# Borel resummation and Exact results in supersymmetric gauge theories

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## References:

[1] M.H., "Borel Summability of Perturbative Series in 4D N=2 and 5D N=1 Supersymmetric Theories", PRL116, 211601(2016) (arXiv: 1603.06207 [hep-th])

[2] M.H., "How to resum perturbative series in 3d N=2 Chern-Simons matter theories", PRD94, 025039 (2016) (arXiv:1604.08653 [hep-th])

[3] M.H., arXiv:1709.nnnnn

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[thanks to localization method: ’07 Pestun ]

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Typically, for supersymmetric quantities,

$$\text{(path integral)} \longrightarrow \int d^{|G|}x \; f(x)$$

(|G|: rank of gauge group G)

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Typically, for supersymmetric quantities,

(path integral) $\longrightarrow$ $\int d^{|G|}x \ f(x)$

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In this talk, I will demonstrate

these exact results are useful for understanding properties of perturbative series in QFT

# Perturbative series of exact results in QFT

Motivaitons:

1. We can practically get much perturbative information
2. We can also study perturbative series around nontrivial saddle points
3. We can check relation between resummation and exact results

# Setup

[cf. some SU(2) cases: Russo, Aniceto-Russo-Schiappa]

- 4d N=2 and 5d N=1 theories on sphere

  —— expansion by $g_{YM}$ around instanton background

- 3d N=2 Chern-Simons theories on $S^3$ & lens sp.

  —— expansion by inverse CS levels

# Summary of main results

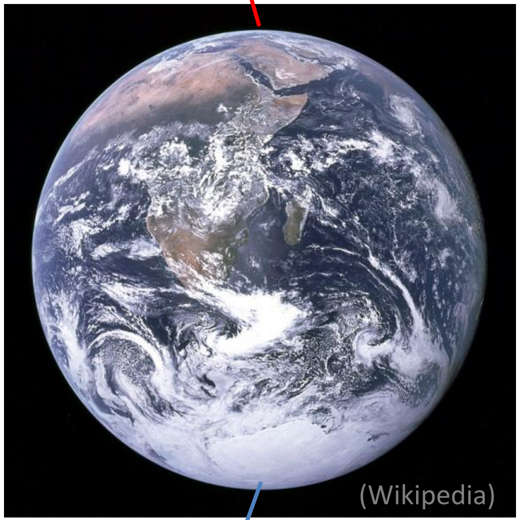
# Results on 4d N=2 SUSY theories (w/ 8 SUSY)

[M.H. '16]

## Set up:

- Theories w/ β≦0 and Lagrangians
  $(Z_{S^4} < \infty)$
- Perturbative expansion by $g_{YM}$
  around fixed # of instanton/anti-inst.

inst.

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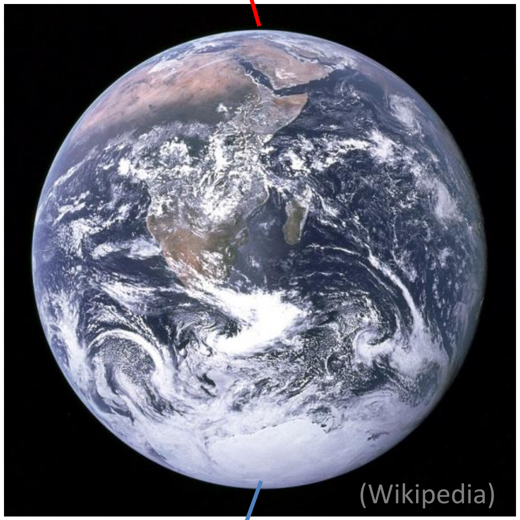


anti-inst.

## Result:

(similar for 5d case)

[cf. some SU(2) theories: Russo, Aniceto-Russo-Schiappa]

- Find explicit finite dimensional integral rep. of Borel trans. for various observables
- $^{\exists}$Singularities only along R- → Borel summable along R+
  (for round $S^4$)
- $(\text{Exact}) = \sum_{\text{instantons}} (\text{Borel resum})$

# Examples

For SU(2) case,

$$Z_{S^4}^{(k,\bar{k})}(g) = \int_{-\infty}^{\infty} da\ e^{-\frac{a^2}{g}} f^{(k,\bar{k})}(a),$$

$$\boxed{\mathcal{B}Z_{S^4}^{(k,\bar{k})}(t) \propto f^{(k,\bar{k})}(a = \sqrt{t})}$$

(anti-)instanton #

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Ex.1) Pure SYM (trivial b.g.):

$$\mathcal{B}Z_{S^4}^{(0,0)}(t) \propto \sqrt{t} \prod_{n=1}^{\infty} \left(1 + \frac{4t}{n^2}\right)^{2n}$$

No singularities $\longleftrightarrow$ Convergent expansion

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Ex.2) SQCD (trivial b.g.):

$$\mathcal{B}Z_{S^4}^{(0,0)}(t) \propto \sqrt{t} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{4t}{n^2}\right)^{2n}}{\left(1 + \frac{t}{n^2}\right)^{2N_f n}}$$

