SELF-RESURGENCE, CHESHIRE CAT RESURGENCE AND THE BENDERWU ALGORITHM

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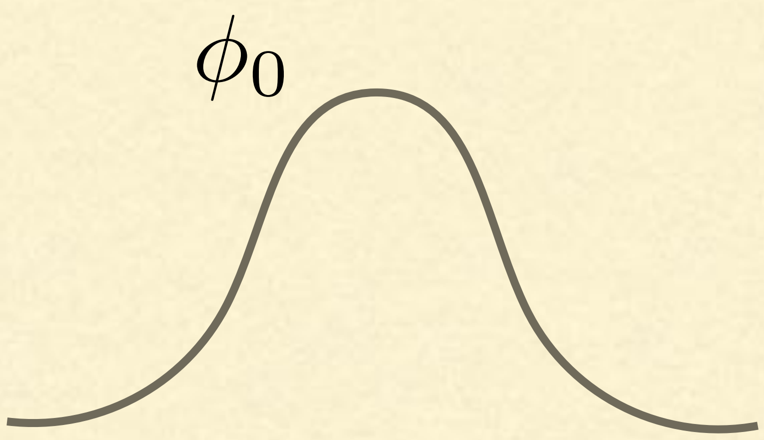
PRL 115 (2015) no.4, 041601/arXiv:1502.06624
JHEP 1511 (2015) 175/arXiv:1507.04063
PRL 116 (2016) no.1, 011601/arXiv:1510.00978
arXiv:1510.03435
arXiv:1709.00854

SEMI-CLASSICS AND GROUND STATE ENERGY

$$Z = e^{-\beta H} = \int \mathcal{D}\phi \, e^{-S[\phi]}$$

path-integral over real paths



Lump of "Φ"

suppressed by $e^{-S[\phi_0]}$

$$Z \approx 1 + V\beta c e^{-S(\phi_0)} + \frac{1}{2}(V\beta c e^{-S(\phi_0)})^2 + \dots$$

$$Z = e^{-\beta E_0^{NP}} + \cdots \approx e^{-\left(-\beta V c e^{-S(\phi_0)}\right)}$$

$$E_0^{NP} \approx -V c e^{-S(\phi_0)}$$

INCONSISTENCY WITH SUSY

$$H = \sum_\alpha \{Q_\alpha^\dagger, Q_\alpha\} \geq 0$$

$$E_n \geq 0$$

Perturbative corrections are vanishing, implying that if the classical semi-classics is true, no non-perturbative semi-classical contributions can exist.

But we know non-perturbative contributions can and do exist in SUSY theories (e.g. spontaneous SUSY breaking). So what is going on?

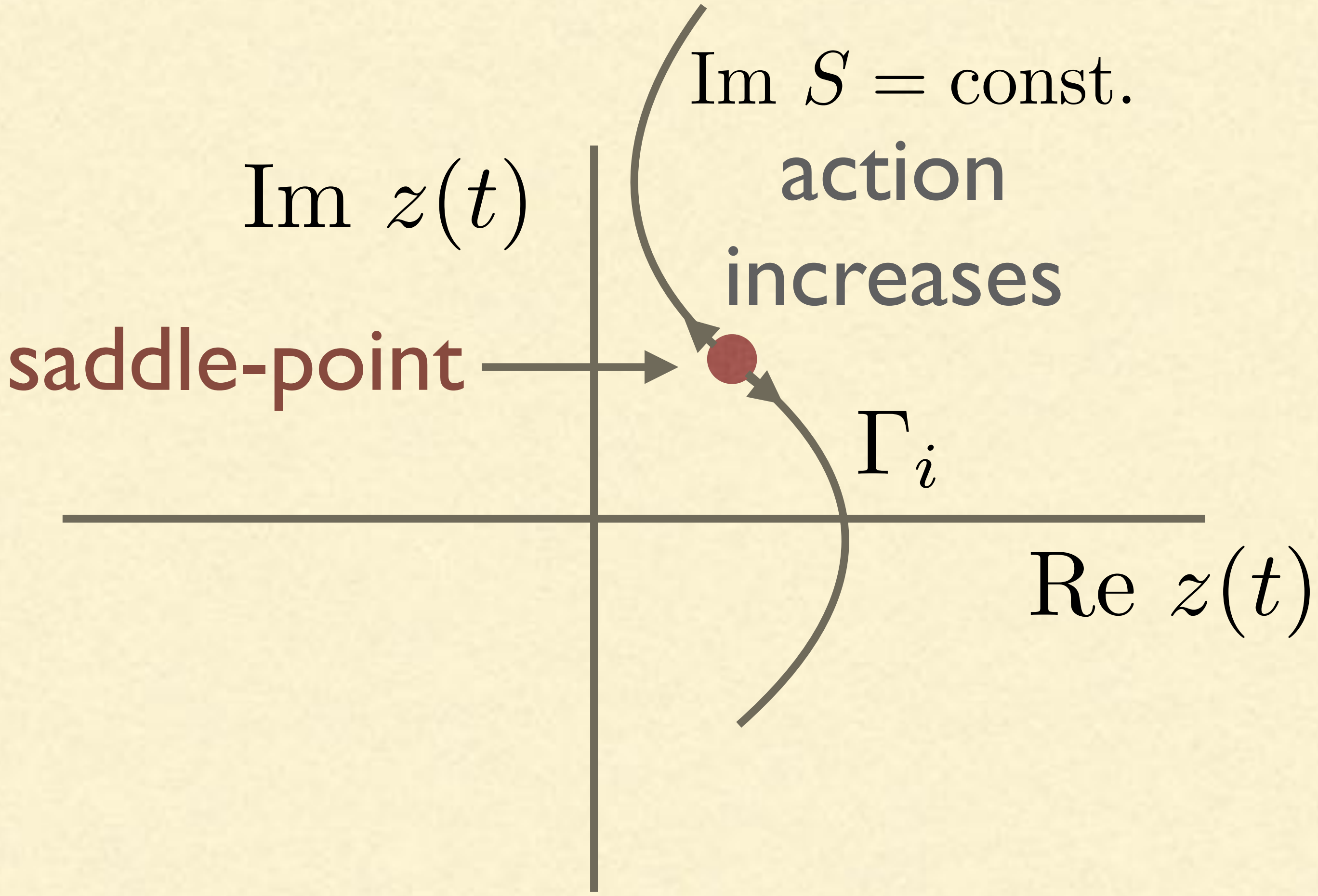
RESOLUTION: PICARD-LEFSCHETZ THEORY

$$\int \mathcal{D}x\, e^{-S[x]} \to \sum_i n_i \int_{\Gamma_i} \mathcal{D}z\, e^{-S[z]}$$

$$\Gamma_i : \quad \frac{dz}{ds} = \frac{\delta \bar{S}[\bar{z}]}{\delta \bar{z}(t)}$$

$$\frac{d}{ds}\mathrm{Im}\ S[z] = 0$$

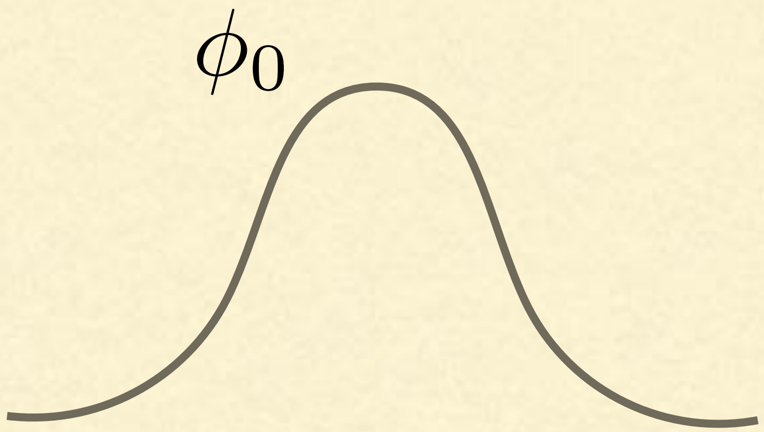
$$\frac{d}{ds}\mathrm{Re}\ S[z] > 0$$



SEMI-CLASSICS AND GROUND STATE ENERGY

$$Z = e^{-\beta H} = \int \mathcal{D}\phi \, e^{-S[\phi]}$$

path-integral over real paths



Lump of "Φ"

suppressed by $e^{-\mathrm{Re}\ S(\phi_0) - i\varphi}$

$$Z \approx 1 + V\beta c e^{-S(\phi_0)} + \frac{1}{2}(V\beta c e^{-S(\phi_0)})^2 + \ldots$$

$$Z = e^{-\beta E_0^{NP}} + \cdots \approx e^{-\left(-\beta V c e^{-S(\phi_0)}\right)}$$

$$E_0^{NP} \approx -V c e^{-\mathrm{Re}\ S(\phi_0) - i\varphi}$$

A ZETA-DEFORMED DOUBLE WELL AND SINE-GORDON

$$g\mathcal{L} = \frac{\dot{x}^2}{2} + \frac{W'(x)^2}{2} + \frac{\zeta g W''(x)}{2}$$

with $\quad W'(x) = x^2 - 1$ or $W'(x) = \sin(x)$

ζ=1 is the SUSY limit!

