\lambda\|_{\mathbb{R}^m}^2 . \end{aligned}$$

- method of multiplier[1]

$$\lambda_{k+1} = \lambda_k + c[\nabla_\lambda d_c](\lambda_k).$$

⋆ method of multiplier is composed of two steps:

$$\begin{aligned} x^k &= \arg\min_x L_c(x, \lambda^k) \\ \lambda_{k+1} &= \lambda_k + c\nabla_\lambda L_c(x^k, \lambda^k) = \lambda_k + cg(x^k) \end{aligned}$$

[1]**def**.: $d_c(\lambda) := \min_x L_c(x,\lambda)$. **fact**: $\nabla d_c(\lambda) = [\nabla_\lambda L_c](x(\lambda),\lambda) = g(x(\lambda))$

A brief history of augmented Lagrangian methods (1/2)

■ equality constrained optimization

$$\min_{x\in\mathbb{R}^n} \quad f(x)$$
$$\text{s.t.} \quad h(x) = 0$$

- 1969 Hestenes[7], Powell[14]

■ inequality constrained optimization

$$\min_{x\in\mathbb{R}^n} \quad f(x)$$
$$\text{s.t.} \quad g(x) \leq 0$$

- 1973 Rockafellar[15, 16]

■ equality and inequality constrained optimization

$$\min_{x\in\mathbb{R}^n} \quad f(x)$$
$$\text{s.t.} \quad h(x) = 0$$
$$\quad g(x) \leq 0$$

- 1982 Berstekas[2]

■ nonsmooth convex optimization

$$\min_{x\in\mathbb{R}^n} \quad f(x) + \phi(Ex)$$

- 1975 Glowinski-Marroco[6]
- 1975 Fortin[4]
- 1976 Gabay-Mercier[5]
- 2000 Ito-Kunisch[9]
- 2009 Tomioka-Sugiyama[18]

A brief history of augmented Lagrangian methods (2/2)

- Augmented Lagrangian Methods (Method of Multipliers)

year	authors	problem	algorithm
1969	Hestenes[7], Powell[14]	equality constraint	
1973	Rockafellar[15, 16]	inequality constraint	
1975	Glowinski-Marroco[6]	nonsmooth convex (FEM)	ALG1
1975	Fortin[4]	nonsmooth convex (FEM)	
1976	Gabay-Mercier[5]	nonsmooth convex (FEM)	ADMM[1]
1982	Berstekas[2]	equality and inequality constraints	
2000	Ito-Kunisch[9]	nonsmooth convex (PDE Opt.)	
2009	Tomioka-Sugiyama[18]	nonsmooth convex (Machine Learning)	DAL[2]
2013	Patrinos-Bemporad [12]	convex composite	Proximal Newton
2014	Patrinos-Stella -Bemporad[13]	(Optimal Control)	FBN[3]

[1] Alternating Direction Method of Multipliers
[2] Dual-augmented Lagrangian method
[3] Forward-Backward-Newton

Augmented Lagrangian Methods [Glowinski+,75]

- nonsmooth convex optimization problem

$$\min_{x,v} \; f(x) + \phi(v), \quad \text{subject to} \quad v = Ex.$$

- augmented Lagrangian

$$L_c(x, v, \lambda) = f(x) + \phi(v) + (\lambda, Ex - v) + (c/2)\|Ex - v\|^2_{\mathbb{R}^m}$$

- dual function

$$d_c(\lambda) := \min_{x,v} L_c(x, v, \lambda)$$

- method of multipliers (ALG1)

$$\lambda_{k+1} = \lambda_k + c\nabla_\lambda d_c(\lambda_k).$$

⋆ method of multiplier is composed of two steps:

step 1	(x_{k+1}, v_{k+1})	$=$	$\arg\min_{x,v} L_c(x, v, \lambda_k)$
step 2	λ_{k+1}	$=$	$\lambda_k + c(Ex_{k+1} - v_{k+1})$

Fact (gradient): $\nabla_\lambda d_c(\lambda) = [\nabla_\lambda L_c](x(\lambda), v(\lambda), \lambda)$

Augmented Lagrangian Methods [Gabay+,76]

- nonsmooth convex optimization problem

$$\min_{x,v} \; f(x) + \phi(v), \quad \text{subject to} \quad v = Ex.$$

- augmented Lagrangian

$$L_c(x, v, \lambda) = f(x) + \phi(v) + (\lambda, Ex - v) + (c/2)\|Ex - v\|^2_{\mathbb{R}^m}$$

