Thermodynamic Entropy as a Noether Invariant

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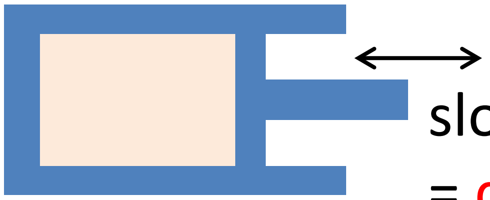
What is Thermodynamic Entropy? as a Noether Invariant

Thermodynamics

- 0th law: existence of equilibrium state
- 1st law: $dE = \underset{\text{heat}}{d'Q} + \underset{\text{work}}{d'W}$
- 2nd law: $S_f \geq S_i$
- Entropy (by Clausius):

$$S = S_0 + \int \frac{dQ}{T}$$



adiabatic wall

slowly

= quasi-static

- Adiabatic invariance

$$S_f = S_i$$

Thermodynamic Entropy as a Noether Invariant

What is ?

Noether theorem in Mechanics

Symmetry ⇔ Conservation law

$\delta_G I = 0$ for any $\hat{q}$ ⇔ $\frac{d}{dt} O_G \Big|_* = 0$ for solutions $\hat{q}_*$

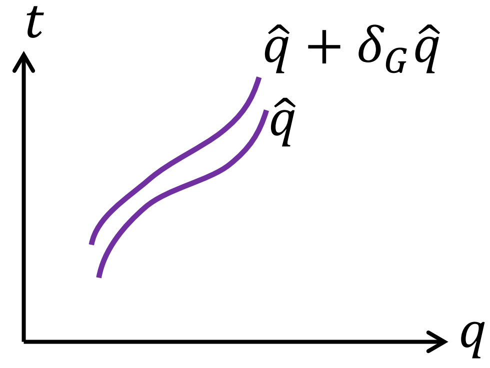
under $\hat{q} \to \hat{q} + \delta_G \hat{q}$

$$I[\hat{q}] = \int_{t_i}^{t_f} dt\, L\big(q(t), \dot{q}(t)\big)$$

Example

$t \to t + \epsilon \Leftrightarrow E$

$x \to x + \epsilon \Leftrightarrow p_x$



Adiabatic invariance: $S_i = S_f$

Thermodynamic Entropy as a Noether Invariant

today's question

What is

Symmetry

?

⇔ Conservation law

Adiabatic invariance: $S_i = S_f$

Thermodynamic Entropy as a Noether Invariant

today's question

What is ***Symmetry*** ⇔ Conservation law **?**

Black hole Entropy

[1973 Bekenstein, 1974 Hawking]

$$S_{BH} = \frac{c^3 A}{4G\hbar}$$

Black-hole entropy ⇔ diffeomorphism invariance

[1993 Wald]

$$S_{BH} = \frac{2\pi}{\hbar}\int_{Area} dA\, Q_{\xi}.$$

Adiabatic invariance: $S_i = S_f$ ←macroscopic

Thermodynamic Entropy as a Noether Invariant

today's question

What is

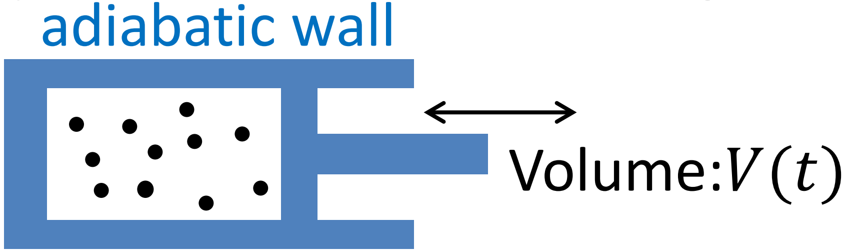
Symmetry ⇔ Conservation law

↑microscopic ?

⇒Let's first find the microscopic description of the adiabatic invariance of entropy.

Setup

- Consider N classical particles with short-range interaction in a box of volume V.



- The action

$$I[\hat{q}] = \int_{t_i}^{t_f} dt\, L\big(q(t), \dot{q}(t), V(t)\big)$$

$q(t) \in \mathbb{R}^{3N}$: a collection of coordinates of N particles

$V(t)$: time-dependent volume ⇒A protocol (functional form of $V(t)$) is fixed.

