SELF-RESURGENCE, CHESHIRE CAT RESURGENCE AND THE BENDER-WU ALGORITHM

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SEMI-CLASSICS AND GROUND STATE ENERGY

\[ Z = e^{-\beta H} = \int D\phi \ e^{-S[\phi]} \]

path-integral over real paths

\[ Z \approx 1 + V \beta ce^{-S(\phi_0)} + \frac{1}{2}(V \beta ce^{-S(\phi_0)})^2 + \ldots \]

\[ Z = e^{-\beta E_0^{NP}} + \ldots \approx e^{-(-\beta V ce^{-S(\phi_0)})} \]

\[ E_0^{NP} \approx -V ce^{-S(\phi_0)} \]
INCONSISTENCY WITH SUSY

\[ H = \sum_{\alpha} \{ Q^\dagger_{\alpha}, Q_{\alpha} \} \geq 0 \]

\[ E_n \geq 0 \]

Perturbative corrections are vanishing, implying that if the classical semi-classics is true, no non-perturbative semi-classical contributions can exist.

But we know non-perturbative contributions can and do exist in SUSY theories (e.g. spontaneous SUSY breaking). So what is going on?
RESOLUTION: PICARD-LEFSCHETZ THEORY

\[ \int \mathcal{D}x \ e^{-S[x]} \to \sum_i n_i \int_{\Gamma_i} \mathcal{D}z \ e^{-S[z]} \]

\[ \Gamma_i : \frac{dz}{ds} = \frac{\delta \bar{S}[\bar{z}]}{\delta \bar{z}(t)} \]

\[ \frac{d}{ds} \text{Im} \ S[z] = 0 \]

\[ \frac{d}{ds} \text{Re} \ S[z] > 0 \]
SEMI-CLASSICS AND GROUND STATE ENERGY

\[ Z = e^{-\beta H} = \int \mathcal{D}\phi \ e^{-S[\phi]} \]

path-integral over real paths

suppressed by

\[ Z \approx 1 + V \beta ce^{-S(\phi_0)} + \frac{1}{2} (V \beta ce^{-S(\phi_0)})^2 + \ldots \]

\[ Z = e^{-\beta E_0^{NP}} + \ldots \approx e^{-(-\beta V c e^{-S(\phi_0)})} \]

\[ E_0^{NP} \approx -V c e^{-\text{Re} S(\phi_0) - i\varphi} \]
A ZETA-DEFORMED DOUBLE WELL AND SINE-GORDON

\[ g\mathcal{L} = \frac{\dot{x}^2}{2} + \frac{W'(x)^2}{2} + \frac{\zeta gW''(x)}{2} \]

with \[ W'(x) = x^2 - 1 \quad \text{or} \quad W'(x) = \sin(x) \]

\[ \zeta = 1 \] is the SUSY limit!

However when \( \zeta \) is any positive integer, these systems are special.

These are related to:

\[ \mathcal{L}_{\text{ferm}} = \frac{1}{2} \dot{x}^2 + \frac{1}{2} W'^2 + \sum_{i=1}^{\zeta} \bar{\psi}_i (\partial_t + W'') \psi_i \]
\[ H = \frac{gp^2}{2} + \frac{1}{2g} (W')^2 + \frac{\zeta}{2} W'' \]

\[ \psi = u(x) e^{-W(x)/g} \quad H\psi = E\psi \rightarrow hu(x) = Eu(x) \]

\[ h = -\frac{g}{2} \frac{d^2}{dx^2} + \sin x \frac{d}{dx} - \frac{\zeta - 1}{2} \cos x \quad \text{For sine-Gordon} \]

\[ h = -\frac{g}{2} \frac{d^2}{dx^2} + \left( \frac{1}{4} - x^2 \right) \frac{d}{dx} + x(\zeta - 1) \quad \text{For DW pot.} \]

(see also the related talk by Hideaki AOYAMA)
The sine-Gordon case

Define

\[ J_+ = e^{-ix}(j - i \frac{d}{dx}) \], \hspace{1cm} J_- = e^{ix}(j + i \frac{d}{dx}) \], \hspace{1cm} J_3 = i \frac{d}{dx} \]