- ADMM, alternating direction method of multiplier

$$\begin{array}{lrcl} \text{step 1} & x_{k+1} & = & \arg\min_x L_c(x, v_k, \lambda_k) \\ \text{step 2} & v_{k+1} & = & \arg\min_v L_c(x_{k+1}, v, \lambda_k) = \text{prox}_{\phi/c}(Ex_{k+1} + \lambda_k/c) \\ \text{step 3} & \lambda_{k+1} & = & \lambda_k + c(Ex_{k+1} - v_{k+1}) \end{array}$$

Assumption: $\text{prox}_{\phi/c}(z) := \arg\min_v(\phi(v) + c/2\|v - z\|^2)$ has a closed-form representation (explicitly computable).

Augmented Lagrangian Methods [Fortin,75]

- nonsmooth convex optimization problem

$$\min_x \; f(x) + \phi(Ex)$$

- (Fortin's) augmented Lagrangian

$$L_c(x,\lambda) = \min_v L_c(x,v,\lambda) = f(x) + \phi_c(Ex + \lambda/c) - 1/(2c)\|\lambda\|^2_{\mathbb{R}^m}$$

where $\phi_c(z)$ is Moreau's envelope

- method of multiplier

$$\lambda_{k+1} = \lambda_k + c\nabla_\lambda d_c(\lambda_k)$$

def.(dual function): $d_c(\lambda) := \min_x L_c(x,\lambda)$.

⋆ method of multiplier is composed of two steps:

$$x_k = \arg\min_x L_c(x,\lambda_k)$$

$$\lambda_{k+1} = \lambda_k + c[Ex^k - \text{prox}_{\phi/c}(Ex^k + \lambda^k/c)]$$

fact (gradient): $\nabla_\lambda d_c(\lambda) = [\nabla_\lambda L_c](x(\lambda),\lambda)$

Assumption: $\text{prox}_{\phi/c}(z) := \arg\min_v(\phi(v) + c/2\|v-z\|^2)$ has a closed-form representation (explicitly computable).

Comparison

- augmented Lagrangian for ALG1[Glowinski+,75], ADMM[Gabay+,76]

$$L_c(x, v, \lambda) = f(x) + \phi(v) + (\lambda, Ex - v) + (c/2)\|Ex - v\|^2_{\mathbb{R}^m}$$

- (Fortin's) augmented Lagrangian for ALM[Fortin,75]

$$\begin{aligned} L_c(x, \lambda) &= \min_v L_c(x, v, \lambda) \\ &= f(x) + \phi_c(Ex + \lambda/c) - 1/(2c)\|\lambda\|^2_{\mathbb{R}^m} \end{aligned}$$

- Fortin's Augmented Lagrangian method has been ignored in the subsequent literature.
- The method was re-discovered by [Ito-Kunisch,2000] and by [Tomioka-Sugiyama,2009].

Moreau's envelope, proximal operator (1/3)

- Moreau's envelope (Moreau-Yosida regularization) of a closed proper convex function ϕ

$$\phi_c(z) := \min_v(\phi(v) + c/2\|v - z\|^2)$$

- The proximal operator $\mathsf{prox}_\phi(z)$ is defined by

$$\mathsf{prox}_\phi(z) := \arg\min_v(\phi(v) + 1/2\|v - z\|^2)$$

$$\begin{aligned}\mathsf{prox}_{\phi/c}(z) &= \arg\min_v(\phi(v) + c/2\|v - z\|^2)\\ &= \arg\min_v(\phi(v)/c + 1/2\|v - z\|^2)\end{aligned}$$

- Moreau's envelope is expressed in terms of the proximal operator

$$\phi_c(z) = \phi(\mathsf{prox}_{\phi/c}(z)) + c/2\|\mathsf{prox}_{\phi/c}(z) - z\|^2$$

Moreau's envelope, proximal operator (2/3)

•ℓ_1 norm

$$\phi(x) = \|x\|_1,\ x \in \mathbb{R}^n$$

$$\text{prox}_\phi(x)_i = \max(0, |x_i| - 1)\text{sign}(x_i),$$

$$G \in \partial_B(\text{prox}_\phi)(x)$$

$$G_{ii} = \begin{cases} 1 & |x_i| > 1, \\ 0 & |x_i| < 1, \\ \in \{0,1\} & |x_i| = 1 \end{cases}$$