However when ζ is any positive integer, these systems are special

These are related to: $\quad \mathcal{L}_{ferm} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}W'^2 + \sum_{i=1}^{\zeta} \bar{\psi}_i(\partial_t + W'')\psi_i$

$$H = \frac{gp^2}{2} + \frac{1}{2g}(W')^2 + \frac{\zeta}{2}W''$$

$$\psi = u(x)e^{-W(x)/g} \qquad H\psi = E\psi \to hu(x) = Eu(x)$$

$$h = -\frac{g}{2}\frac{d^2}{dx^2} + \sin x \frac{d}{dx} - \frac{\zeta - 1}{2}\cos x$$ For sine-Gordon

$$h = -\frac{g}{2}\frac{d^2}{dx^2} + (\frac{1}{4} - x^2)\frac{d}{dx} + x(\zeta - 1)$$ For DW pot.

(see also the related talk by Hideaki AOYAMA)

The sine-Gordon case

Define

$$J_+ = e^{-ix}(j - i\frac{d}{dx})\ ,\ \ J_- = e^{ix}(j + i\frac{d}{dx})\ ,\ \ J_3 = i\frac{d}{dx}$$

Which obey the SU(2) algebra and for which

$$J_+J_- + J_-J_+ + J_3^2 = j(j+1)$$

$$h = \frac{g}{2}J_3^2 - \frac{1}{2}(J_+ + J_-) - \frac{\zeta - 1 - 2j}{2}\cos(x)$$

Vanishes if ζ=1+2j

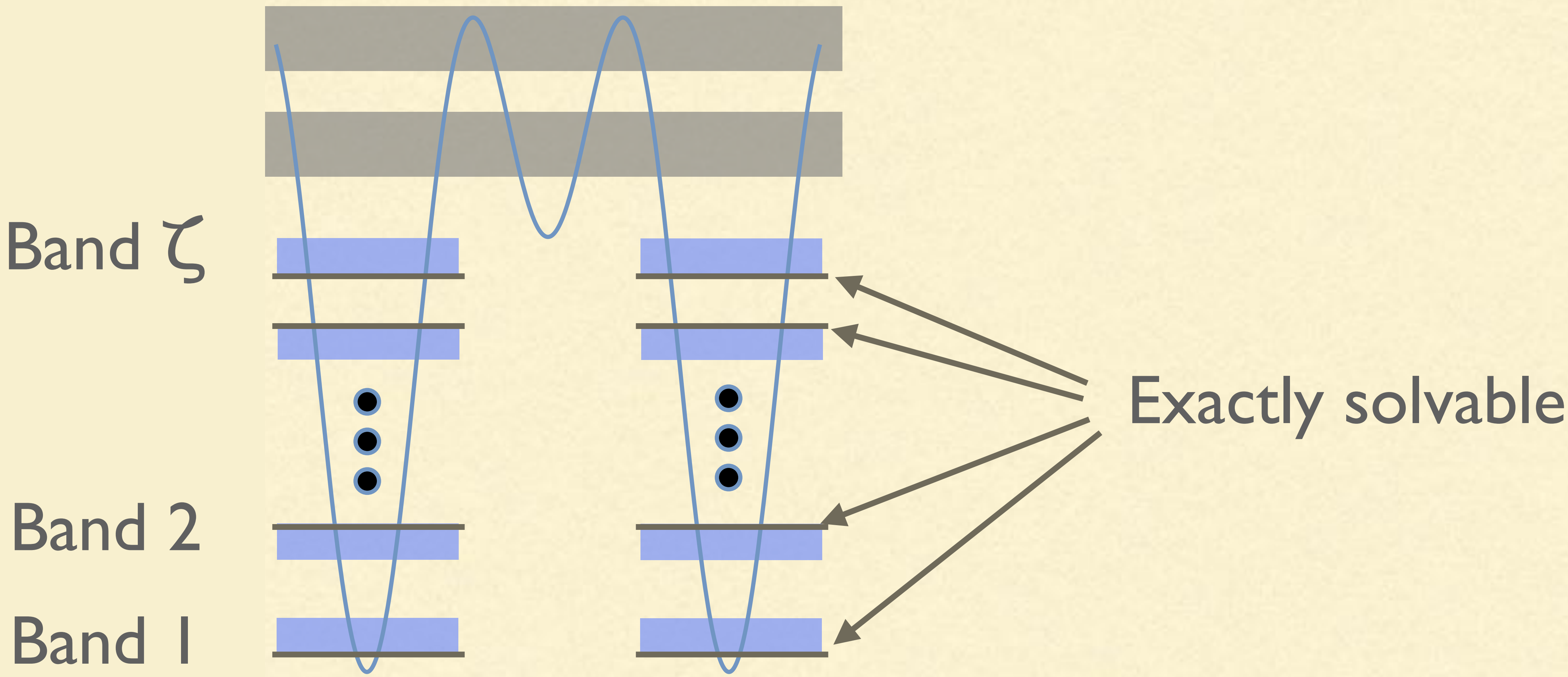
But j is either positive integer or half integer,
so ζ=1,2,… are special

- So when ζ are positive integers we can reduce the eigenvalue equation to the equation involving the J-operators which obey the SU(2) algebra.
- These J-operators are special because they leave a subspace of functions invariant. In particular plane-waves

$$u_m = e^{-imx} \ , \qquad m = -j, -j+1, \ldots, j$$

Transform into each other under the action of J-operators like the states of the SU(2) group in the 2j+1 dimensional representation. In other words u_m act like spin-j basis states

This decomposes the solutions into solutions in this subspace and the rest. Since this subspace is finite the Schrodinger equation can be solved within it exactly algebraically.

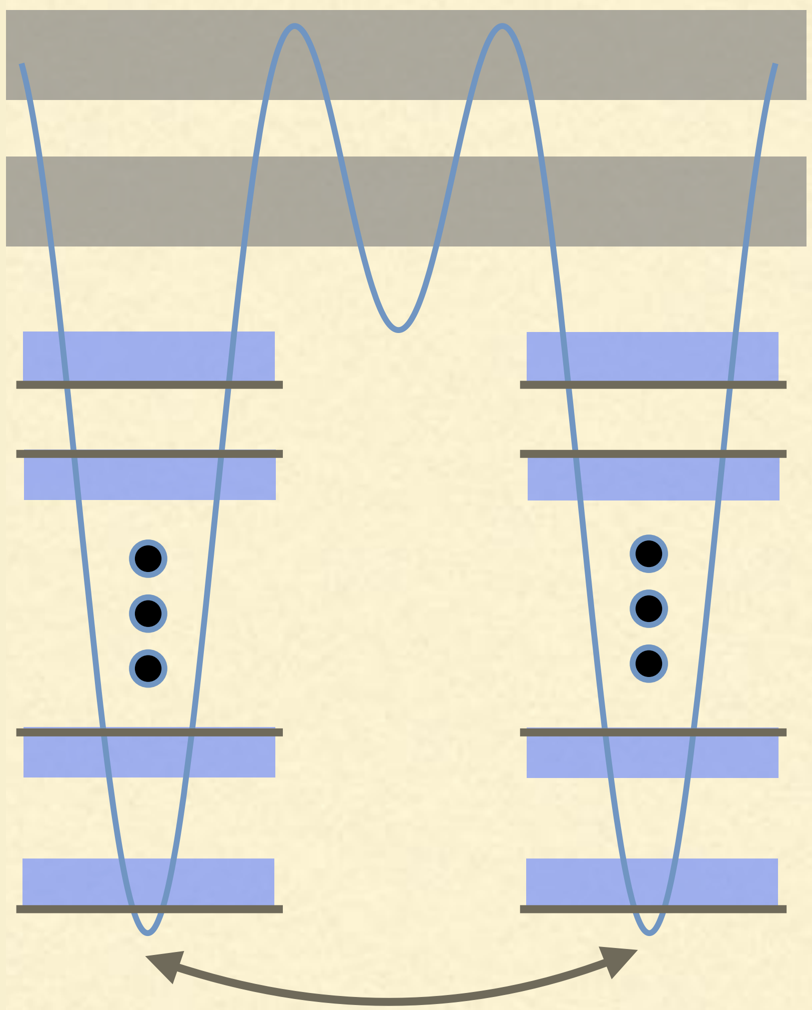


Perfectly convergent series expansion in g

e.g. ζ=4

$$E_0 = -\frac{1}{2} + \frac{5}{8}g - \frac{\sqrt{4+2g+g^2}}{2}\,, \qquad E_1 = +\frac{1}{2} + \frac{5}{8}g - \frac{\sqrt{4-2g+g^2}}{2}\,,$$