Which obey the SU(2) algebra and for which

\[ J_+ J_- + J_- J_+ + J_3^2 = j(j + 1) \]

\[ h = \frac{g}{2} J_3^2 - \frac{1}{2}(J_+ + J_-) - \frac{\zeta - 1 - 2j}{2} \cos(x) \]

Vanishes if \( \zeta = 1 + 2j \)

But \( j \) is either positive integer or half integer, so \( \zeta = 1, 2, \ldots \) are special
So when ζ are positive integers we can reduce the eigenvalue equation to the equation involving the J-operators which obey the SU(2) algebra.

These J-operators are special because they leave a subspace of functions invariant. In particular plane-waves

\[ u_m = e^{-imx}, \quad m = -j, -j + 1, \ldots, j \]

Transform into each other under the action of J-operators like the states of the SU(2) group in the 2j+1 dimensional representation. In other words \( u_m \) act like spin-j basis states

This decomposes the solutions into solutions in this subspace and the rest. Since this subspace is finite the Schrodinger equation can be solved within it exactly algebraically.
Exactly solvable

Perfectly convergent series expansion in $g$

$$E_0 = -\frac{1}{2} + \frac{5}{8}g - \frac{\sqrt{4 + 2g + g^2}}{2}$$,
$$E_2 = -\frac{1}{2} + \frac{5}{8}g + \frac{\sqrt{4 + 2g + g^2}}{2}$$,
$$E_1 = +\frac{1}{2} + \frac{5}{8}g - \frac{\sqrt{4 - 2g + g^2}}{2}$$,
$$E_3 = +\frac{1}{2} + \frac{5}{8}g - \frac{\sqrt{4 - 2g + g^2}}{2}$$,

e.g. $\zeta=4$
There is always an instanton. What happened with it? Why doesn’t it contribute with $e^{-1/g}$ corrections?
Classically degenerate

Complex bion

Real bion
THE REAL BION CONTRIBUTION

\[ D x(t) \rightarrow D x'(t) \times (\text{quasi-})\text{zeromodes} \]

\[ S_{eff}'(\tau_0) = 0 \]

\[ E_{CB} \approx -C e^{-2S_{inst}} \int_0^\infty d\tau \ e^{-\frac{A}{g} e^{-\tau}} e^{-\zeta \tau} \]

\[ E_{RB} \approx -C e^{-2S_{inst}} \Gamma(\zeta) \left( \frac{g}{A} \right)^\zeta \]
THE COMPLEX BION CONTRIBUTION

\[ E_{CB} \approx -Ce^{-2S_{\text{inst}}} \int_0^\infty d\tau \, e^{A/g} e^{-\tau} e^{-\zeta \tau} \]

Integral saturated at \( \tau=0 \) where all approximations fail

We can continue \( g\to-g \) like in the talk of Zinn-Justin — the so-called Bogomolny—Zinn-Justin (BZJ) prescription. This makes both perturbation theory well defined as well as the above integral saturate at \( \tau \gg \log(1/g) \).
However, there are conceptual problems with this because

1. Analytic continuation produces terms $e^{1/g}$ which are small when $g<0$ but large when $g$ is continued back to $g>0$. These need to be dropped before continuing back. There is no rational for doing this

2. Since the Borel sum of the perturbation theory can be defined for infinitesimal continuation of $g$ to avoid a pole on the positive real axis of the Borel plane, one cannot help wonder why an infinitesimal transformation is not sufficient to cure the instanton—anti-instanton amplitude
**ALTERNATIVE RATIONAL #1**

\[ \mathcal{D} x(t) \rightarrow \mathcal{D} x'(t) \times \text{(quasi-)zeromodes} \]

Treat with Picard-Lefschetz theory

\[ E_{CB} \approx -Ce^{-2S_{inst}} \int d\tau \frac{A}{g} e^{-\tau} e^{-\zeta \tau} \]

The Balitsky-Yung Cycle

\[ S'_{eff}(\tau_0) = 0 \]

\[ E_{CB} \approx -Ce^{-2S_{inst}} \Gamma(\zeta) \left( \frac{g}{A} \right)^{\zeta} e^{\pm i\zeta \pi} \]
The above prescription seems to work beautifully, and gives wonderful agreement with the asymptotic of the perturbation theory, while not causing conceptual problems which I mentioned.