# Interpretations

**Result:**

$^{\exists}$Borel singularities only along R- in 4d N=2 SUSY theories

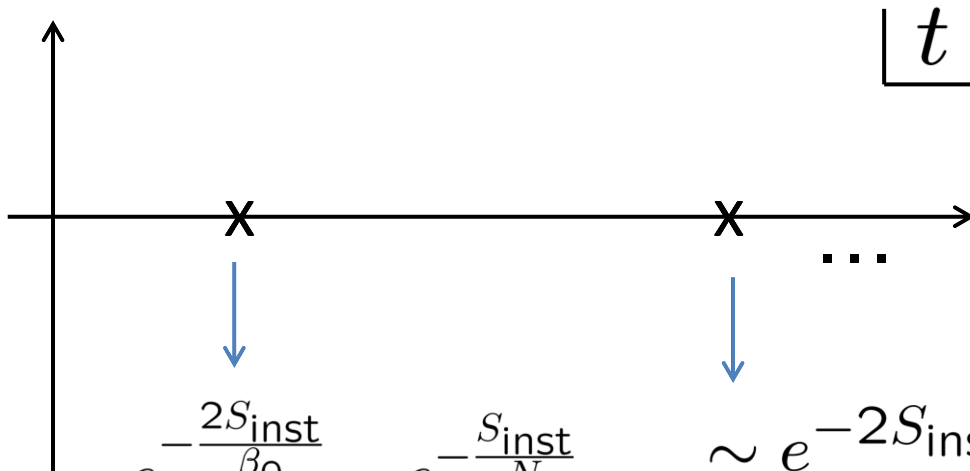
$^{\exists}$Agreement w/ recent conjecture on QCD-like theory

&

Confusion compared w/ usual story of resummation

# Nontrivial consistency w/ a conjecture on QCD

## Borel plane in typical gauge theory :




$\sim e^{-\frac{2S_{\mathrm{inst}}}{\beta_0}} \sim e^{-\frac{S_{\mathrm{inst}}}{N}}$ $\sim e^{-2S_{\mathrm{inst}}}$

IR renormalon Inst.-anti-inst.

# Nontrivial consistency w/ a conjecture on QCD

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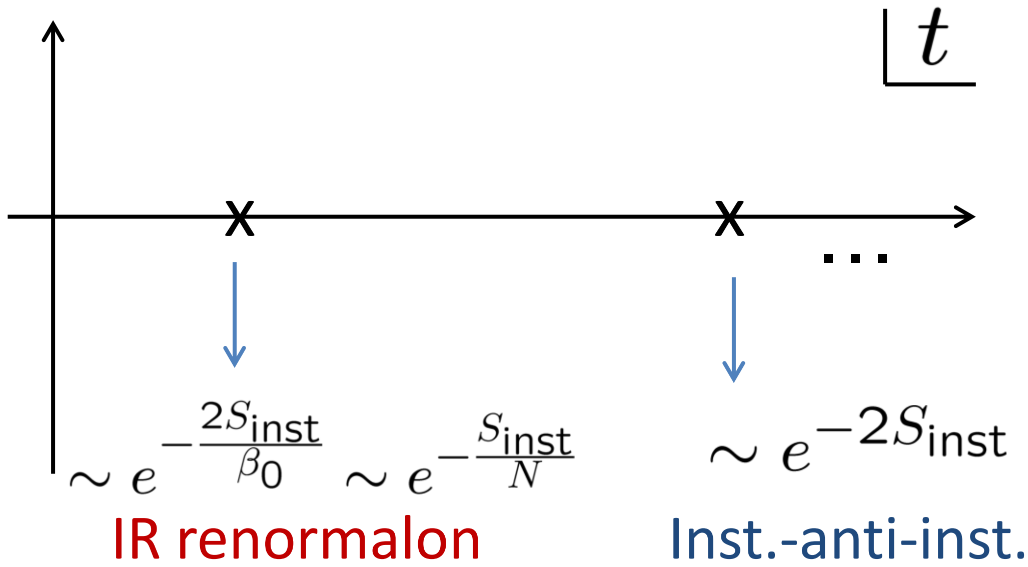


Conjecture: (IR renormalon) = (Combination of monopoles)

[Argyres-Unsal '12]

# Nontrivial consistency w/ a conjecture on QCD

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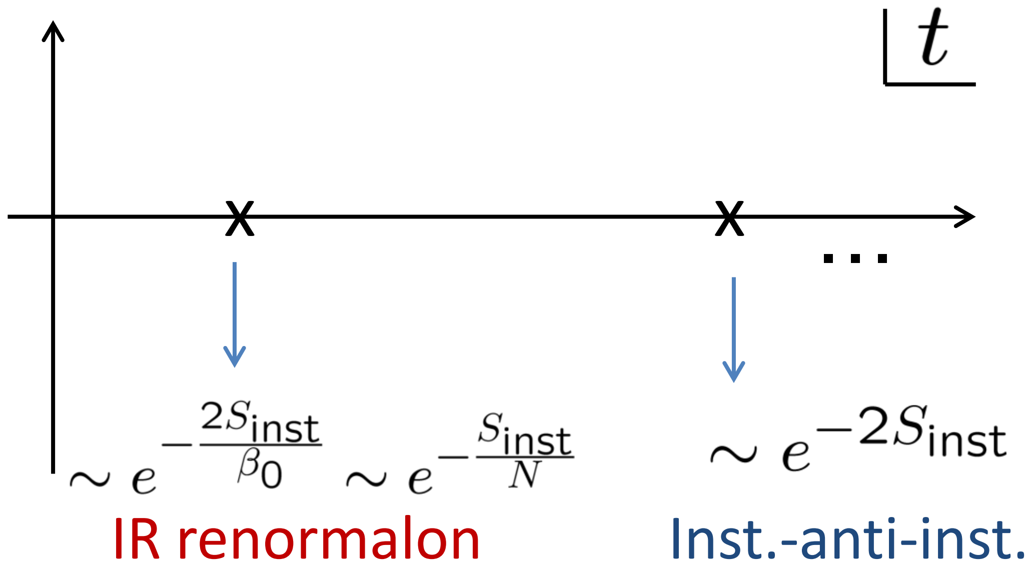