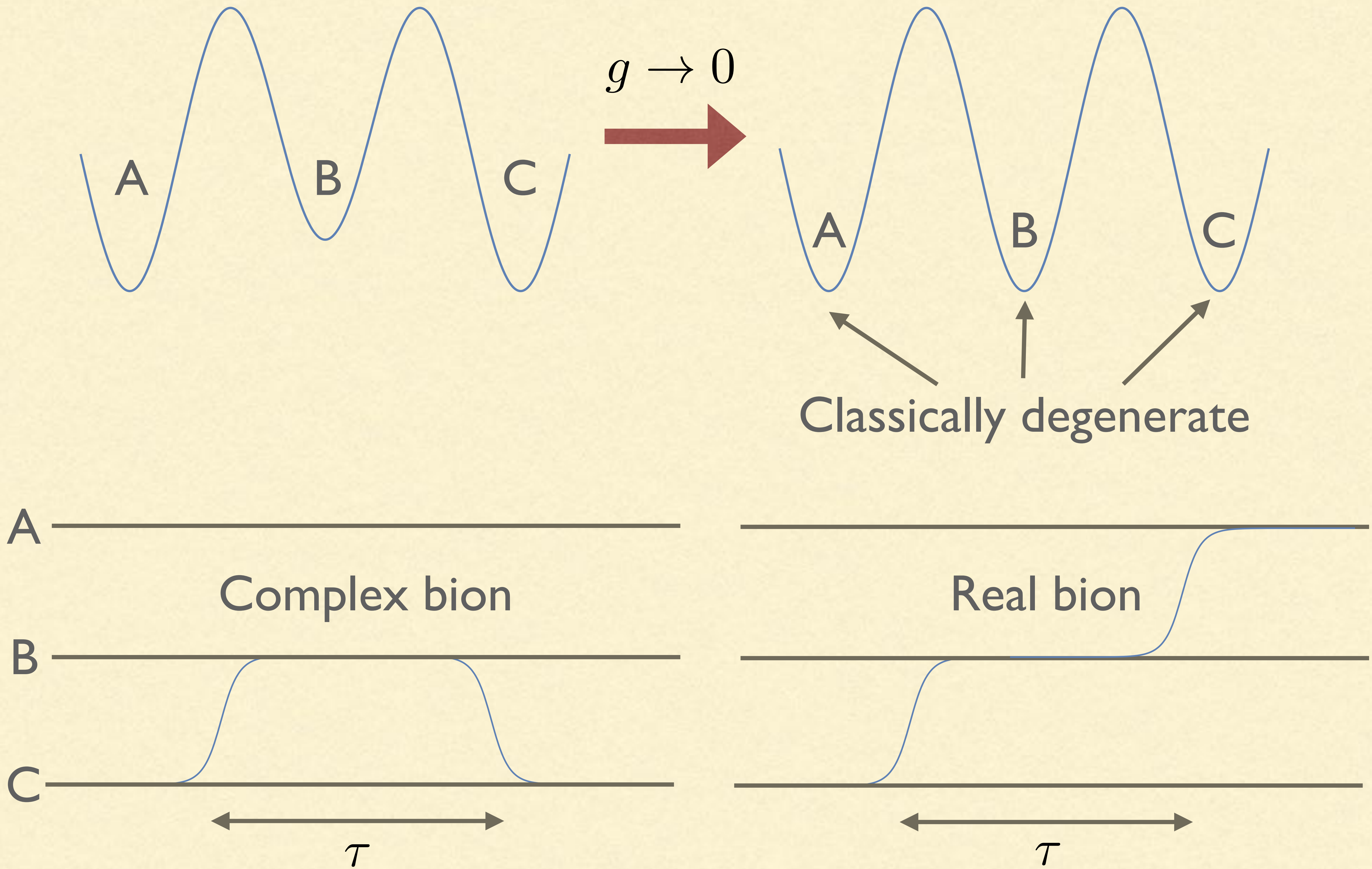
$$E_2 = -\frac{1}{2} + \frac{5}{8}g + \frac{\sqrt{4+2g+g^2}}{2}\,, \qquad E_3 = +\frac{1}{2} + \frac{5}{8}g - \frac{\sqrt{4-2g+g^2}}{2}\,,$$



There is always an instanton

What happened with it?

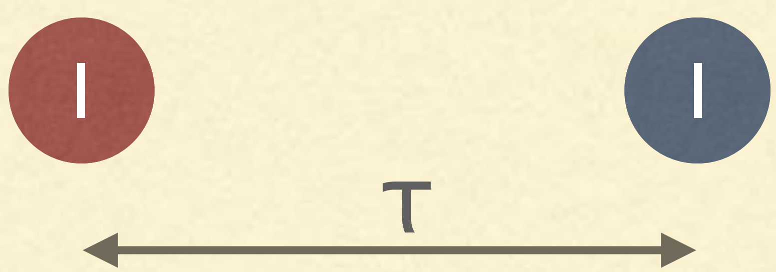
Why doesn't it contribute with $e^{-1/g}$ corrections?



A
B
C
$g \to 0$
A
B
C
Classically degenerate
A
B
C
Complex bion
τ
Real bion
τ

THE REAL BION CONTRIBUTION

$$\mathcal{D}x(t) \to \mathcal{D}x'(t) \times \text{(quasi-)zeromodes}$$

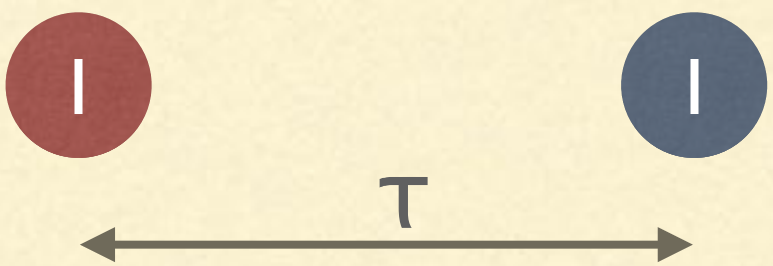


$$S'_{eff}(\tau_0) = 0$$

$$E_{CB} \approx -Ce^{-2S_{inst}} \int_0^\infty d\tau\; e^{-\frac{A}{g}e^{-\tau}} e^{-\zeta\tau}$$

$$E_{RB} \approx -Ce^{-2S_{inst}} \Gamma(\zeta) \left(\frac{g}{A}\right)^{\zeta}$$

THE COMPLEX BION CONTRIBUTION



$$E_{CB} \approx -C e^{-2S_{inst}} \int_0^\infty d\tau \; e^{\frac{A}{g} e^{-\tau}} e^{-\zeta\tau}$$

Integral saturated at τ=0 where all approximations fail

We can continue g->-g like in the talk of Zinn-Justin — the so-called Bogomolny—Zinn-Justin (BZJ) perscription. This makes both perturbation theory well defined as well as the above integral saturate at τ>>log(1/g).

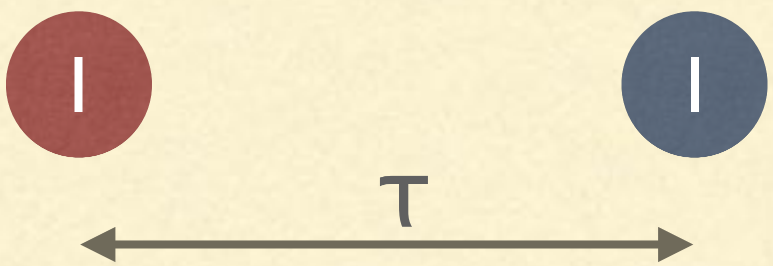
However, there are conceptual problems with this because

1. Analytic continuation produces terms $e^{1/g}$ which are small when g<0 but large when g is continued back to g>0. These need to be dropped before continuing back. There is no rational for doing this
2. Since the Borel sum of the perturbation theory can be defined for infinitesimal continuation of g to avoid a pole on the positive real axis of the Borel plane, one cannot help wonder why an infinitesimal transformation is not sufficient to cure the instanton—anti-instanton amplitude

ALTERNATIVE RATIONAL #1

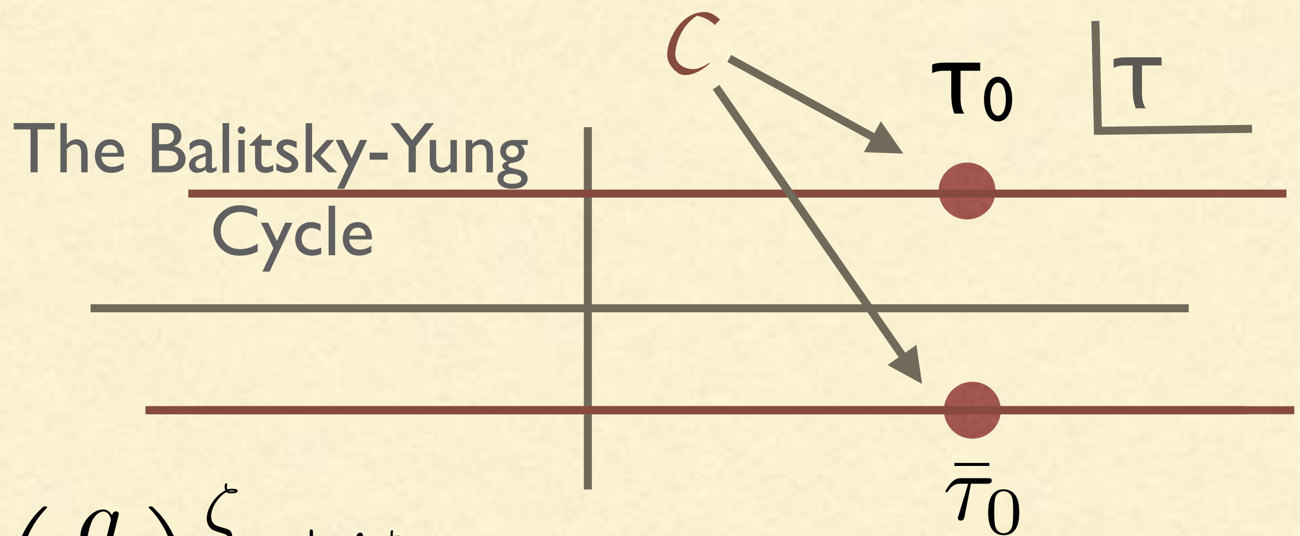
$$\mathcal{D}x(t) \to \mathcal{D}x'(t) \times \text{(quasi-)zeromodes}$$