These instanton—anti-instanton configurations are really solutions of the complexified equations of motion, giving the following heuristic picture of the two problems:

\[
E^{NP} \approx -Ce^{-2S_{\text{inst}}} e^{\pm i\zeta\pi}
\]

This played a vital role to show consistency of semi-classics in N=1 SUSY QM (with Bahtash, Schafer, Dunne & Unsal), N=2 SUSY QM (with Bahtash, Poppitz & Unsal), in SUSY Yang-Mills (with Bahtash, Schafer, Unsal), in CP(N-1) models (works of T. Fujimori, S. Kamata, Tatsuhiro Misumi, Muneto Nitta, Norisuke Sakai), in Gross-Witten large N expansion (Buividovich, Dunne, Valgushev), etc.
BUT...

There are still conceptual problems with this

- This logic works for the $\zeta$-deformed systems, but not for $\zeta=0$ (although it does work in this limit, but conceptually this shouldn’t be necessary)

- The extremized action is complex, and since the imaginary part of the action is conserved along both the upward flow (the thimble) and the downward flow, the latter cannot intersect the original integration cycle

- The action which is extremized has $g$-dependent pieces which should be dropped when extremizing the action
Resolution: the critical point at “infinity”, the $\zeta=0$ case revisited

\[ E_{NP} \approx -C e^{-2S_{inst}} \int d(\omega \tau) e^{\frac{A}{g}[e^{-\omega \tau} + e^{-\omega(\beta-\tau)}]} \]

Periodic boundary condition:

$\tau = \tau_0 + \tilde{\tau}$

After subtracting the uncorrelated-instantons term:

$E_{NP}^{II} \approx + \log(g/A) \pm i\pi - \gamma$
\[ E_{NP} \approx -Ce^{-2S_{inst}} \int d(\omega \tau) e^{ \frac{A}{g} \left[ e^{-\omega \tau} + e^{-\omega (\beta-\tau)} \right] } \]

Periodic boundary condition:
\[ \tau = \tau_0 + \tilde{\tau} \]

After subtracting the uncorrelated-instantons term:
\[ E_{NP}^{\text{II}} \approx + \log(g/A) \pm i\pi - \gamma \]
THE BEHAVIOR OF PERTURBATION THEORY AND QUASI-EXACT SOLVABILITY
with Can Kozcas, Yuya Tanizaki, Mithat Unsal

\[ V_{\pm}(x) = \frac{W'(x)^2}{2} \pm \zeta g \frac{W''(x)}{2} \]

\[ E^{NP} \approx -Ce^{-2S_{inst}} e^{\pm i\pi} \]

- The integer \( \zeta \) theories are special
- The perturbation theory is CONVERGENT for the first \( \zeta \) states
- In the case of Double Sine Gordon, a part of a spectrum is exactly solvable (Turbiner 1988), and the exact solution is reproduced by the perturbation theory
- In the case of the Tilted Double Well potential, the perturbation theory, although convergent for lowest \( \zeta \) states, does not give the correct answer, i.e. it is missing the non-perturbative contribution which is unambiguous.
THE INTEGER $\zeta$ THEORY

First $\zeta$ states/bands convergent for $\zeta \in \mathbb{N}^+$ and asymptotic otherwise.
TILTED DOUBLE WELL \( \zeta=3 \)
(see also talk by Hideaki AOYAMA)
TILTED DOUBLE WELL
(see also talk by Hideaki AOYAMA)

\[ \zeta = 20 \]

are complex for sufficiently large \( g \)

\[ \text{Re} E_{PT}(g) \]
The Cheshire-Cat Resurgence

\[ E_{NP} = -\frac{1}{\pi} \left( \frac{g}{8} \right)^{\zeta-1} \Gamma(\zeta) (\cos \theta + e^{\pm i\pi \zeta}) e^{-2S_0/g} \mathcal{P}_{\text{fluc}}(g, \zeta) \]

Due to real saddle

“The Real Bion”

Due to complex saddle

“The Complex Bion”

At \( \zeta=1,2,3,\ldots \) the “complex bion” is unambiguous and therefore, so must the perturbation theory be unambiguous as well. In fact it is convergent, as we already discussed.

