What is QFT:

Resurgence, transseries, and Lefschetz thimbles

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Some of the work presented here is done in collaboration with :
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Motivation: Can we make sense out of QFT? When is there a continuum definition of QFT?

Dyson(50s),
't Hooft (77),

Quoting from M. Douglas comments, in Foundations of QFT, talk at String-Math 2011

"A good deal of mathematical work starts with the Euclidean functional integral. There is no essential difficulty in rigorously defining a Gaussian functional integral, in setting up perturbation theory, and in developing the BRST and BV formulations (see e.g. K. Costello's work).

A major difficulty, indeed many mathematicians would say the main reason that QFT is still "not rigorous," is that standard perturbation theory only provides an asymptotic (divergent) expansion. There is a good reason for this, namely exact QFT results are not (often) analytic in a finite neighborhood of zero coupling.

The situation is actually worse than described by Douglas.
In fact, this is only first and artificially isolated item in a longer list of problems.
For example,

Yang-Mills/QCD and standard/old problems

1) Perturbation theory is an asymptotic *(divergent) expansion even after regularization and renormalization.* Is there a meaning to perturbation theory?

2) Invalidity of the semi-classical dilute instanton gas approximation on R4.
DIG assumes inter-instanton separation is much larger than the instanton size, but the latter is a moduli, hence no meaning to the assumption.

3) ``Infrared embarrassment",e.g., large-instanton contribution to vacuum energy is IR-divergent, see Coleman's lectures.

4) A resolution of 2) was put forward by considering the theory in a small thermal box. But in the weak coupling regime, the theory always lands on the deconfined "regime". So, *no semi-classical approximation for the confined regime* until recently.

5) Incompatibility of large-N results with instantons.(better be so!)

6) The renormalon ambiguity, ('t Hooft,79), deeper, to be explained.

You may be surprised to hear that all of the above may very well be interconnected according to the resurgence theory.

Recently, few people are attempting to answer and reinvigorate the question: whether/ when a N.P. continuum definition of QFT/string theory may exist.

Dunne, Cherman, Sulejmanpasic, Argyres, Basar, Dorigoni, Sakai, Nitta, Misumi, Fujimori, Kamata, Tanizaki, Kozcaz, Gukov, Marino, Petrov, Milekhin, MÜ,.....
Resurgence in QFTs, QM, and path integrals, analytic continuation

Schiappa, Marino, Aniceto,Vonk: Realized first resurgence in string theory

Kontsevich: recent talks at PI, Simons, Resurgence from the path integral perspective.
Garoufalidis, Costin: Math and Topological QFTs
Witten: Analytic continuation of path integral, Lefschetz thimbles

The common concept unknown in physics community, is a recent mathematical progress:
Resurgence Theory, [Jean Ecalle (80s)]
and applied to QM by Pham, Delabaere, Voros, Zinn-Justin, (related Dingle-Berry-Howls)

Ecalle's theory may very well change the overall perspective on asymptotic analysis, for both mathematicians and physicists alike for good.

Main promise: NP-data (or NP-completion) can be extracted from P-data!

Current status of work in QFT

It is, in my opinion, disappointing to see that there is almost no concerted effort addressing these problems within HEP community. These are deep problems, worthy problems, but ”given up” problems for ~30 years.

- I take the stand-point put forward by Marcel Proust:

- “The real voyage of discovery consists not in seeking new landscapes, but in having new eyes.”

- Or Berry’s perspective “I like new things in old things”

- I believe if we have sufficiently good techniques/ideas to understand QFT, it should not matter if the theory is supersymmetric or not.

In order to say something new on an old problem, we must have both new physical perspective and new mathematical tools. We must be in search of new ideas and tools. Here are two recent ideas from physics and two from mathematics, and better ones are needed:

- Resurgence theory and Trans-series
- Complex Morse Theory (or Picard-Lefschetz theory) and complexification of path integral
- Adiabatic Continuity (analyticity) (Avatar of large-N Volume independence)
- Reliable Semi-classics (calculability)
- New 't Hooft anomalies

Under the spell of large-N volume independence

I will review a set of connected ideas:

Large-N volume independence

Adiabatic continuity

Reliable semi-classics (Sorry for the word adjective. I use it because the concept of semi-classics is awfully abused in the past.)

Resurgence and trans-series

Lefschetz thimbles

Discrete mixed 't Hooft anomalies

Goal: A useful dynamical framework for asymptotically free QFTs and more general QFTs.

Simpler question: Can we make sense of the semi-classical expansion of QFT?