Conjecture: (IR renormalon) = (Combination of monopoles)

[Argyres-Unsal '12]

But there is no such solution for $\mathcal{N}=2$ [Popitz-Unsal ]

$\longrightarrow$ No IR renormalon type singularities for $\mathcal{N}=2$ ?

# Confusion?

Usually Borel singularities come from nontrivial saddles w/ the same topological numbers [cf. Lipatov '77]

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Now we have $\int_{S^4} F \wedge F \propto k - \bar{k}$

For example, around trivial saddle, we expect

- No Borel singularities from $k \neq \bar{k}$
- ∃ Borel singularities from $k = \bar{k}$ (namely, at t=2k)

But we do not have such singularities.

# Results on 3d N=2 SUSY Chern-Simons theories

(w/ 4 SUSY)

[M.H. '16]

**Set up:**

- General Chern-Simons (CS) theories coupled to matters
  $(Z_{S^3} < \infty)$
- Perturbative expansion by inverse CS levels

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**Set up:**

- General Chern-Simons (CS) theories coupled to matters
  $(Z_{S^3} < \infty)$

- Perturbative expansion by inverse CS levels

**Result:**

$$S_\theta I(g) = \int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}I(t)$$

- Find finite dimensional integral rep. of Borel trans.

- Usually non-Borel summable along R+

- But always Borel summable along (half-)imaginary axis

- (Borel resum. w/ θ=±π/2) = (exact result)

# Examples

For SU(2) case,

$$Z_{S^3}(g) = \int_{-\infty}^{\infty} d\sigma \ e^{\frac{i\text{sgn}(k)}{g}\sigma^2} f(\sigma), \qquad \boxed{\mathcal{B}Z_{S^3}(t) \propto f(\sigma = \sqrt{i\text{sgn}(k)t})}$$

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Ex.1) Pure SUSY CS:

$$\mathcal{B}Z_{S^3}(t) \propto \sigma \cdot \sinh^2(\sigma)\Big|_{\sigma=\sqrt{i\mathrm{sgn}(k)t}}$$

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Ex.2) SQCD w/ hypers and real mass:

$$\mathcal{B}Z_{S^3}(t) \propto \frac{\sigma \cdot \sinh^2(\sigma)}{\left(\cosh\frac{\sigma-m}{2}\cosh\frac{\sigma+m}{2}\right)^{N_f}}\Bigg|_{\sigma=\sqrt{i\mathrm{sgn}(k)t}}$$