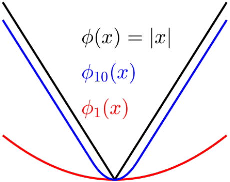
•Euclidean norm

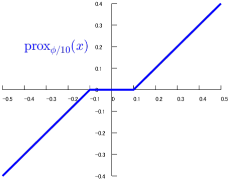
$$\phi(x) = \|x\|,\ x \in \mathbb{R}^n$$

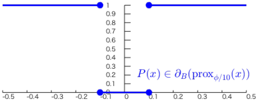
$$\text{prox}_\phi(x) = \max(0, \|x\| - 1)\frac{x}{\|x\|},$$

$$\partial_B(\text{prox}_\phi)(x) =$$

$$\begin{cases} I - \frac{1}{\|x\|}(I - \frac{xx^\top}{\|x\|^2}) & \|x\| > 1, \\ 0 & \|x\| < 1, \\ \{0, I - \frac{1}{\|x\|}(I - \frac{xx^\top}{\|x\|^2})\} & \|x\| = 1 \end{cases}$$







Ref. Fixed-Point Algorithms for Inverse Problems in Science and Engineering Springer Optimization and Its Applications, Chapter 10.

Moreau's envelope, proximal operator (3/3)

- Properties[1]
  - $\phi_c(z) \le \phi(z)$, $\lim_{c\to\infty} \phi_c(z) = \phi(z)$
  - $\phi_c(z) + (\phi^*)_{1/c}(cz) = (c/2)\|z\|^2$

def.(convex conjugate):
$\phi^*(x^*) := \sup_x(\langle x^*, x\rangle - \phi(x))$.

  - $\nabla_z \phi_c(z) = c(z - \mathsf{prox}_{\phi/c}(z))$
  - $\|\mathsf{prox}_{\phi/c}(z) - \mathsf{prox}_{\phi/c}(w)\|^2 \le \langle \mathsf{prox}_{\phi/c}(z) - \mathsf{prox}_{\phi/c}(w), z - w)$
  - $\mathsf{prox}_{\phi/c}(z) + (1/c)\mathsf{prox}_{c\phi^*}(cz) = z$
  - $\mathsf{prox}_{\phi/c}$ is globally Lipschitz, almost everywhere differentiable
  - $P \in \partial_B(\mathsf{prox}_{\phi/c})(x)$ is a symmetric positive semidefinite that satisfies $\|P\| \le 1$.
  - $\partial_B(\mathsf{prox}_{c\phi^*})(x) = \{I - P \colon P \in \partial_B(\mathsf{prox}_{\phi/c})(x/c)\}$

[1] [17, Bauschke+11],[13, Partrions+14]

Differentiability, Optimality condition

- Fortin's augmented Lagrangian

$$L_c(x,\lambda) = f(x) + \phi_c(Ex + \lambda/c) - 1/(2c)\|\lambda\|^2_{\mathbb{R}^m}$$

- Fortin's augmented Lagrangian L_c is convex and continuously differentiable with respect to x, and is concave and continuously differentiable with respect to λ.

$$\nabla_x L_c(x,\lambda) = \nabla_x f(x) + cE^\top(Ex + \lambda/c - \mathrm{prox}_{\phi/c}(Ex + \lambda/c)),$$
$$\nabla_\lambda L_c(x,\lambda) = Ex - \mathrm{prox}_{\phi/c}(Ex + \lambda/c).$$

- The following conditions on a pair (x^*, λ^*) are equivalent.

(a) $\quad \nabla_x f(x^*) + E^\top\lambda^* = 0 \quad$ and $\quad \lambda^* \in \partial\phi(Ex^*)$.

(b) $\quad \nabla_x f(x^*) + E^\top\lambda^* = 0 \quad$ and $\quad Ex^* - \mathrm{prox}_{\phi/c}(Ex^* + \lambda^*/c) = 0$.

(c) $\quad \nabla_x L_c(x^*, \lambda^*) = 0 \quad$ and $\quad \nabla_\lambda L_c(x^*, \lambda^*) = 0$.

(d) $\quad L_c(x^*, \lambda) \le L_c(x^*, \lambda^*) \le L_c(x, \lambda^*), \qquad \forall x \in X,\ \forall\lambda \in H$.

- If there exists a pair (x^*, λ^*) that satisfies (d), then x^* is the minimizer of (1).