Argyres, MÜ, Dunne, MÜ, 2012

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)} \, (\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n} \, e^{-n \, A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)} \, (\lambda\hbar)^k$$

pert. th. | n-instanton factor | pert. th. around n-instanton

All series appearing above are asymptotic, i.e., divergent as $c_{(0,k)} \sim k!$. The combined object is called trans-series following resurgence literature

Borel resummation idea: If $P(\lambda) \equiv P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$ has convergent Borel transform

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$

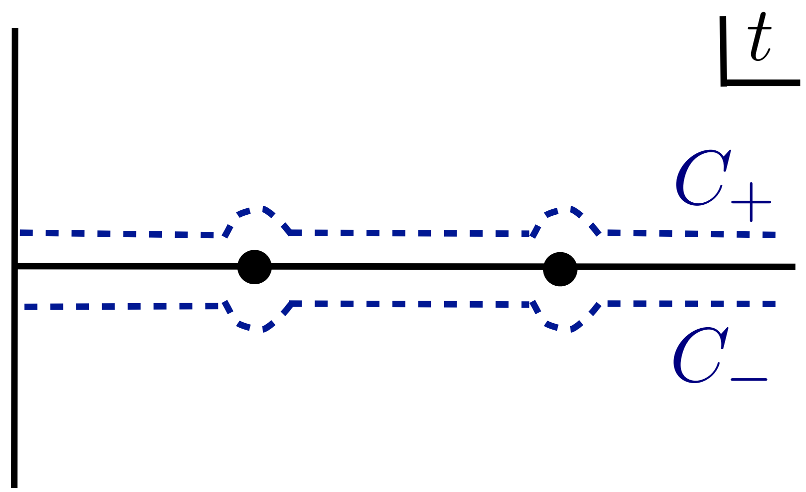
in neighborhood of $t = 0$, then

$$\mathbb{B}(g^2) = \frac{1}{g^2} \int_0^{\infty} BP(t) e^{-t/g^2} dt \, .$$

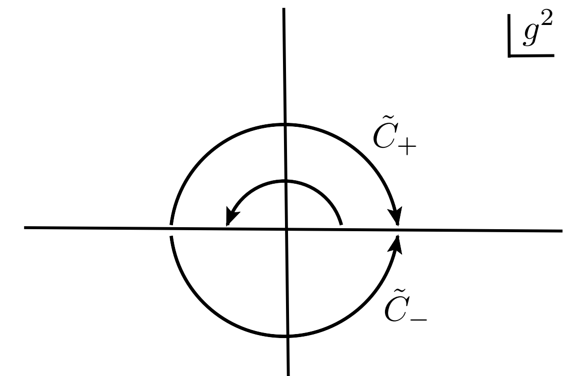
formally gives back $P(g^2)$, but is ambiguous if $BP(t)$ has singularities at $t \in \mathbb{R}^+$:

Borel plane and lateral (left/right) Borel sums

Directional (sectorial) Borel sum. $\mathcal{S}_\theta P(g^2) \equiv \mathbb{B}_\theta(g^2) = \frac{1}{g^2}\int_0^{\infty\cdot e^{i\theta}} BP(t)\, e^{-t/g^2} dt$



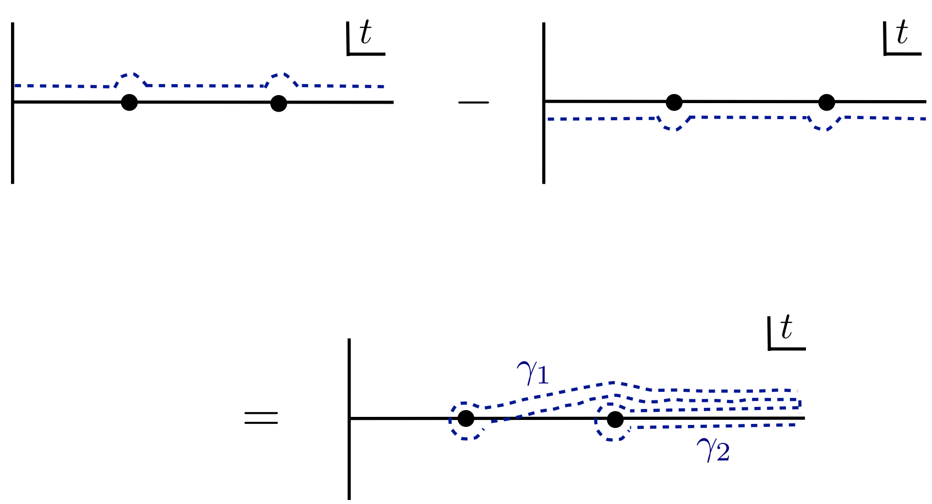
left and right Borel sums



Same structure by analytic continuation

$$\mathbb{B}_{0\pm}(|g^2|) = \mathrm{Re}\,\mathbb{B}_0(|g^2|) \pm i\,\mathrm{Im}\,\mathbb{B}_0(|g^2|), \qquad \mathrm{Im}\,\mathbb{B}_0(|g^2|) \sim e^{-2S_I} \sim e^{-2A/g^2}$$