Treat with Picard-Lefschetz theory



$$E_{CB} \approx -Ce^{-2S_{inst}} \int_{C} d\tau\; e^{\frac{A}{g} e^{-\tau}} e^{-\zeta\tau}$$

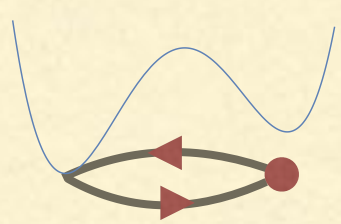
$$S'_{eff}(\tau_0) = 0$$



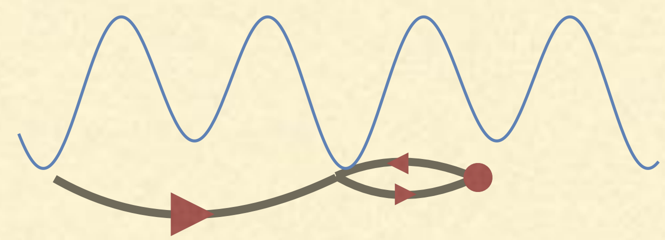
$$E_{CB} \approx -Ce^{-2S_{inst}} \Gamma(\zeta) \left(\frac{g}{A}\right)^{\zeta} e^{\pm i\zeta\pi}$$

THIS SEEMS TO BE CORRECT, BECAUSE IT WORKS

- The above prescription seems to work beautifully, and gives wonderful agreement with the asymptotic of the perturbation theory, while not causing conceptual problems which I mentioned.
- These instanton—anti-instanton configurations are really solutions of the complexified equations of motion, giving the following heuristic picture of the two problems



$$E^{NP} \approx -Ce^{-2S_{inst}}e^{\pm i\zeta\pi}$$



$$E^{NP} \approx -Ce^{-2S_{inst}}(\cos\theta + e^{\pm i\zeta\pi})$$

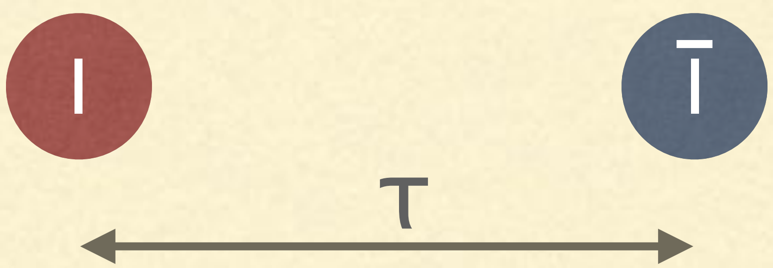
- This played a vital role to show consistency of semi-classics in N=1 SUSY QM (with Bahtash, Schafer, Dunne & Unsal), N=2 SUSY QM (with Bahtash, Poppitz & Unsal), in SUSY Yang-Mills (with Bahtash, Schafer, Unsal), in CP(N-1) models (works of T. Fujimori, S. Kamata, Tatsuhiro Misumi, Muneto Nitta, Norisuke Sakai), in Gross-Witten large N expansion (Buividovich, Dunne, Valgushev), etc.

BUT…

There are still conceptual problems with this

- This logic works for the ζ-deformed systems, but not for $\zeta=0$ (although it does work in this limit, but conceptually this shouldn't be necessary)
- The extremized action is complex, and since the imaginary part of the action is conserved along both the upward flow (the thimble) and the downward flow, the latter cannot intersect the original integration cycle
- The action which is extremized has g-dependent pieces which should be dropped when extremizing the action

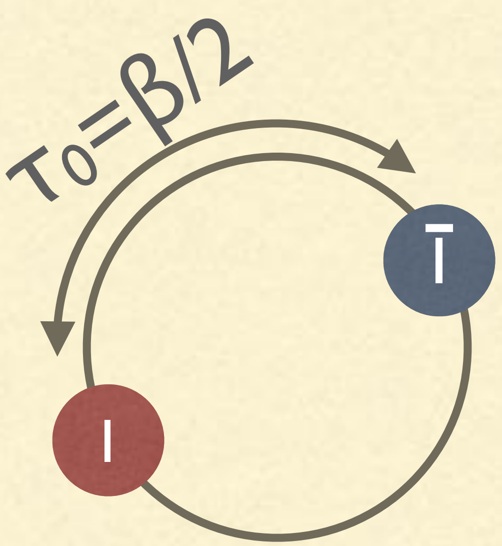
Resolution: the critical point at "infinity", the ζ=0 case revisited

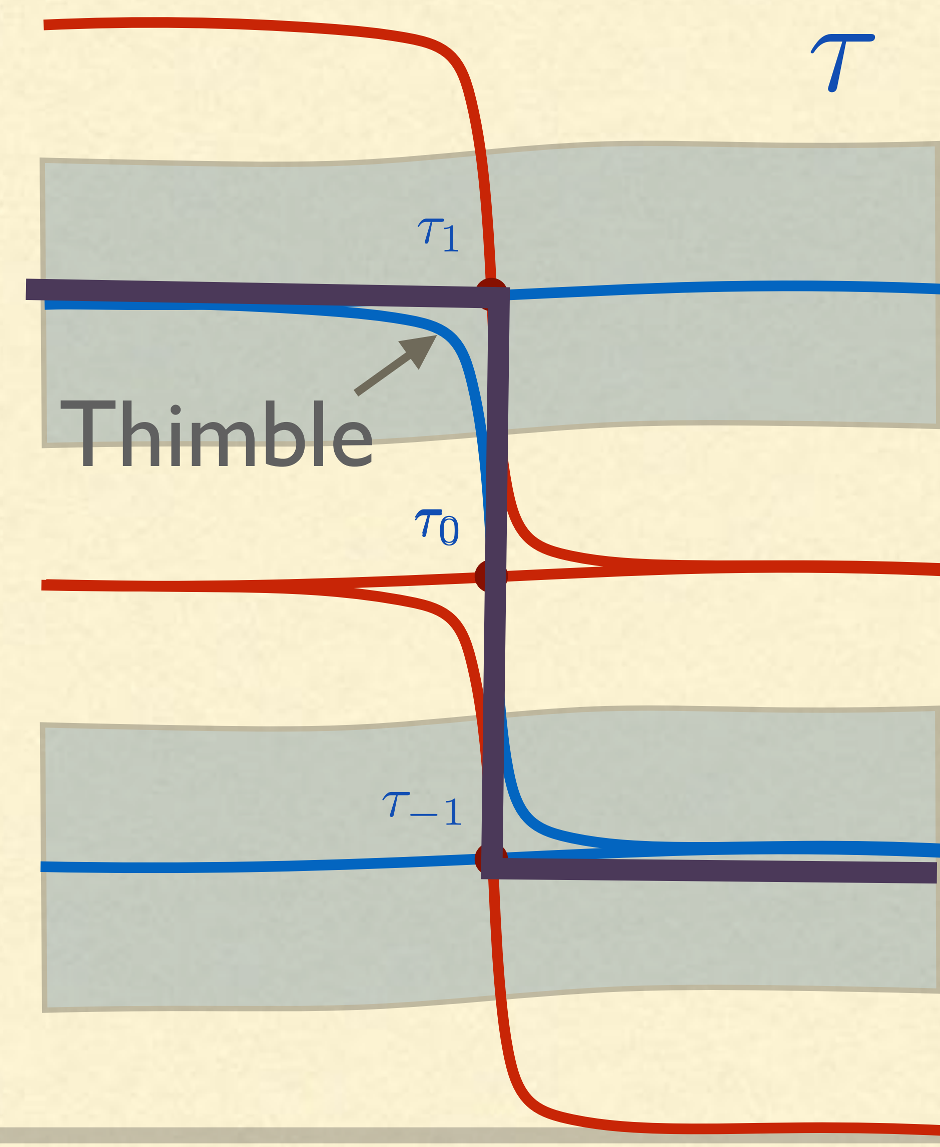


$$E_{NP} \approx -Ce^{-2S_{inst}} \int d(\omega\tau) e^{\frac{A}{g}\left[e^{-\omega\tau} + e^{-\omega(\beta-\tau)}\right]}$$