Newton Methods

Newton methods

- Problem

$$\min_x f(x) + \phi(Ex)$$

- Approaches

(1-1) Lagrange-Newton: Solve the optimality system by Newton method

$$\nabla_x L_c(x,\lambda) = 0 \quad \text{and} \quad \nabla_\lambda L_c(x,\lambda) = 0$$

(1-2) A special case ($E = I$). Solve the optimality system by Newton method

$$x = \mathrm{prox}_{\phi/c}(x - \nabla_x f(x)/c)$$

(2) Apply Newton method to Fortin's augmented Lagrangian method

step 1 Solve for x_k : $\nabla_x L_c(x_k, \lambda_k) = 0$

step 2 $\lambda_{k+1} = \lambda_k + c(Ex_k - \mathrm{prox}_{\phi/c_k}(Ex_k + \lambda_k/c)).$

⋆ For fixed λ_k

$$x_k = \arg\min_x L_c(x, \lambda_k) \quad \Longleftrightarrow \quad \nabla_x L_c(x_k, \lambda_k) = 0.$$

Lagrange-Newton method

- Lagrange-Newton method: Solve the optimality system by Newton method[1]

$$\nabla_x L_c(x,\lambda) = 0 \quad \text{and} \quad \nabla_\lambda L_c(x,\lambda) = 0$$

- Newton system:

$$\left[\begin{array}{c|c} \nabla_x^2 f(x) + cE^\top (I-G)E & ((I-G)E)^\top \\ \hline (I-G)E & -c^{-1}G \end{array}\right] \begin{bmatrix} d_x \\ d_\lambda \end{bmatrix} = -\begin{bmatrix} \nabla_x L_c(x,\lambda) \\ \nabla_\lambda L_c(x,\lambda) \end{bmatrix}$$

where $G \in \partial(\mathrm{prox}_{\phi/c})(Ex + \lambda/c)$

- or equivalently

$$\left[\begin{array}{c|c} \nabla_x^2 f(x) & E^\top \\ \hline (I-G)E & -c^{-1}G \end{array}\right] \begin{bmatrix} x^+ \\ \lambda^+ \end{bmatrix} = \begin{bmatrix} \nabla_x^2 f(x)x - \nabla_x f(x) \\ \mathrm{prox}_{\phi/c}(z) - Gz \end{bmatrix}$$

where $z = Ex + \lambda/c$. $d_x = x^+ - x$, $d_\lambda = \lambda^+ - \lambda$.

[1] When E is not surjective, the Newton system becomes inconsistent, i.e., there may be no solution to the Newton system. In this case, a proximal algorithm is employed to guarantee the solvability of the Newton system

Newton method for composite convex problem

- Problem

$$\min_x f(x) + \phi(x)$$

- Apply Newton method to the optimality condition

$$\nabla_x f(x) + \lambda = 0, \quad x - \mathrm{prox}_{\phi/c}(x + \lambda/c) = 0$$
$$\iff x - \mathrm{prox}_{\phi/c}(x - \nabla_x f(x)/c) = 0$$

- Define $F_c(x) = L_c(x, -\nabla_x f(x))$.
- Algorithm

step 1: $G \in \partial(\mathrm{prox}_{\phi/c})(x - \nabla_x f(x)/c)$.
step 2: $\left(I - G(I - c^{-1}D^2 f(x))\right) d = -(x - \mathrm{prox}_{\phi/c}(x - \nabla_x f(x)/c))$.
step 3: Armijo's rule: Find the smallest nonnegative integer j such that

$$F_c(x + 2^{-j}d) \le F_c(x) + \mu 2^{-j}\langle \nabla_x F_c(x), d\rangle \quad (0 < \mu < 1)$$
$$x^+ = x + 2^{-j}d$$

⋆ step 2: a variable metric gradient method on $F_c(x)$:

$$c(I - c^{-1}\nabla_x^2 f(x))(I - G(I - c^{-1}\nabla_x^2 f(x)))d = -\nabla_x F_c(x).$$

- The algorithm is proposed in [12, Patrinos-Bemporad].

Newton method applied to the priaml update

- Apply Newton method to solve the nonlinear equation arising from Fortin's augmented Lagrangian method

$$\text{step 1} \quad \nabla_x L_{c_k}(x, \lambda_k) = 0$$

$$\text{step 2} \quad \lambda_{k+1} = \lambda_k + c_k(Ex_k - \mathrm{prox}_{\phi/c_k}(Ex_k + \lambda_k/c)).$$

- Newton system

$$(\nabla_x^2 f(x) + cE^\top (I - G)E)d_x = -\nabla_x L_{c_k}(x, \lambda)$$

Numerical Test

Numerical Test

- ℓ^1 least square problems

$$\min_x \frac{1}{2}\|Ax - y\|^2 + \gamma\|x\|_1.$$

- The data set (A, y, x_0): L1-Testset (mat files) available at `http://wwwopt.mathematik.tu-darmstadt.de/spear/`
- Used 200 datasets. (dimension of $x : 1,024 \sim 49,152$).
- Windows surface pro 3, Intel i5 / 8GB RAM.