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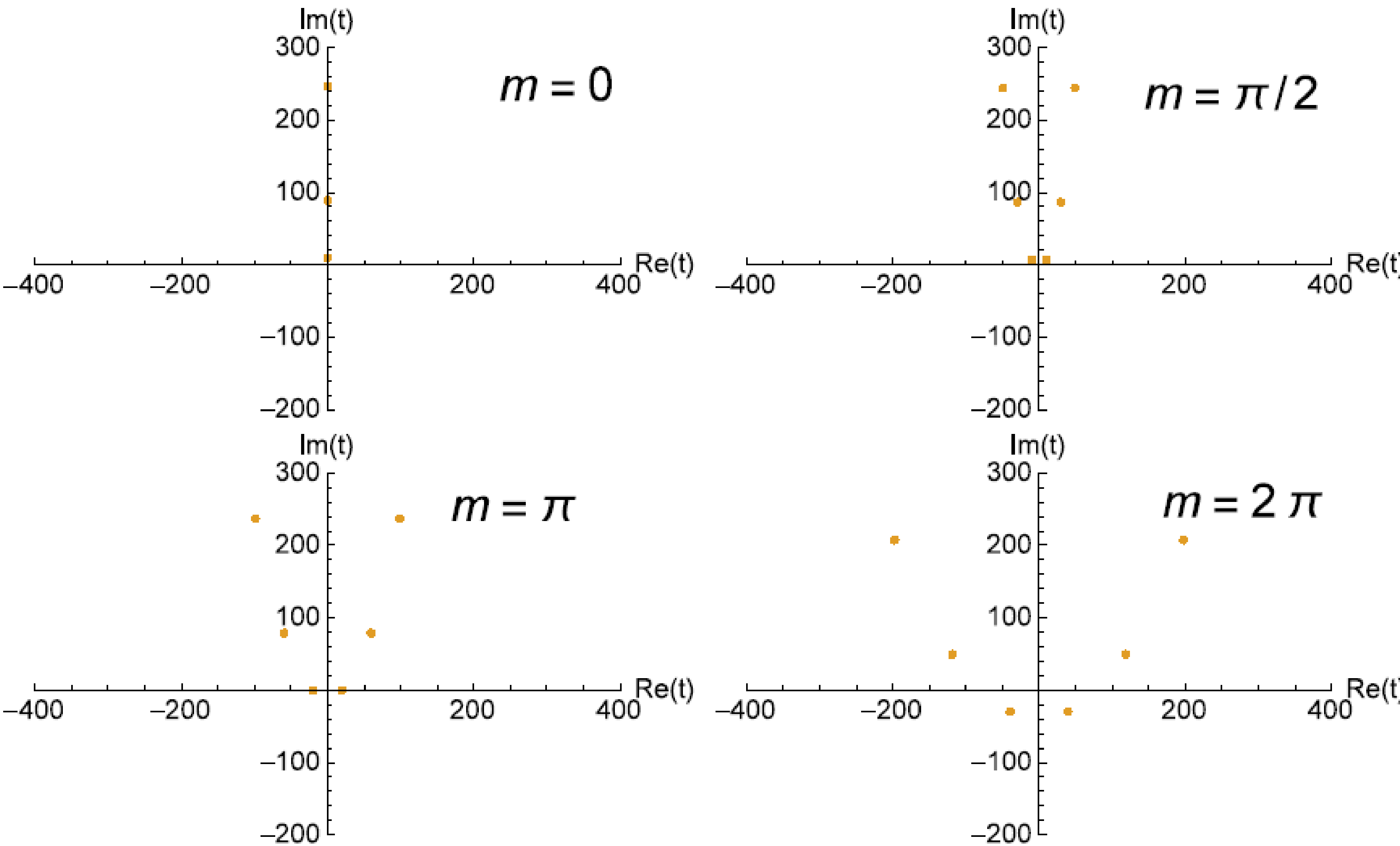

# Interpretation of Borel singularities (3d)

[M.H., to appear]

All the singularities can be explained by

## complexified SUSY solutions

which are not on original contour of path integral
but formally satisfy SUSY conditions: $Q\lambda = 0, \ Q\psi = 0$

# Interpretation of Borel singularities (3d)

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All the singularities can be explained by

**complexified SUSY solutions**

which are not on original contour of path integral
but formally satisfy SUSY conditions: $Q\lambda = 0, \ Q\psi = 0$

Indeed their actions agree residues:

$$e^{-S} \sim \mathrm{Res}\left[\mathcal{BO}(t)\right]$$

The numbers also agree if we follow the rule:

| | | |
|---|---|---|
| 1 solution | $\longleftrightarrow$ | Simple pole |
| n sols. w/ the same S | $\longleftrightarrow$ | Degree-n pole |

# Contents

# Partition function of Superconformal QCD on $S^4$

SU(N) SQCD w/ 2N-fundamental hypermultiplets

**Exact result by localization method:**

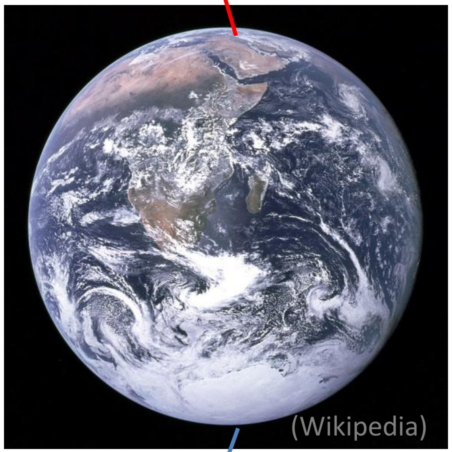
[Pestun '07]

$$Z_{\rm SQCD}(g,\theta) = \int_{-\infty}^{\infty} d^N a \; e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2} \tilde{Z}(a) Z_{\rm inst}(g,\theta;a)$$

$$Z_{\rm inst}(g,\theta;a) = \sum_{k,\bar{k}=0}^{\infty} e^{-\frac{k+\bar{k}}{g}+i(k-\bar{k})\theta} Z_{\rm inst}^{(k,\bar{k})}(a)$$

$$\boxed{Z_{\rm SQCD}^{(k,\bar{k})}(g) = \int_{-\infty}^{\infty} d^N a \; e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2} \tilde{Z}(a) Z_{\rm inst}^{(k,\bar{k})}(a)}$$

inst.




anti-inst.