Periodic boundary condition:

$$\tau = \tau_0 + \tilde{\tau}$$

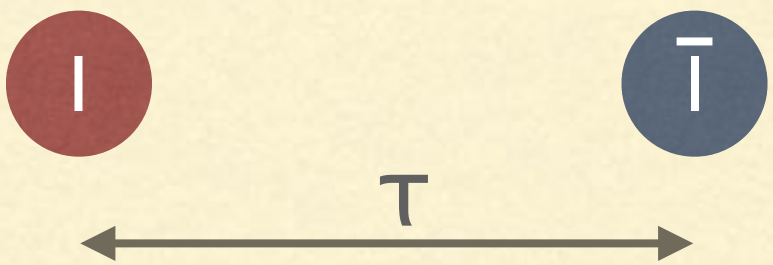




After subtracting the uncorrelated-instantons term:

$$E_{NP}^{I\bar{I}} \approx + \log(g/A) \pm i\pi - \gamma$$

COMMENDT ON THE BOGOMOLNY-ZINN-JUSTIN PERSCRIPTION



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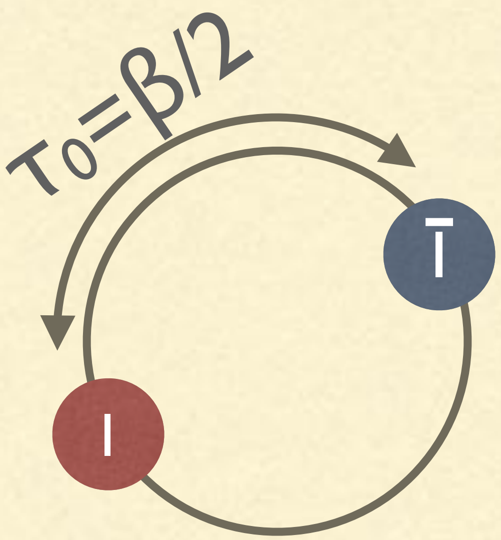
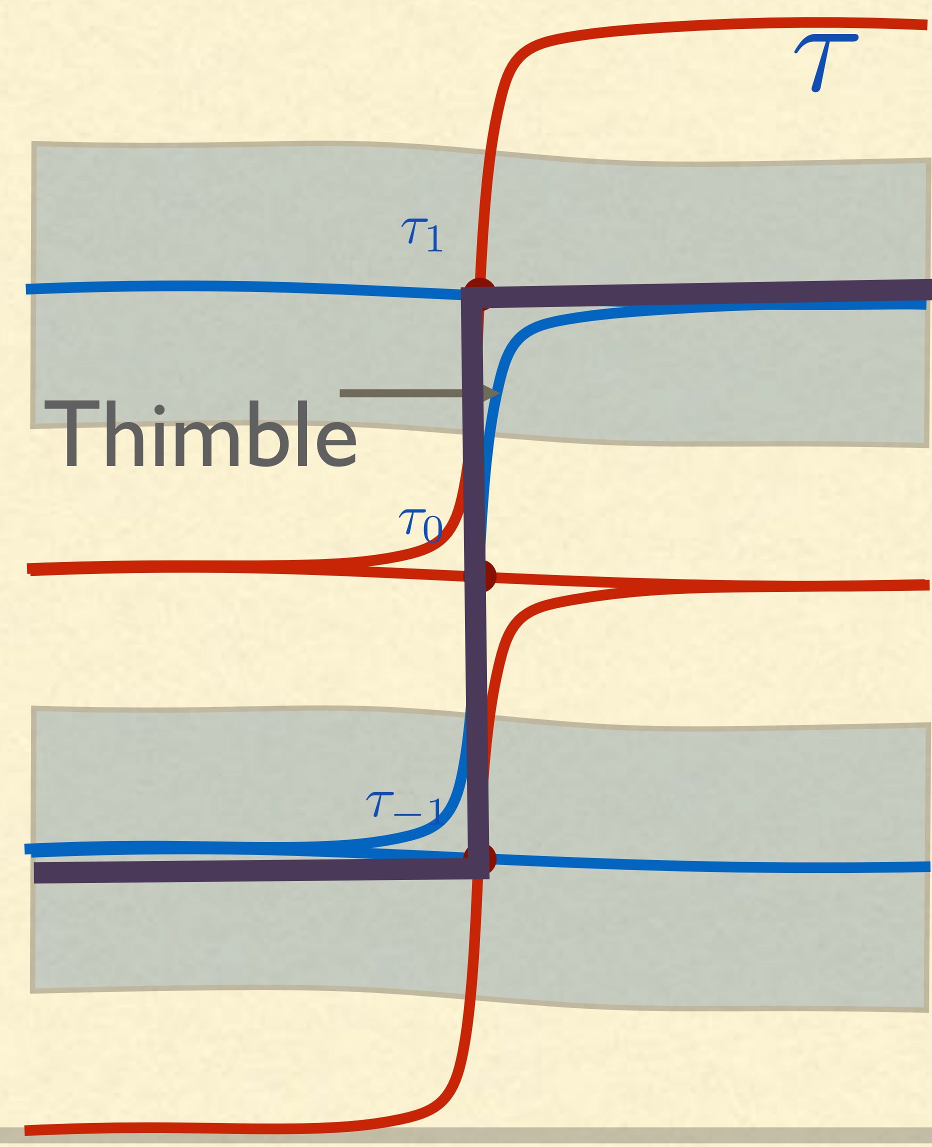




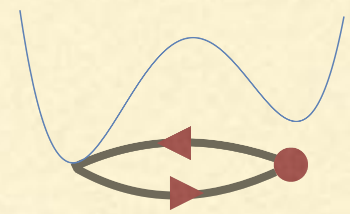
After subtracting the uncorrelated-instantons term:

$$E_{NP}^{I\bar{I}} \approx +\log(g/A) \pm i\pi - \gamma$$

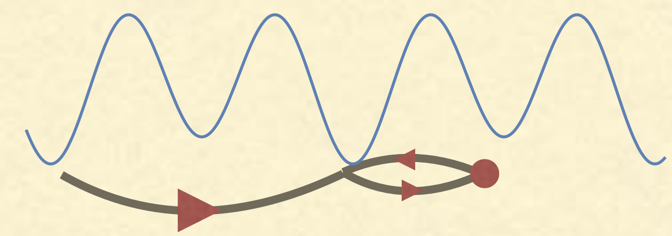
THE BEHAVIOR OF PERTURBATION THEORY AND QUASI-EXACT SOLVABILITY

with Can Kozcas, Yuya Tanizaki, Mithat Unsal

$$V_{\pm}(x) = \frac{W'^2}{2} \pm \zeta g \frac{W''(x)}{2}$$



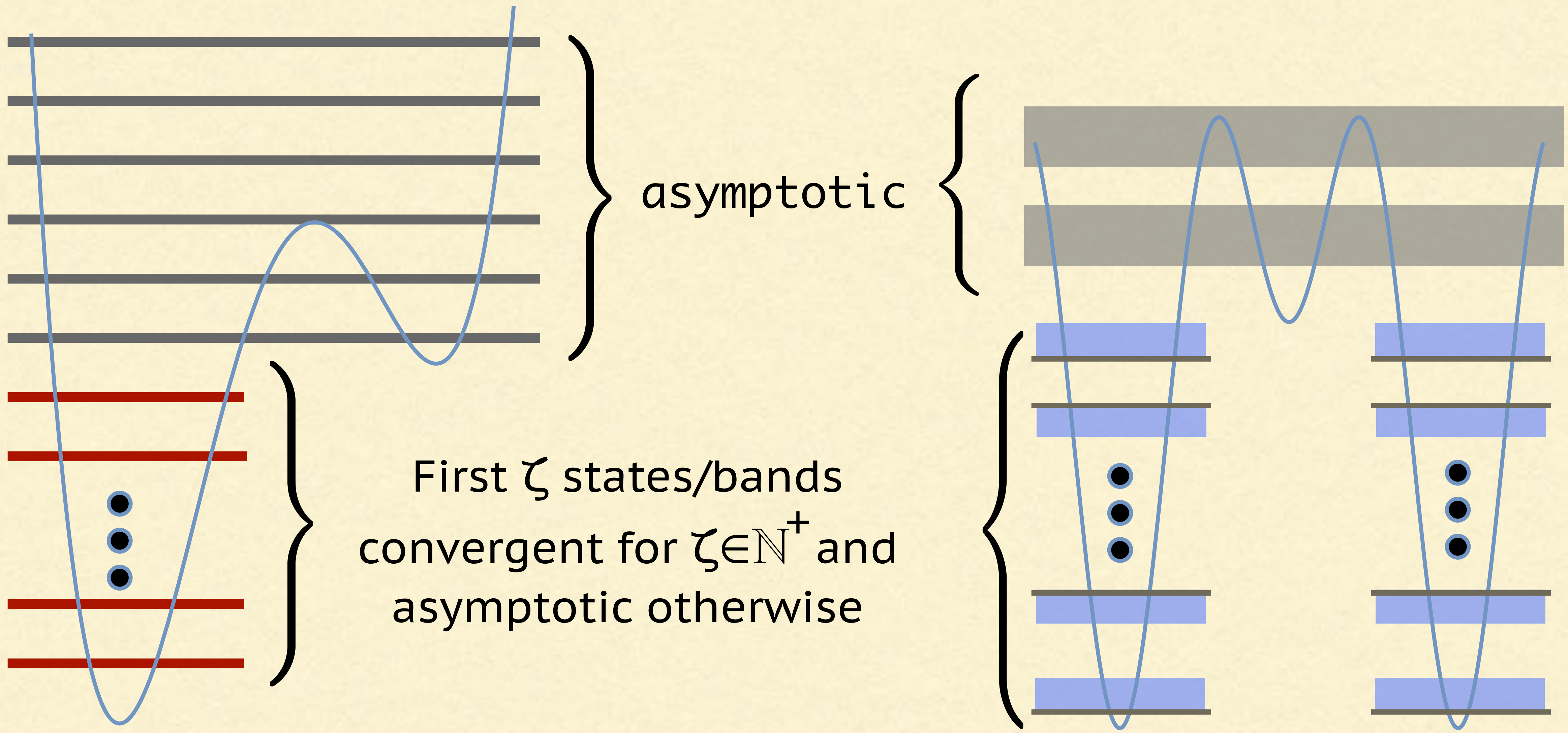
$$E^{NP} \approx -Ce^{-2S_{inst}} e^{\pm i\zeta\pi}$$