Example: $L = \sum_{i=1}^{N} \frac{1}{2} m \dot{\boldsymbol{r}}_i - \sum_{i<j} U_{int}\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) - \sum_{i=1}^{N} U_{Wall}(\boldsymbol{r}_i; V(t))$

- The energy

$$E(q, \dot{q}, V) = \dot{q}\,\frac{\partial L}{\partial \dot{q}} - L$$

Boltzmann Entropy

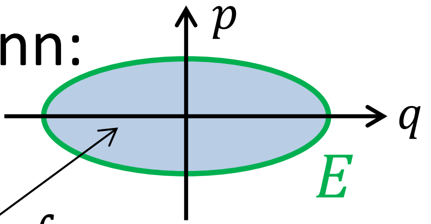
- Consider the Hamiltonian formulation:

$$\left\{\begin{array}{l}\text{a phase space coordinate } \Gamma = (q,p) \text{ with } p \equiv \dfrac{\partial L}{\partial \dot{q}} \\ \text{Hamiltonian } \; H(\Gamma,V) = E(q,\dot{q}(q,p,V),V)\end{array}\right.$$

- The microscopic definition of entropy by Boltzmann:

$$\boxed{S(E,V) = \log \frac{\Omega(E,V)}{N!}}$$

∼microscopic randomness of a system



$$\Omega(E,V) \equiv \int d\Gamma \theta\big(E - H(\Gamma,V)\big)$$

- The identity of $S(E,V)$:

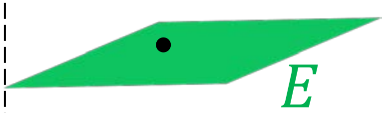
The first law $\boxed{dS = \beta dE + \beta P dV}$

Inverse temperature

$$\beta(E,V) \equiv \frac{\Sigma(E,V)}{\Omega(E,V)} = \frac{\partial S}{\partial E}$$

micro-canonical ensemble ~ typical behavior

$$\rho = \frac{\delta}{\Sigma}$$



$$\langle A \rangle_{E,V}^{mc} \equiv \frac{1}{\Sigma(E,V)} \int d\Gamma A(\Gamma) \delta\big(E - H(\Gamma,V)\big)$$

$$\Sigma(\mathrm{E},V) \equiv \int d\Gamma \delta\big(E - H(\Gamma,V)\big)$$

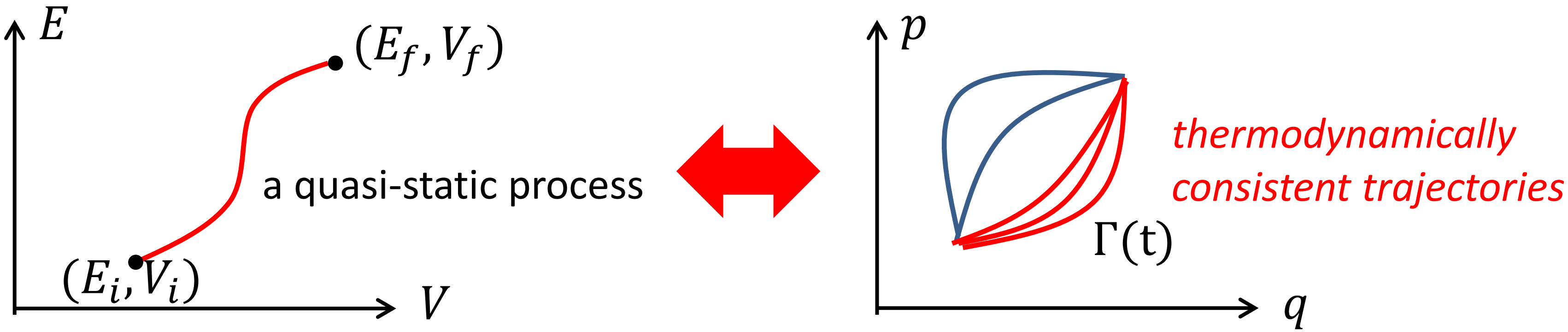
Thermodynamic pressure

$$P \equiv -\left\langle \frac{\partial H}{\partial V} \right\rangle_{E,V}^{mc}$$