We are interested in small-g expansion of this

We would like to study small-g expansion of

$$Z_{\mathrm{SQCD}}^{(k,\bar{k})}(g) = \int_{-\infty}^{\infty} d^N a \ e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2} \tilde{Z}(a) Z_{\mathrm{inst}}^{(k,\bar{k})}(a)$$

## A naïve way:

1. Compute perturbative expansion at all orders
2. Compute Borel transformation
3. Look at its analytic property

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~~A naïve way:~~ Our method:

~~1. Compute perturbative expansion at all orders~~

~~2. Compute Borel transformation~~ Find Borel trans. hidden in localization formula

3. Look at its analytic property

# Borel trans. hidden in localization formula

$$Z_{\mathrm{SQCD}}^{(k,\bar{k})}(g) = \int_{-\infty}^{\infty} d^N a \ e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2} \tilde{Z}(a) Z_{\mathrm{inst}}^{(k,\bar{k})}(a)$$

Taking polar coordinate $a_i = \sqrt{t}\hat{x}_i$ w/ $(\hat{x}^i)^2 = 1,$

# Borel trans. hidden in localization formula

$$Z_{\text{SQCD}}^{(k,\bar{k})}(g) = \int_{-\infty}^{\infty} d^N a \ e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2} \tilde{Z}(a) Z_{\text{inst}}^{(k,\bar{k})}(a)$$

Taking polar coordinate $a_i = \sqrt{t}\hat{x}_i$ w/ $(\hat{x}^i)^2 = 1$,

$$\boxed{Z_{\text{SQCD}}^{(k,\bar{k})}(g) = \int_0^{\infty} dt \ e^{-\frac{t}{g}} f^{(k,\bar{k})}(t)}$$

Similar to the Borel resummation formula!

Is this Borel transformation?

$$\left( f^{(k,\bar{k})}(t) = \int_{S^{N-1}} d^{N-1}\hat{x} \ h^{(k,\bar{k})}(t,\hat{x}), \ \ h^{(k,\bar{k})}(t,\hat{x}) = \tilde{Z}(a) Z_{\text{inst}}^{(k,\bar{k})}\Big|_{a^i=\sqrt{t}\hat{x}^i} \right)$$

$$\boxed{Z^{(k,\bar{k})}_{\text{SQCD}}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}} f^{(k,\bar{k})}(t)}$$

Is this Borel trans.?

More precisely, given $Z^{(k,\bar{k})}_{\text{SQCD}}(g) \sim \sum_{\ell=0}^{\infty} c_\ell^{(k,\bar{k})} g^{\sharp+\ell}$,

$$\boxed{f^{(k,\bar{k})}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell^{(k,\bar{k})}}{\Gamma(\sharp+\ell)} t^{\sharp+\ell-1} \ ??}$$

(analytic continuation)

We can prove that this is actually true.

# Analytic property of Borel trans.

$$Z^{(k,\bar{k})}_{\mathrm{SQCD}}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}} f^{(k,\bar{k})}(t),\ f^{(k,\bar{k})}(t) = \int_{S^{N-1}} d^{N-1}\hat{x}\ h^{(k,\bar{k})}(t,\hat{x})$$

For trivial b.g.,

$$h^{(0,0)}(t,\hat{x}) = \delta\left(\sum_j \hat{x}_j\right) \prod_{i<j} (\hat{x}_i - \hat{x}_j)^2 \prod_{n=1}^{\infty} \frac{\prod_{i<j}\left(1 + \frac{t(\hat{x}_i-\hat{x}_j)^2}{n^2}\right)^{2n}}{\prod_j \left(1 + \frac{t(\hat{x}_j)^2}{n^2}\right)^{2Nn}}$$

No singularities for $t \in R_+$ $\Rightarrow$ Borel summable!!

# General theory w/ Lagrangians (&β≦0)

Suppose a theory w/ gauge group: $G = G_1 \times \cdots \times G_n$

$$Z_{S^4}(g,\theta) = \int_{-\infty}^{\infty} d^{|G|}a \; Z_{\text{cl}}(g;a)\tilde{Z}(a)Z_{\text{inst}}(g,\theta;a)$$

$$Z_{\text{cl}}(g;a) = \exp\left[-\sum_{p=1}^{n} \frac{1}{g_p}\text{tr}(a^{(p)})^2\right]$$

Taking polar coordinate $a_i^{(p)} = \sqrt{t_p}\hat{x}_i^{(p)}$,

$$\boxed{Z_{S^4}^{(\{k\},\{\bar{k}\})}(g) = \int_0^{\infty} d^n t \; e^{-\sum_p \frac{t_p}{g_p}} f^{(\{k\},\{\bar{k}\})}(t_1,\cdots,t_n)}$$

Borel trans.

Borel summable!!

# Remark on non-conformal case

- $g_{YM}$ is running
- Here $g_{YM}$ is at scale $1/R_{sphere}$

For example, in pure SYM case,

[cf. Pestun '07]

$$e^{-\frac{8\pi^2}{g_{\rm YM}^2}{\rm tr}a^2} \cdot Z_{1-{\rm loop}}^{\mathcal{N}=2^*}(a,m) \xrightarrow{mR_{S^4} \gg 1} e^{-\frac{8\pi^2}{\tilde{g}_{\rm YM}^2}{\rm tr}a^2} \cdot Z_{1-{\rm loop}}^{{\rm pure}\mathcal{N}=2}(a)$$

$$\frac{1}{\tilde{g}_{\rm YM}^2} = \frac{1}{g_{\rm YM}^2} - \frac{C_2}{8\pi^2}\log\left(mR_{S^4}\right)$$

# Relation to the exact result

We have shown

(Borel resum. in sector w/ fixed inst./anti-inst. #)

$$||$$

(Truncation of whole exact result to the same sector)

Thus,

$$\boxed{\begin{array}{c}\text{(Exact result including full instanton corrections)} \\ || \\ \displaystyle\sum_{k,\bar{k}} \text{ (Borel resummation)}\end{array}}$$

# Resurgence triangle and our result

(instanton #, anti-inst.#)

(0,0)

?