$$E^{NP} \approx -Ce^{-2S_{inst}} (\cos\theta + e^{\pm i\zeta\pi})$$

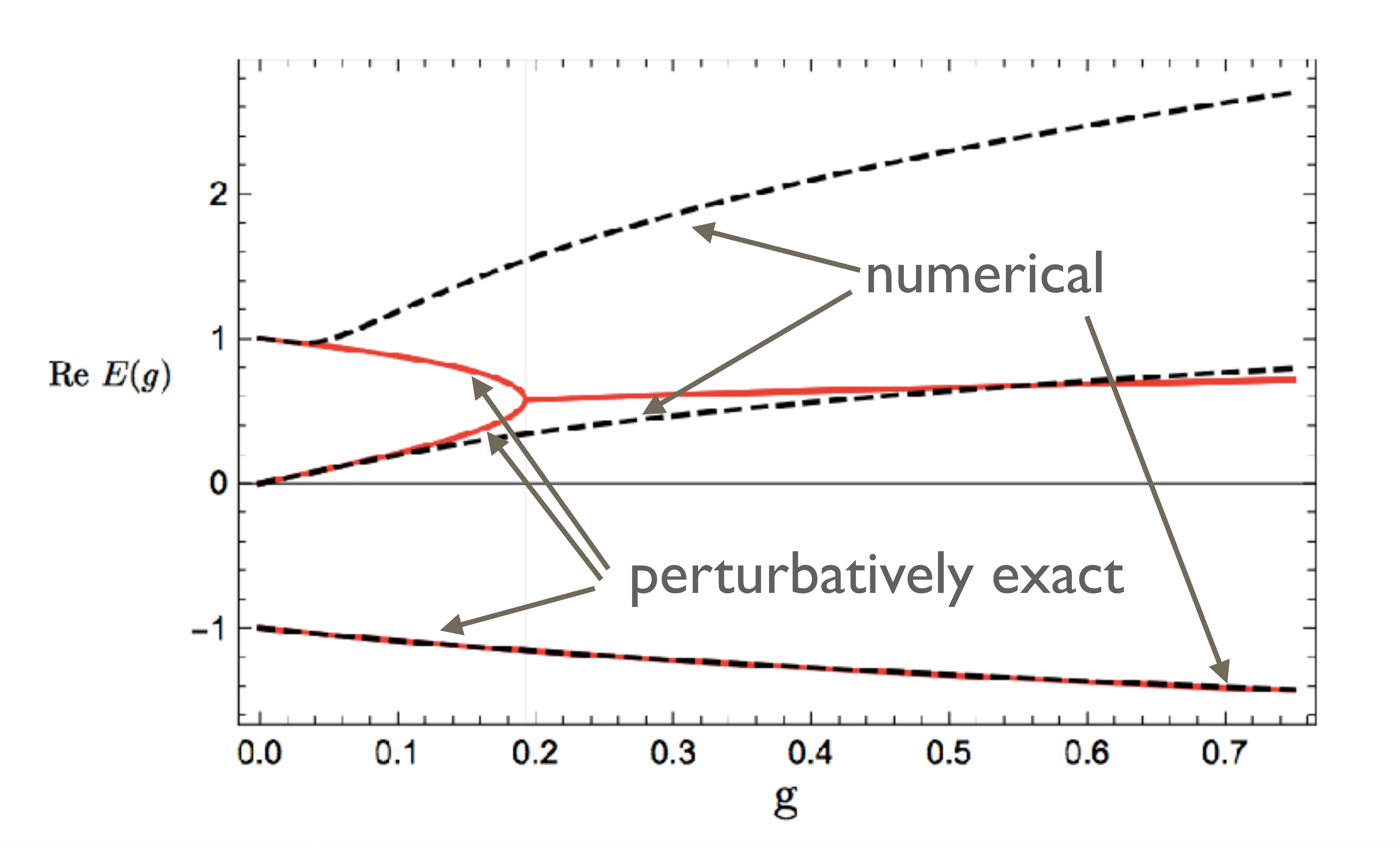
- The integer ζ theories are special
- The perturbation theory is CONVERGENT for the first ζ states
- In the case of Double Sine Gordon, a part of a spectrum is exactly solvable (Turbiner 1988), and the exact solution is reproduced by the perturbation theory
- In the case of the Tilted Double Well potential, the perturbation theory, although convergent for lowest ζ states, does not give the correct answer, i.e. it is missing the non-perturbative contribution which is unambiguous.

THE INTEGER ζ THEORY



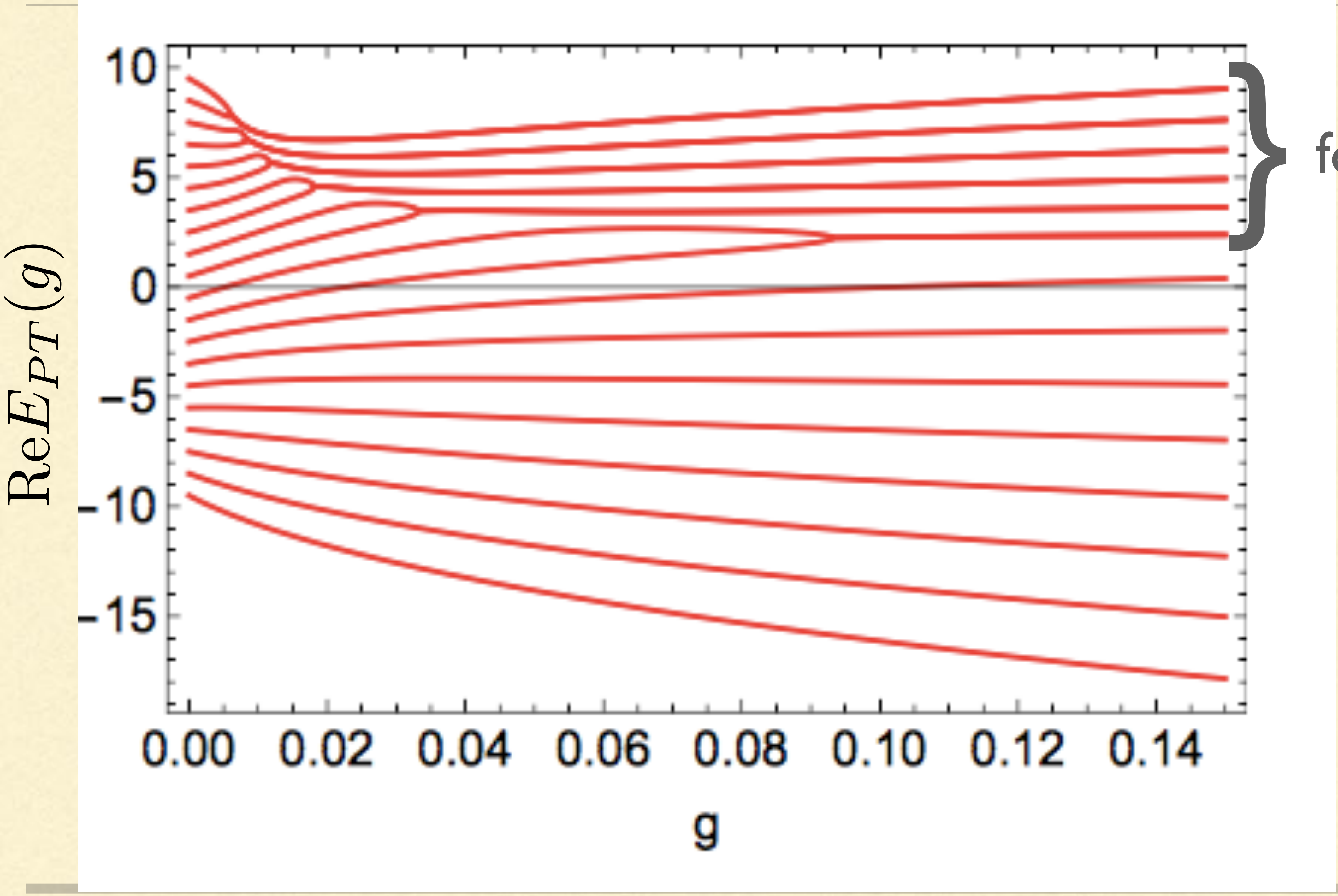
TILTED DOUBLE WELL $\zeta=3$

(see also talk by Hideaki AOYAMA)



TILTED DOUBLE WELL ζ=20

(see also talk by Hideaki AOYAMA)



are complex for sufficiently large g

THE CHASHIRE-CAT RESURGENCE

$$E_{NP} = -\frac{1}{\pi}\left(\frac{g}{8}\right)^{\zeta-1}\Gamma(\zeta)(\cos\theta + e^{\pm i\pi\zeta})e^{-2S_0/g}\mathcal{P}_{\text{fluc}}(g,\zeta)$$

Due to real saddle
"The Real Bion"

Due to complex saddle
"The Complex Bion"

At ζ=1,2,3,… the "complex bion" is unambiguous and therefore, so must the perturbation theory be unambiguous as well. In fact it is convergent, as we already discussed.