Bridge between micro and macro

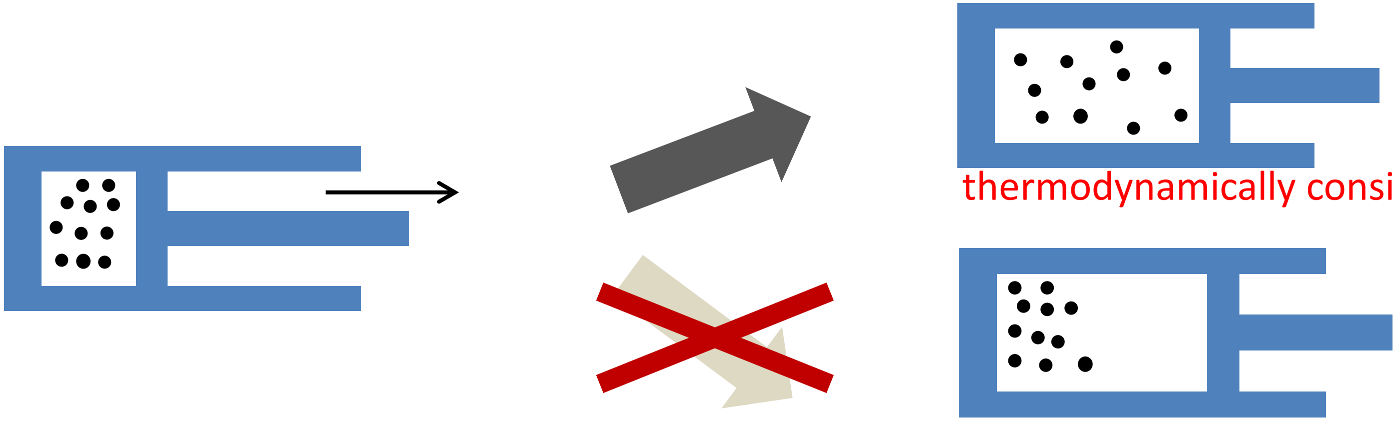
- We are interested in quasi-static time evolution of the macroscopic system.

⇒Consider a class of phase-space trajectories consistent with quasi-static processes in thermodynamics.



<thermodynamic state space>

<phase space>



thermodynamically consistent!

non-consistent

Thermodynamically consistent trajectory

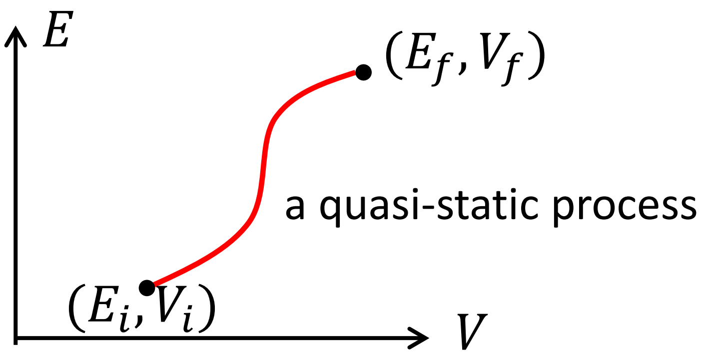
quasi-static limit

Consider a parameter $V(t) = \bar{V}(\epsilon t)$,

introduce $\tau = \epsilon t$,

and take $\epsilon \to 0$ with $\tau_i = \epsilon t_i, \tau_f = \epsilon t_f$ fixed.

$$\Rightarrow \frac{dV}{dt} = \epsilon \frac{d\bar{V}}{d\tau} = \epsilon \mathcal{O}(1) \ll 1$$



The conditions of thermodynamically consistent trajectory Γ(t):

$$(1)\quad E(t) = \bar{E}(\tau)$$

$$E(t) = H\big(\Gamma(t), V(t)\big)$$

$$(2)\quad \int_{\tau_i}^{\tau_f} d\tau \frac{d\bar{V}}{d\tau}\left[\frac{\partial H}{\partial V} - \left\langle \frac{\partial H}{\partial V} \right\rangle^{mc}_{\bar{E}(\tau),\bar{V}(\tau)}\right] = 0$$

mechanical work = thermodynamic work

in the quasi-static limit.