(1,0) (0,1)

? ?

(2,0) (1,1) (0,2)

? ? ?

(3,0) (2,1) (1,2) (0,3)

(4,0) (3,1) (2,2) (1,3) (0,4)

Every sector is Borel summable, unambiguous

➡ Every sector is isolated in some sense (at least from this viewpoint)

# Other observables

- Supersymmetric Wilson loop on $S^4$

$$W = P\,\mathrm{exp}\Big[\oint ds\,(iA_\mu \dot{x}^\mu + \Phi)\Big]$$

- Bremsstrahrung function in SCFT on $R^4$ [cf. Fiol-Gerchkovitz-Komargodski '15]

$$(\text{Energy of quark}) = B \int dt\ \dot{a}^2$$

- Extremal correlator in SCFT on $R^4$ [cf. Gerchkovitz-Gomis-Ishtiaque -Karasik-Komargodski-Pufu '16]

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \bar{\mathcal{O}} \rangle$$

- Partition function on squashed $S^4$ ～SUSY Renyi entrory

[cf. Hama-Hosomichi, Nosaka-Terashima]

[cf. Nishioka-Yaakov '13, Crossley-Dyer-Sonner, Huang-Zhou]

# 3d N=2 SUSY CS matter theory

# Partition function of CS adjoint SQCD on $S^3$

$U(N)_k$ SQCD w/ $\left\{ \begin{array}{l} \text{fundamental} \\ \text{anti-fundamental} \\ \text{adjoint} \end{array} \right\}$ chiral multiplets

By localization method,

[Kapustin-Willett-Yaakov, Jafferis
Hama-Hosomichi-Lee]

$$Z_{\mathrm{SQCD}}(g) = \int_{-\infty}^{\infty} d^N\sigma \; e^{\frac{i\cdot \mathrm{sgn}(k)}{g}\sum_{j=1}^{N}\sigma_j^2}\, \tilde{Z}(\sigma)$$

$$g \propto \frac{1}{|k|}$$

We are interested in small-g (large level) expansion of this

# Borel trans. hidden in localization formula

$$Z_{\text{SQCD}}(g) = \int_{-\infty}^{\infty} d^N\sigma \ e^{\frac{i\cdot \text{sgn}(k)}{g}\sum_{j=1}^{N}\sigma_j^2} \tilde{Z}(\sigma)$$

Taking polar coordinate $\sigma_i = \sqrt{\tau}\widehat{x}_i$

$$\begin{aligned} Z_{\text{SQCD}}(g) &= \int_0^\infty d\tau \ e^{\frac{i\text{sgn}(k)}{g}\tau} f(\tau) \\ &= i\text{sgn}(k) \int_0^{-i\text{sgn}(k)\infty} dt \ e^{-\frac{t}{g}} f(i\text{sgn}(k)t) \end{aligned}$$

Similar to the Borel resummation formula but w/ different integral contour!

$$\left( f(\tau) = \int d^{N-1}\widehat{x} \ h(\tau,\widehat{x}), \qquad h(\tau,\widehat{x}) = \tilde{Z}(\sigma)\Big|_{\sigma^i=\sqrt{\tau}\widehat{x}^i} \right)$$

$$\boxed{Z_{\text{SQCD}}(g) = i\text{sgn}(k)\int_0^{-i\text{sgn}(k)\infty} dt\, e^{-\frac{t}{g}} f(i\text{sgn}(k)t)}$$

Borel transformation?

By using the technique in 4d, we can actually prove

$$\boxed{i\text{sgn}(k) f(\tau) = \mathcal{B}Z_{\text{SQCD}}(-i\text{sgn}(k)\tau)}$$