Further cosθ+exp(iπζ)=0, there are no non-perturbative contributions. This is consistent with the exact solution of the low-lying spectrum.

SELF-RESURGENCE

$$\mathcal{P}_{fluc}(\nu, g; \zeta) = \frac{\partial E_{PT}(\nu, g)}{\partial \nu} \exp\left[2S_0 \int_0^g dg\; g^{-2} \left[\frac{\partial E_{PT}(\nu, g)}{\partial \nu} - \left(1 - \frac{g(2\nu + 1 - \zeta))}{2S_0}\right)\right]\right]$$

- a version of Alvarez-Dünne-Unsal (or A-Dün relation) 2014
- relates petrubation theory around trivial vacuum to the "complex bion" (instanton—anti-instanton) fluctuation
- But a complex bion dictates late orders of perturbation theory
- Hence we have a relation between early terms of PT and late terms of PT

THE SELF-RESURGENCE OR ECHO-RESURGENCE

$$b_0 + b_1 g + b_2 g^2 + \dots$$

$$e^{-2S_{inst}}$$

Alvarez-Dunne-Unsal relation

Traditional resurgence

$$E_{PT} = a_0 + a_1 g + a_2 g^2 + \dots \qquad \dots + a_{124} g^{124} + a_{125} g^{125} + \dots$$

THE SELF-RESURGENCE

$$\mathcal{P}_{fluct}(\nu, g; \zeta) = (1 + b_1 g + b_2 g + \dots)$$

Asymptotic growth of PT required by resurgence:

$$a_n \approx \frac{1}{2\pi} \frac{1}{8^{\zeta-1}} \frac{(n-\zeta)!}{\Gamma(1-\zeta)(2S_0)^{n-\zeta+1}}$$
$$\times \left(1 + b_1 \frac{2S_0}{n-\zeta} + b_2 \frac{(2S_0)^2}{(n-\zeta)(n-\zeta-1)} + \dots\right)$$

THE **BenderWu** PACKAGE: STUDYING LARGE ORDERS

http://library.wolfram.com/infocenter/MathSource/9479/

Bender-Wu (1973): anharmonic oscillator

. J. J. M. Verbaarschot, P. C. West, and T. T. Wu, *Large order behavior of the supersymmetric anharmonic oscillator*, *Phys. Rev.* D42 (1990) 1276–1284.

. C. M. Bender and G. V. Dunne, *Large order perturbation theory for a nonHermitian PT symmetric Hamiltonian*, *J. Math. Phys.* 40 (1999) 4616–4621, [quant-ph/9812039].

. C. M. Bender, G. V. Dunne, P. N. Meisinger, and M. Simsek, *Quantum complex Henon-Heiles potentials*, *Phys. Lett.* A281 (2001) 311–316, [quant-ph/0101095].

. M. Stone and J. Reeve, *Late Terms in the Asymptotic Expansion for the Energy Levels of a Periodic Potential*, *Phys. Rev.* D18 (1978) 4746.

THE **BenderWu** PACKAGE: STUDYING LARGE ORDERS

Untitled-6

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In[1153]:= BenderWu[x^2/2 + x^4, x, 0, 100, Output → "Energy"]
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Out[1153]=

$$\Big\{\frac{1}{2},\ \frac{3}{4},\ -\frac{21}{8},\ \frac{333}{16},\ -\frac{30\,885}{128},\ \frac{916\,731}{256},\ -\frac{65\,518\,401}{1024},\ \frac{2\,723\,294\,673}{2048},$$

$$-\frac{1\,030\,495\,099\,053}{32\,768},\ \frac{54\,626\,982\,511\,455}{65\,536},\ -\frac{6\,417\,007\,431\,590\,595}{262\,144},\ \frac{413\,837\,985\,580\,636\,167}{524\,288},$$

$$-\frac{116\,344\,863\,173\,284\,543\,665}{4\,194\,304},\ \frac{8\,855\,406\,003\,085\,477\,228\,503}{8\,388\,608},\ -\frac{1\,451\,836\,748\,576\,538\,293\,163\,705}{33\,554\,432},$$

$$\frac{127\,561\,682\,802\,713\,500\,067\,360\,049}{67\,108\,864},\ -\frac{191\,385\,927\,852\,560\,927\,887\,828\,084\,605}{2\,147\,483\,648},$$

$$\frac{19\,080\,610\,783\,320\,698\,048\,964\,226\,601\,511}{4\,294\,967\,296},\ -\frac{4\,031\,194\,983\,593\,309\,788\,607\,032\,686\,292\,335}{17\,179\,869\,184},$$

$$\frac{449\,820\,604\,540\,765\,836\,160\,529\,697\,491\,458\,635}{34\,359\,738\,368},\ -\frac{211\,491\,057\,584\,560\,795\,425\,148\,309\,663\,914\,344\,715}{274\,877\,906\,944},$$

$$\frac{26\,120\,222\,383\,762\,781\,149\,654\,970\,754\,934\,417\,034\,805}{549\,755\,813\,888},$$

$$-\frac{6\,763\,740\,228\,794\,489\,740\,596\,394\,161\,822\,714\,805\,728\,135}{2\,199\,023\,255\,552},$$

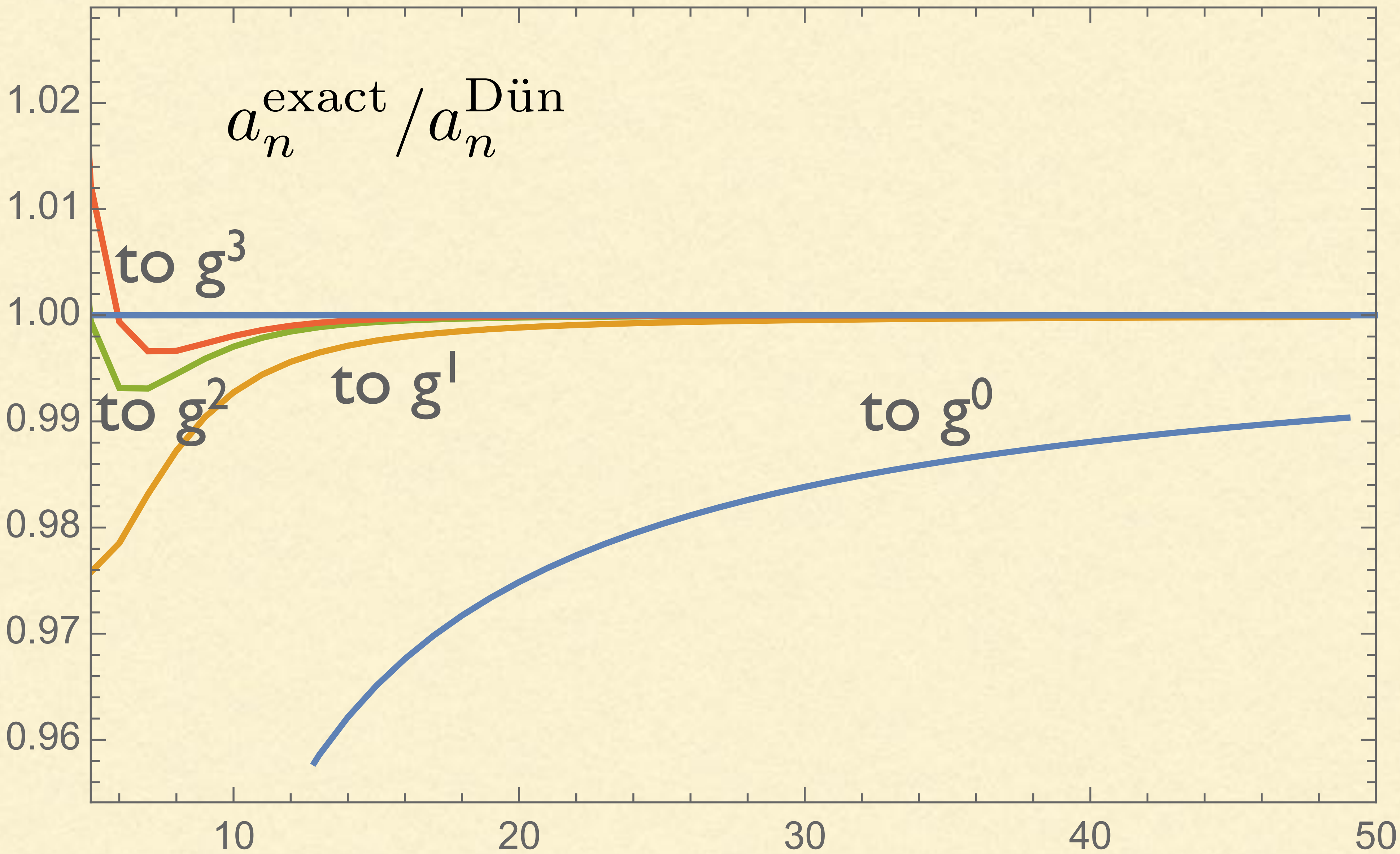
$$\frac{916\,117\,527\,799\,746\,577\,984\,641\,800\,484\,328\,417\,887\,454\,935}{4\,398\,046\,511\,104},$$

$$-\frac{1\,036\,464\,696\,105\,720\,129\,803\,188\,216\,619\,901\,855\,131\,610\,583\,785}{70\,368\,744\,177\,664},$$

$$\frac{152\,773\,774\,949\,844\,438\,983\,482\,018\,830\,957\,022\,324\,392\,611\,587\,683}{140\,737\,488\,355\,328},$$

$$-\frac{46\,864\,661\,934\,282\,542\,604\,967\,271\,035\,084\,212\,028\,364\,938\,136\,115\,903}{562\,949\,953\,421\,312},$$