Microscopic description of adiabatic invariance of entropy

Let's consider time evolution of

$$S(t)=S\Big(\mathrm{E}\big(\Gamma(t),V(t)\big),V(t)\Big)$$

in the quasi-static limit $\epsilon\to 0$.

$$S(t_f)-S(t_i)=\int_{t_i}^{t_f} dt\,\frac{dS(t)}{dt}$$

$$dS=\beta dE-\beta\left\langle\frac{\partial H}{\partial V}\right\rangle_{E,V}^{mc} dV$$

$$=\int_{t_i}^{t_f} dt\,\beta\left[\frac{dE}{dt}-\dot V\left\langle\frac{\partial H}{\partial V}\right\rangle_{E(t),V(t)}^{mc}\right]$$

$$\frac{dE(t)}{dt}=\overset{\text{“heat”}}{\frac{\partial H}{\partial\Gamma}\dot\Gamma}+\overset{\text{“work”}}{\frac{\partial H}{\partial V}\dot V}=\frac{\partial H}{\partial V}\dot V$$

Adiabatic condition $\fallingdotseq 0$

$$dE=\delta Q+\delta W$$

$$=\int_{\tau_i}^{\tau_f} d\tau\,\beta\frac{d\bar V}{d\tau}\left[\frac{\partial H}{\partial V}-\left\langle\frac{\partial H}{\partial V}\right\rangle_{\bar E(\tau),\bar V(\tau)}^{mc}\right]$$

$$=0$$

Thermodynamically consistent trajectory

$$\int_{\tau_i}^{\tau_f} d\tau\,\frac{d\bar V}{d\tau}\left[\frac{\partial H}{\partial V}-\left\langle\frac{\partial H}{\partial V}\right\rangle_{\bar E(\tau),\bar V(\tau)}^{mc}\right]=0$$

⇒We will find the symmetry corresponding to this conservation law in the Noether theorem.

A generalized Noether theorem

- The Noether theorem in text books

$$\delta_G I = 0 \text{ for any } \hat{q} \Leftrightarrow \frac{d}{dt} O_G \Big|_* = 0 \text{ for solutions } \hat{q}_*$$

under $\hat{q} \to \hat{q} + \delta_G \hat{q}$

generalized

- A generalized Noether theorem

[Trautman 1967, Sarlet-Cantrijin 1981]

$$\delta_G I = \int_{t_i}^{t_f} dt \, \frac{df(q,\dot{q},t)}{dt} \text{ for any } \hat{q}$$

under $\hat{q} \to \hat{q} + \delta_G \hat{q}$

$$\Leftrightarrow \frac{d}{dt} O_G \Big|_* = 0 \text{ for solutions } \hat{q}_*$$

A non-uniform time translation

- Consider a non-uniform time translation

$$t \to t' = t + \eta\xi(q,\dot{q},t)$$

η: infinitesimal parameter

$$\Rightarrow \begin{cases} q(t) \to q'(t') = q(t) \\ V(t) \to V'(t') = V(t') \end{cases}$$

(←just relabeling)

(←The protocol is fixed.)

$$\Rightarrow \delta_G I = \eta \int_{t_i}^{t_f} dt \left[-\mathcal{E}\dot{q} + \frac{d}{dt}(\xi E) \right]$$

$$\boxed{\mathcal{E} \equiv \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}}$$

- Suppose that there exist $\xi(q,\dot{q},t)$ and $\psi(q,\dot{q},t)$ s.t.

$$\delta_G I = \eta \int_{t_i}^{t_f} dt \frac{d\psi}{dt}$$

$$\Longrightarrow -\int_{t_i}^{t_f} dt\, \mathcal{E}\, \dot{q}\, \xi = [\psi + \xi E]_{t_i}^{t_f}$$

$\Rightarrow \psi + \xi E$
is conserved on $q_*(t)$!