Namely,

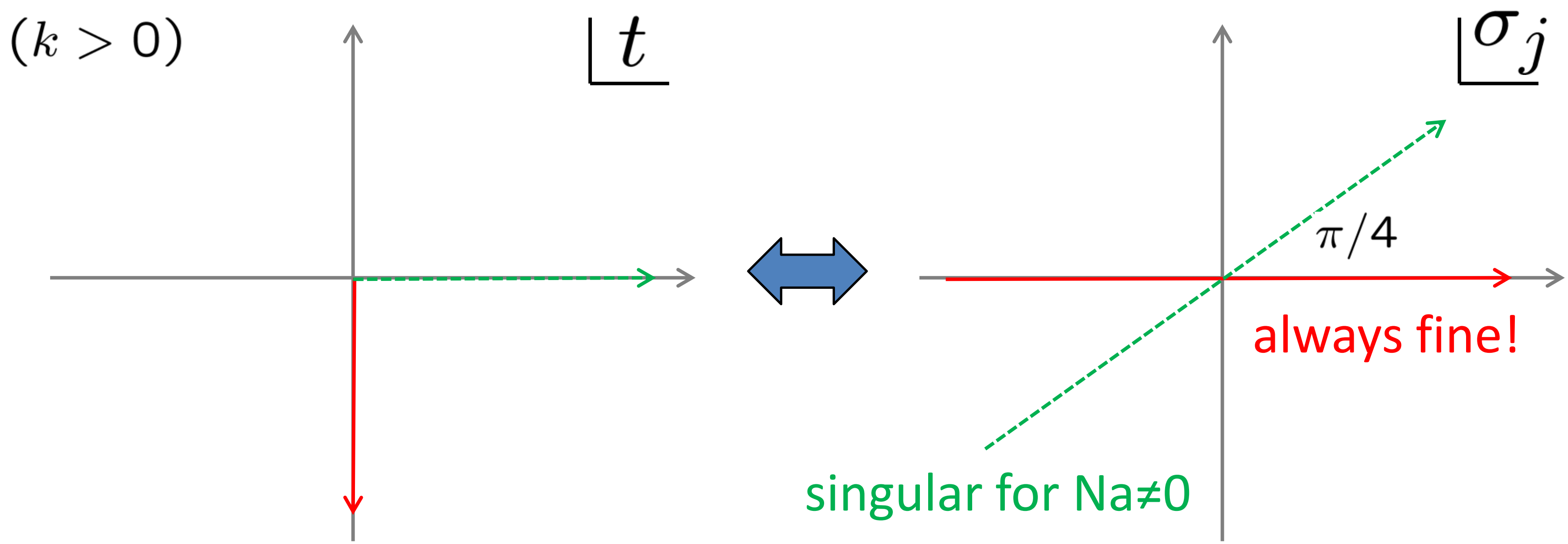
$$\boxed{Z_{\text{SQCD}}(g) = \int_0^{-i\text{sgn}(k)\infty} dt\, e^{-\frac{t}{g}} \mathcal{B}Z_{\text{SQCD}}(t)}$$

# Analytic property of Borel trans.

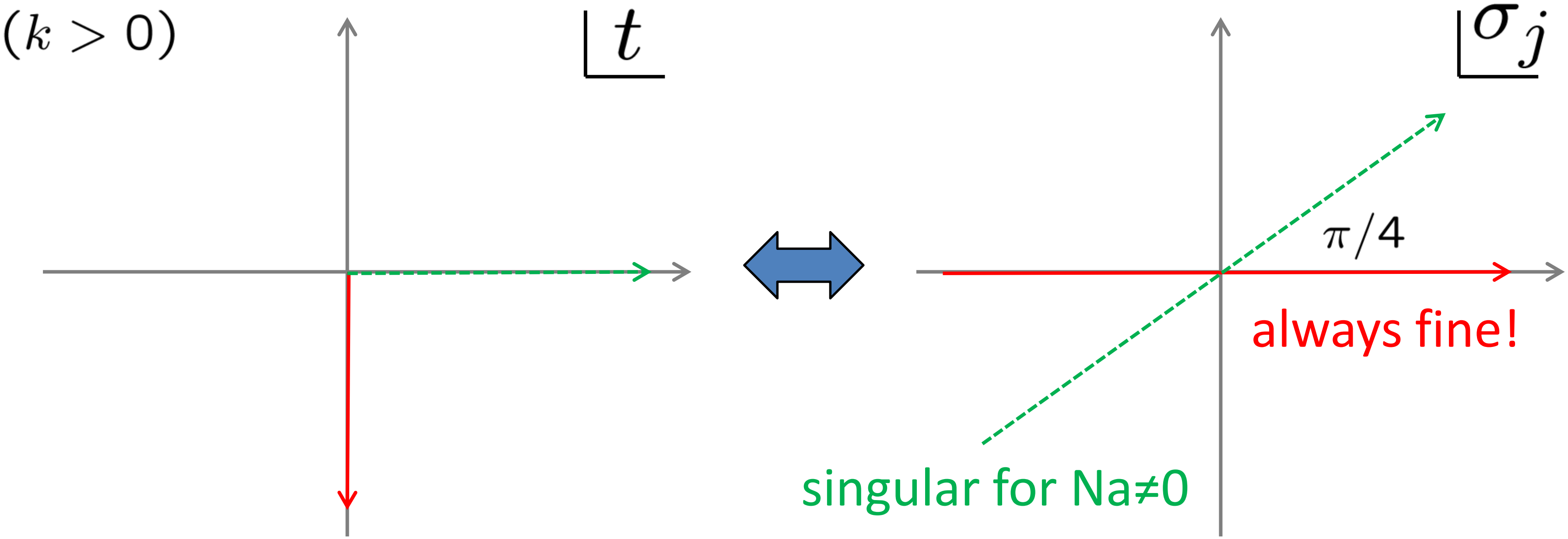
$$Z_{\rm SQCD}(g)=\int_0^{-i{\rm sgn}(k)\infty}dt\ e^{-\frac{t}{g}}\mathcal{B}Z_{\rm SQCD}(t),\quad \mathcal{B}Z_{\rm SQCD}(t)=\int_{S^{N-1}}d^{N-1}\hat{x}\ \tilde{Z}\left(\sigma=\sqrt{i{\rm sgn}(k)t}\hat{x}\right)$$

$$\tilde{Z}(\sigma)=\prod_{j=1}^N\frac{s_1^{\bar{N}_f}\left(\sigma_j+i(1-\bar{\Delta}_f)\right)}{s_1^{N_f}\left(\sigma_j-i(1-\Delta_f)\right)}\frac{\prod_{i<j}4\sinh^2(\pi(\sigma_i-\sigma_j))}{\prod_{i,j}s_1^{N_a}\left(\sigma_i-\sigma_j-i(1-\Delta_a)\right)},\qquad s_1(z)=\prod_{n=1}^\infty\left(\frac{n-iz}{n+iz}\right)^n$$