SELF-RESURGENCE



THE DIFFERENCE EQUATIONS

With Jie Gu arXiv:1709.00854 Also look at the talks by R. Schiappa and Y. Hatsuda

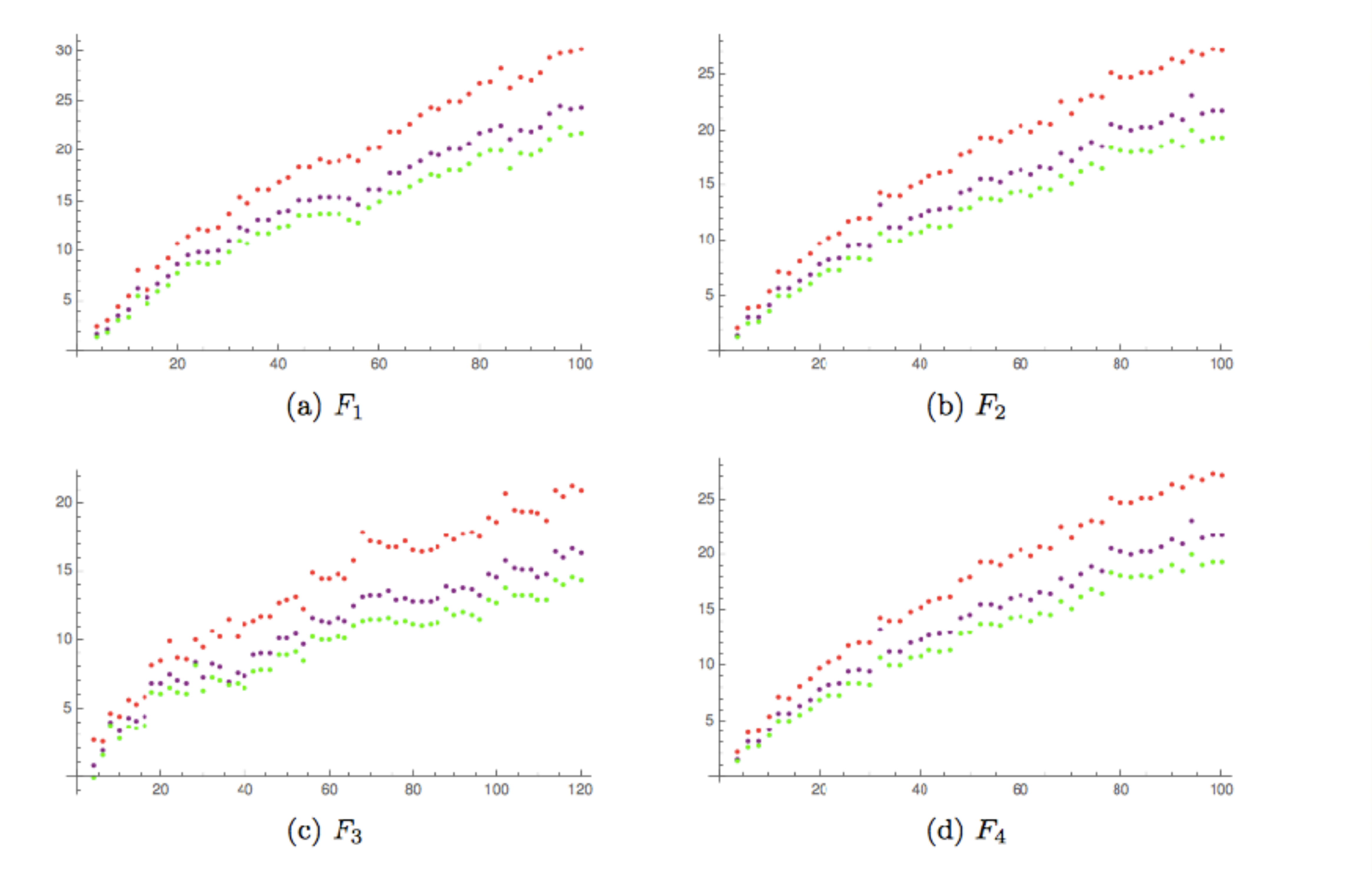
The BenderWu for difference equations

All described genus-1 curves!

geometry	Hamiltonian operator
F_1	$\mathcal{H} = e^{x} + e^{-x/2+p} + e^{-x/2-p}$
F_2	$\mathcal{H} = e^{x} + m_1 e^{-x} + e^{p} + e^{-p}$
F_3	$\mathcal{H} = e^{x} + e^{-x/2+p} + e^{-x/2-p} + m_1 e^{-x}$
F_4	$\mathcal{H} = e^{x} + e^{-x+p} + e^{-x-p} + m_1 e^{-x}$
F_5	$\mathcal{H} = e^{x/2-p} + e^{x/2+p} + e^{-x} + m_1 e^{-x/2+p} + m_2 e^{-x/2-p}$
F_6	$\mathcal{H} = e^{x} + e^{p} + e^{-x-p} + m_1 e^{-x} + m_2 e^{-x+p}$
F_7	$\mathcal{H} = e^{x/2-p} + e^{x/2+p} + e^{-x} + m_1 e^{x} + m_2 e^{-x/2+p} + m_3 e^{-x/2-p}$
F_8	$\mathcal{H} = e^{x} + e^{p} + e^{-x-p} + m_1 e^{x+p} + m_2 e^{-x} + m_3 e^{-x+p}$
F_9	$\mathcal{H} = e^{x+p} + e^{x-p} + e^{-x} + m_1 e^{x} + m_2 e^{-p} + m_3 e^{p}$
F_{10}	$\mathcal{H} = e^{x} + e^{p} + e^{-x-p} + m_1 e^{-x} + m_2 e^{-x+p} + m_3 e^{-x+2p}$
F_{11}	$\mathcal{H} = e^{x} + e^{p} + e^{-x-p} + m_1 e^{-x} + m_2 e^{-x+p} + m_3 e^{-x+2p} + m_4 e^{-p}$
F_{12}	$\mathcal{H} = e^{x/2-p} + e^{x/2+p} + e^{-x} + m_1 e^{-x/2+p} + m_2 e^{-x/2-p} + m_3 e^{2p} + m_4 e^{-2p}$
F_{13}	$\mathcal{H} = e^{x} + e^{-x-2p} + e^{-x+2p} + m_1 e^{p} + m_2 e^{-p} + m_3 e^{-x-p} + m_4 e^{-x+p} + m_5 e^{-x}$
F_{14}	$\mathcal{H} = e^{x+p/2} + m_1 e^{x-p/2} + e^{-x-3p/2} + e^{-x+3p/2} + m_2 e^{-p} + m_3 e^{p} + m_4 e^{-x-p/2} + m_5 e^{-x+p/2}$
F_{15}	$\mathcal{H} = e^{x/2-p} + e^{x/2+p} + e^{-x} + m_1 e^{x} + m_2 e^{2p} + m_3 e^{-2p} + m_4 e^{-x/2+p} + m_5 e^{-x/2-p}$
F_{16}	$\mathcal{H} = e^{x/2-p} + e^{x/2+p} + e^{-x} + m_1 e^{-x/2+p} + m_2 e^{-x/2-p} + m_3 e^{2p} + m_4 e^{-2p} + m_5 e^{x/2+3p} + m_6 e^{x/2-3p}$

Up to 36 orders in hbar considered
by Y. Hatsuda
arXiv:1507.04799

Roughly number of digits of agreement between numerical eigenvalue the Borel-Pade sum v.s. the order of Borel-Pade



CONCLUSIONS

- The nature of semi-classics is inextricably linked to the complexification of the path-integrals
- The machinery of resurgence guarantees the reality of all real physical observables
- Self-resurgent behavior: early-terms—late-terms relation in the same saddle-sector
- SUSY and integer ζ-deformed theories are special, with resurgent cancellation not needed for certain observables (i.e. energy-levels)
- The resurgence mechanism is not lost, and is restored with slight deformation of such theories
- Potential connection with emergent symmetries in QCD(adj) (Cherman, Unsal…)
- Application of BenderWu to the quantum mirror curves
- Is there self-resurgence in the quantum mirror curves?