Derivation of the symmetry 1

- Let’s derive the symmetry for adiabatic invariance of entropy.
- The condition of symmetry is non-trivial:

$$-\int_{t_i}^{t_f} dt\, \mathcal{E}\dot{q}\xi = \int_{t_i}^{t_f} dt \frac{d}{dt}(\psi + \xi E)$$

- Consider now

$$\xi = \Xi(E(q,\dot{q},V),V), \qquad \psi = \Psi(E(q,\dot{q},V),V)$$

and restrict trajectories to thermodynamically consistent ones.

⇒ The condition becomes

$$\Xi\left[dE - dV\left\langle\frac{\partial H}{\partial V}\right\rangle_{E,V}^{mc}\right] = d(\Psi + \mathrm{E}\Xi)$$

⇒ The solution is

$$\boxed{\Xi = \mathcal{F}(S)\beta}$$

←Symmetry emerges for thermodynamically consistent trajectories!

Derivation of the symmetry 2

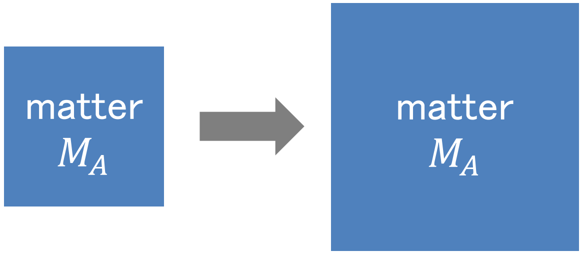
- The Noether invariant is

$$\boxed{\Psi + E\Xi = \int^{S} dS' \, \mathcal{F}(S')}$$

- Assume that this is extensive.

scaled copy

$$V \to \lambda V$$
$$N \to \lambda N$$



➡ $$\begin{cases} \Psi + E\Xi \to \lambda(\Psi + E\Xi) \\ E \to \lambda E \end{cases}$$

➡ $$\begin{cases} \Psi \to \lambda\Psi \\ \Xi \to \Xi \end{cases}$$

intensive

➡ $\beta \to \beta$

$\Xi\beta^{-1} = \mathcal{F}(S)$ is intensive and independent of V.

Derivation of the symmetry 3

- Here we can express

$$\boxed{\mathcal{F}(S) = \bar{\bar{\mathcal{F}}}(s; M).}$$

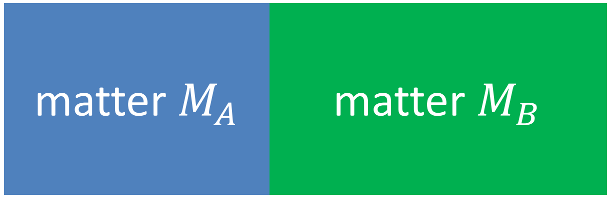
$s \equiv \frac{S}{N}$

material dependence

- Consider a composite system.

$\Xi\beta^{-1} = \mathcal{F}(S)$

$$\beta_A = \beta_B$$
$$\Xi_A = \Xi_B$$



$$\Rightarrow \boxed{\bar{\bar{\mathcal{F}}}(s_A; M_A) = \bar{\bar{\mathcal{F}}}(s_B; M_B)}$$

- If $M_A = M_B = M$

$\Rightarrow \bar{\bar{\mathcal{F}}}(s_A; M) = \bar{\bar{\mathcal{F}}}(s_B; M)$ for any s_A, s_B

$\Rightarrow \bar{\bar{\mathcal{F}}}(s; M) = c(M)$

- If $M_A \neq M_B$

$\Rightarrow c(M_A) = c(M_B)$ for any M_A, M_B

$\Rightarrow c(M) = c_*$

Derivation of the symmetry 4

- Thus, we obtain

$$\Xi\beta^{-1} = \mathcal{F} = c_*$$

[time] × [energy]
= [action]

independent of state and material
= universal constant

$$\Rightarrow \boxed{c_* \propto \hbar}$$

⇒Our framework based on classical theory has led to the existence of the Planck constant.