Sufficient condition for Borel summability
= Absence of singularities along the contour in $\tilde{Z}(\sigma)$



# Analytic property of Borel trans. (Cont'd)




- When we have adjoint matters,
  it would be non-Borel summable along R+

- But it is always Borel summable along θ=-π/2

# General 3d N=2 CS matter theory

Suppose a theory w/ gauge group: $G = G_1 \times \cdots \times G_n$

$$Z_{S^3}(g) = \int_{-\infty}^{\infty} d^{|G|}\sigma \ Z_{\text{cl}}(g;\sigma)\tilde{Z}(\sigma)$$

$$Z_{\text{cl}}(g;a) = \exp\left[\sum_{p=1}^{n} \frac{i \cdot \text{sgn}(k_p)}{g_p} \text{tr}(\sigma^{(p)})^2\right]$$

Taking polar coordinate $\sigma_i^{(p)} = \sqrt{\tau_p}\hat{x}_i^{(p)}$,

$$\boxed{Z_{S^3}(g) = \left[\prod_{p=1}^{n} \int_0^{-i\text{sgn}(k_p)\infty} d^n t \ e^{-\frac{t_p}{g_p}}\right] \mathcal{B}Z_{S^3}(t)}$$

⇒ Borel summable along $\theta_p = -\frac{\text{sgn}(k_p)\pi}{2}$

# Relation to the exact result

We have shown

$$\boxed{Z_{S^3}(g) = \left[\prod_{p=1}^{n} \int_0^{-i\mathrm{sgn}(k_p)\infty} d^n t \; e^{-\frac{t_p}{g_p}}\right] \mathcal{B}Z_{S^3}(t)}$$

Thus,

(Exact result)

||

(Borel resummation along the directions)

# Other observables

- SUSY Wilson loop on $S^3$:

$$W = P\,\mathrm{exp}\Big[\oint ds\,(iA_\mu \dot{x}^\mu + \Phi)\Big]$$

- Bremsstrahrung function in SCFT on $R^3$ [cf. Lewkowycz-Maldacena '13]

- 2-pt. function of U(1) flavor current in SCFT

- 2-pt. function of stress tensor in SCFT

- Partition function on squashed $S^3$～SUSY Renyi entropy

- Partition function on squashed lens space

# Interpretation of singularities (3d)

[M.H., to appear]

# Interpretation of poles (3d ellipsoid case)

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All the poles are explained by complexified SUSY solutions:

$$\begin{cases} 0 = Q\lambda = \left(\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho} - D_\mu\sigma\right)\gamma^\mu\epsilon - iD\epsilon - \frac{i}{f(\vartheta)}\sigma\epsilon \\ 0 = Q\psi = -\gamma^\mu\epsilon D_\mu\phi - \epsilon\sigma\phi - \frac{i\Delta}{f(\vartheta)}\epsilon\phi + i\bar{\epsilon}F, \end{cases}$$

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Especially, $\sigma \in \mathbf{R}$ on the original path integral contour.

If we relax this, we have

$$F_{\mu\nu} = 0, \quad D = -\frac{1}{f(\vartheta)}\sigma, \quad F = 0,$$

$$\sigma = -i\left(mb + nb^{-1} + \frac{b+b^{-1}}{2}\Delta\right), \quad \gamma^\mu\epsilon D_\mu\phi + \epsilon\sigma\phi + \frac{i\Delta}{f(\vartheta)}\epsilon\phi = 0$$

$(m, n \in \mathbf{Z}_{\geq 0},\ b : \text{squashing parameter})$

We can show $\boxed{e^{-S} \sim \mathrm{Res}\left[\mathcal{B}Z_{S_b^3}(t)\right]}$

# Summary & Outlook

# Summary

How to resum perturbative series in SUSY gauge theories

## 4d N=2 theories:

- $^\exists$Singularities only along R- → Borel summable along R+ (for round $S^4$)
- (Exact) = $\sum_{\text{instantons}}$ (Borel resum)

## 3d N=2 CS matter theories:

- Usually non-Borel summable along R+
- Always Borel summable along (half-)imaginary axis
- (Exact result) =(Borel resummation along the direction)
- (Singularities) = (Complexified SUSY solutions)

# Open questions

- Less SUSY case?
- Other observables? [For ‘t Hooft loop, M.H.-D.Yokoyama, in preparation]
- How can we see convergence in planar limit?
- Expansion by other parameters? (such as 1/N)

**4d N=2 theories:**

- Physical interpretation of poles in complex plane?

**3d N=2 CS matter theories:**

- Restriction to 3d N=4 case? [cf. Russo ’12]

Thanks!