- Comparison:
  MOSEK (free academic license available at `https://mosek.com/`)
  MOSEK is a software package for solving large scale sparse problems.
  Algorithm: the interior-point method.
- $c = 0.99/\sigma_{\max}(A^T A)$.
- $\gamma = 0.01\|A^T y\|_\infty$.
- Compute until $\sqrt{\|\nabla_x L_c(x^k, \lambda^k)\|^2 + \|\nabla_\lambda L_c(x^k, \lambda^k)\|^2} < 10^{-12}$ is met.

Newton method for ℓ^1 regularized L.S.

- Newton system:

$$\begin{bmatrix} A^\top A & I \\ I - G & -cG \end{bmatrix} \begin{bmatrix} d_x \\ d_\lambda \end{bmatrix} = - \begin{bmatrix} A^T(Ax - y) \\ x - \mathrm{prox}_{\phi/c}(x + c\lambda) \end{bmatrix}$$

where $G = \partial(\mathrm{prox}_{\phi/c})(x + c\lambda)$:

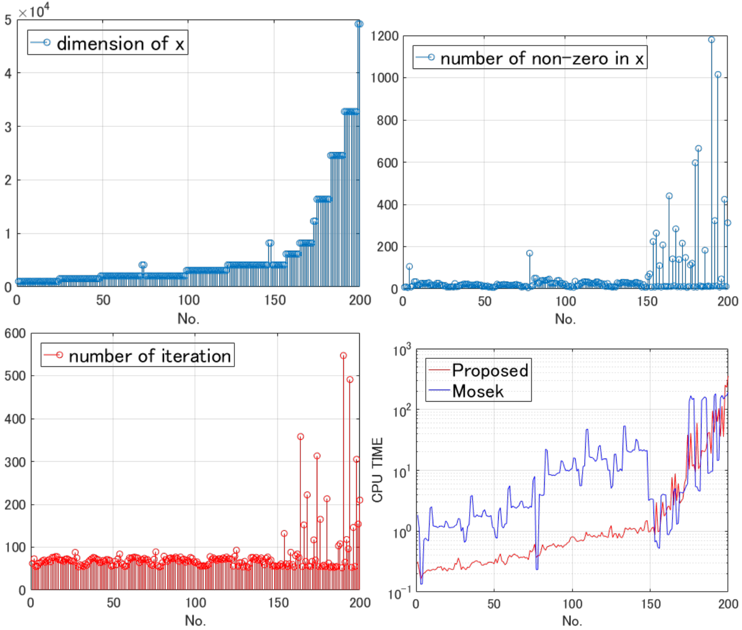
$$G_{ii} = 0 \quad \text{for } i \in \mathbf{o} := \{j \mid |x_j + c\lambda_j| \le \gamma c, j = 1, 2, \ldots, n\}$$

$$G_{ii} = 1 \quad \text{for } i \in \mathbf{o}^\mathsf{c} = \{j \mid |x_j + c\lambda_j| > \gamma c, j = 1, 2, \ldots, n\}$$

- Simply the system.

$$\begin{aligned} d_x(\mathbf{o}) &= -\, x(\mathbf{o}) \\ A(:, \mathbf{o}^\mathsf{c})^\top A(:, \mathbf{o}^\mathsf{c}) d_x(\mathbf{o}^\mathsf{c}) &= -\, [\gamma \mathsf{sign}(x(\mathbf{o}^\mathsf{c}) + c\lambda(\mathbf{o}^\mathsf{c})) \\ &\quad + A(:, \mathbf{o}^\mathsf{c})^\top (A(:, \mathbf{o}^\mathsf{c}) x(\mathbf{o}^\mathsf{c}) - y)] \\ d_\lambda &= -\, A^\top A(x + d_x) + A^T y. \end{aligned}$$

Results



×10^4
dimension of x
No.
number of non−zero in x
No.
number of iteration
No.
Proposed
Mosek
CPU TIME
No.

QUESTIONS ?
COMMENTS ?
SUGGESTIONS ?

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