Further \( \cos \theta + \exp(i\pi \zeta) = 0 \), there are no non-perturbative contributions. This is consistent with the exact solution of the low-lying spectrum.
SELF-RESURGENCE

\[ \mathcal{P}_{fluc}(\nu, g; \zeta) = \frac{\partial E_{PT}(\nu, g)}{\partial \nu} \exp \left[ 2S_0 \int_0^g dg \; g^{-2} \left[ \frac{\partial E_{PT}(\nu, g)}{\partial \nu} - \left( 1 - \frac{g(2\nu + 1 - \zeta)}{2S_0} \right) \right] \right] \]

• a version of Alvarez-Dünne-Unsal (or A-Dün relation) 2014
• relates perturbation theory around trivial vacuum to the “complex bion” (instanton—anti-instanton) fluctuation
• But a complex bion dictates late orders of perturbation theory
• Hence we have a relation between early terms of PT and late terms of PT
THE SELF-RESURGENCE OR ECHO-RESURGENCE

\[ E_{PT} = a_0 + a_1 g + a_2 g^2 + \ldots + a_{124} g^{124} + a_{125} g^{125} + \ldots \]

Alvarez-Dunne-Unsal relation

Traditional resurgence

\[ b_0 + b_1 g + b_2 g^2 + \ldots + e^{-2S_{inst}} \]
THE SELF-RESURGENCE

\[ P_{\text{fluct}}(\nu, g; \zeta) = (1 + b_1 g + b_2 g + \ldots) \]

Asymptotic growth of PT required by resurgence:

\[ a_n \approx \frac{1}{2\pi} \frac{1}{8^{\zeta-1}} \frac{(n - \zeta)!}{\Gamma(1 - \zeta)(2S_0)^{n - \zeta + 1}} \times (1 + b_1 \frac{2S_0}{n - \zeta} + b_2 \frac{(2S_0)^2}{(n - \zeta)(n - \zeta - 1)} + \ldots) \]
THE **BenderWu** PACKAGE: STUDYING LARGE ORDERS

http://library.wolfram.com/infocenter/MathSource/9479/

**Bender-Wu (1973): anharmonic oscillator**


THE **BenderWu** PACKAGE: STUDYING LARGE ORDERS
SELF-RESURGENCE

\[ a_n^{\text{exact}} / a_n^{\text{Dün}} \]

\( \text{to } g^3 \)

\( \text{to } g^2 \)

\( \text{to } g^1 \)