- Therefore, we have

$$\Xi = a\hbar\beta$$

$\mathcal{F} = c_* = a\hbar$

$$\Psi + E\Xi = \int^{S} dS' \mathcal{F}(S') = a\hbar(S + bN)$$

Main result

- In the macroscopic system, the symmetry

$$\boxed{t \to t + \eta\hbar\beta\big(E(t), V(t)\big)}$$

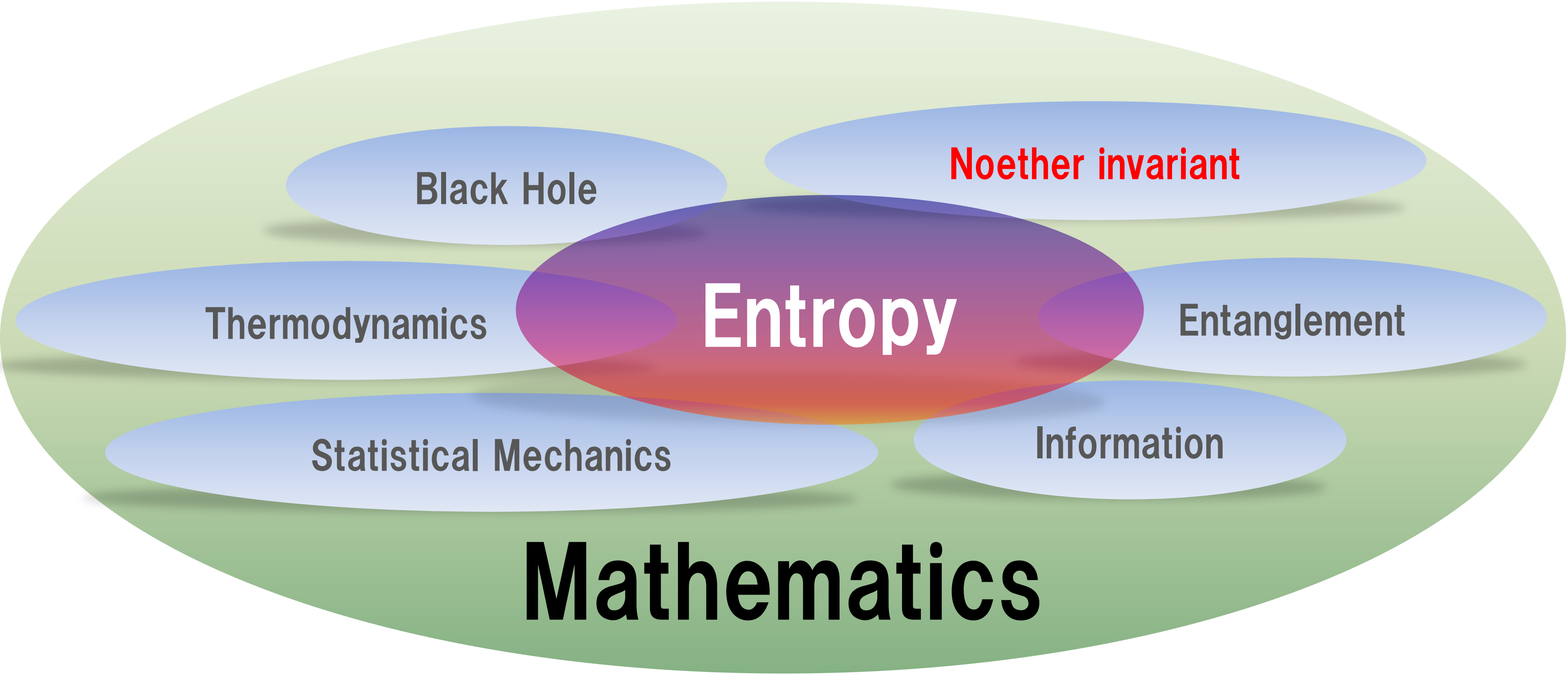
emerges for “thermodynamically consistent trajectories”.

- Then, the Noether invariant is

$$\Psi + E\Xi = \hbar(S + bN).$$

- This provides a new and unique characterization of entropy.

Various aspects of entropy on mathematics



Black Hole

Noether invariant

Thermodynamics

Entropy

Entanglement

Statistical Mechanics

Information

Mathematics

Math connects different physics and makes new ideas.

Thank you very much!