\( \text{to } g^0 \)
### THE DIFFERENCE EQUATIONS

**With Jie Gu arXiv:1709.00854**  
Also look at the talks by R. Schiappa and Y. Hatsuda

The Bender-Wu for difference equations

<table>
<thead>
<tr>
<th>geometry</th>
<th>Hamiltonian operator</th>
<th>All described genus-1 curves!</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>$\mathcal{H} = e^x + e^{-x/2+p} + e^{-x/2-p}$</td>
<td>Up to 36 orders in $\hbar$ considered by Y. Hatsuda arXiv:1507.04799</td>
</tr>
<tr>
<td>$F_2$</td>
<td>$\mathcal{H} = e^x + m_1 e^{-x} + e^p + e^{-p}$</td>
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<tr>
<td>$F_3$</td>
<td>$\mathcal{H} = e^x + e^{-x/2+p} + e^{-x/2-p} + m_1 e^{-x}$</td>
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<tr>
<td>$F_4$</td>
<td>$\mathcal{H} = e^x + e^{-x+p} + e^{-x-p} + m_1 e^{-x}$</td>
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<tr>
<td>$F_5$</td>
<td>$\mathcal{H} = e^{x/2-p} + e^{x/2+p} + e^{-x} + m_1 e^{-x/2+p} + m_2 e^{-x/2-p}$</td>
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<tr>
<td>$F_6$</td>
<td>$\mathcal{H} = e^x + e^p + e^{-x-p} + m_1 e^{-x} + m_2 e^{-x+p}$</td>
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<tr>
<td>$F_7$</td>
<td>$\mathcal{H} = e^{x/2-p} + e^{x/2+p} + e^{-x} + m_1 e^x + m_2 e^{-x/2+p} + m_3 e^{-x/2-p}$</td>
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<tr>
<td>$F_8$</td>
<td>$\mathcal{H} = e^x + e^p + e^{-x-p} + m_1 e^{x+p} + m_2 e^{-x} + m_3 e^{-x+p}$</td>
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<tr>
<td>$F_9$</td>
<td>$\mathcal{H} = e^{x+p} + e^{-p} + e^{-x} + m_1 e^x + m_2 e^{-p} + m_3 e^p$</td>
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<td>$F_{10}$</td>
<td>$\mathcal{H} = e^x + e^p + e^{-x-p} + m_1 e^{-x} + m_2 e^{-x+p} + m_3 e^{-x+2p}$</td>
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<tr>
<td>$F_{11}$</td>
<td>$\mathcal{H} = e^x + e^p + e^{-x-p} + m_1 e^{-x} + m_2 e^{-x+p} + m_3 e^{-x+2p} + m_4 e^{-p}$</td>
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<tr>
<td>$F_{12}$</td>
<td>$\mathcal{H} = e^{x/2-p} + e^{x/2+p} + e^{-x} + m_1 e^{-x/2+p} + m_2 e^{-x/2-p} + m_3 e^{2p} + m_4 e^{-2p}$</td>
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<tr>
<td>$F_{13}$</td>
<td>$\mathcal{H} = e^x + e^{-x-2p} + e^{-x+2p} + m_1 e^p + m_2 e^{-p} + m_3 e^{-x-p} + m_4 e^{-x+p} + m_5 e^{-x}$</td>
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<tr>
<td>$F_{14}$</td>
<td>$\mathcal{H} = e^{x+p/2} + m_1 e^{-x-p/2} + e^{-x-3p/2} + e^{-x+3p/2} + m_2 e^{-p} + m_3 e^3 + m_4 e^{-x-p/2} + m_5 e^{-x+p/2}$</td>
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<tr>
<td>$F_{15}$</td>
<td>$\mathcal{H} = e^{x/2-p} + e^{x/2+p} + e^{-x} + m_1 e^x + m_2 e^{2p} + m_3 e^{-2p} + m_4 e^{-x/2+p} + m_5 e^{-x/2-p}$</td>
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<tr>
<td>$F_{16}$</td>
<td>$\mathcal{H} = e^{x/2-p} + e^{x/2+p} + e^{-x} + m_1 e^{-x/2+p} + m_2 e^{-x/2-p} + m_3 e^{2p} + m_4 e^{-2p} + m_5 e^{x/2+3p} + m_6 e^{x/2-3p}$</td>
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Roughly number of digits of agreement between numerical eigenvalue the Borel-Pade sum v.s. the order of Borel-Pade
CONCLUSIONS

- The nature of semi-classics is inextricably linked to the complexification of the path-integrals
- The machinery of resurgence guarantees the reality of all real physical observables
- Self-resurgent behavior: early-terms—late-terms relation in the same saddle-sector
- SUSY and integer $\zeta$-deformed theories are special, with resurgent cancellation not needed for certain observables (i.e. energy-levels)
- The resurgence mechanism is not lost, and is restored with slight deformation of such theories
- Potential connection with emergent symmetries in QCD(adj) (Cherman, Unsal…)
- Application of BenderWu to the quantum mirror curves
- Is there self-resurgence in the quantum mirror curves?