The *non-equality* of the left and right Borel sum means the series is *non-Borel summable or ambiguous*. The ambiguity has the same form of a 2-instanton factor (not 1). The measure of ambiguity (Stokes automorphism/jump in g-space interpretation):



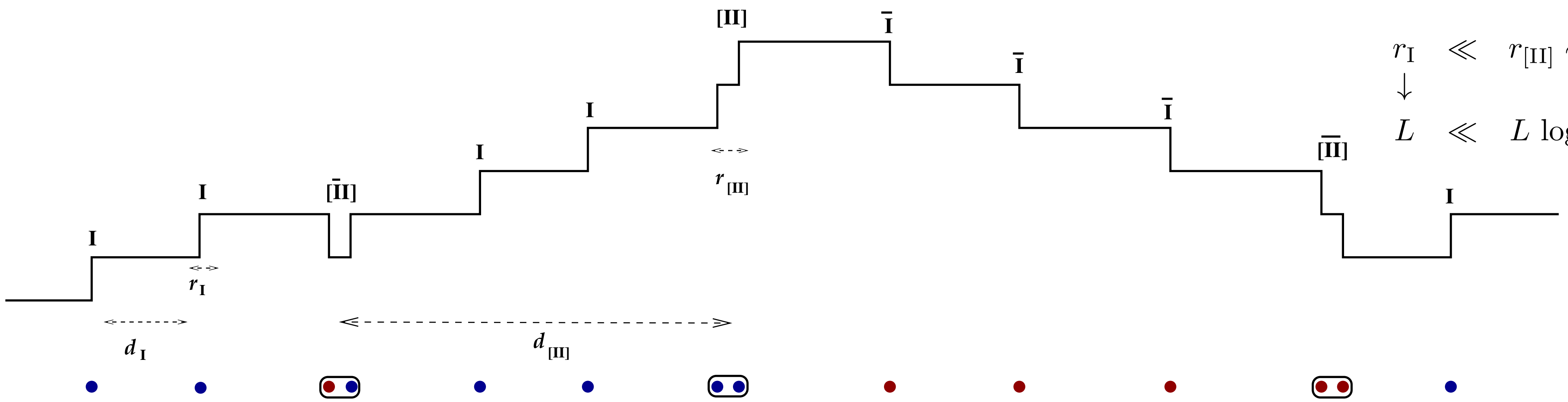
$$\mathcal{S}_{\theta+} = \mathcal{S}_{\theta-} \circ \underline{\mathfrak{S}}_\theta \equiv \mathcal{S}_{\theta-} \circ (\mathbf{1} - \mathrm{Disc}_{\theta-}),$$

$$\mathrm{Disc}_{\theta-}\,\mathbb{B} \sim e^{-t_1/g^2} + e^{-t_2/g^2} + \dots \qquad t_i \in e^{i\theta}\mathbb{R}^+$$

Jean Ecalle, 80s

Bogomolny--Zinn-Justin (BZJ) prescription

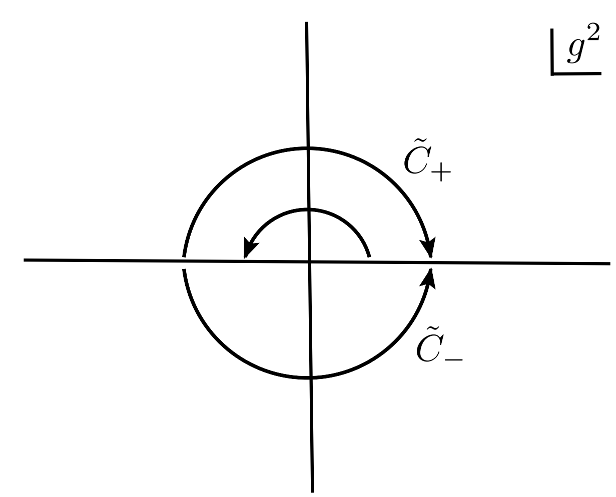
Bogomolny-Zinn-Justin prescription in QM (80s): done for double well potential, but consider a periodic potential. Dilute instanton, molecular instanton gas.



$$\begin{array}{ccccccc} r_{\rm I} & \ll & r_{\rm [II]} \sim \ell_{\rm qzm} & \ll & d_{\rm I} & \ll & d_{\rm [II]}, \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L & \ll & L\,\log\left(\frac{1}{g^2}\right) & \ll & L\,e^{S_0} & \ll & L\,e^{2S_0}. \end{array}$$

How to make sense of topological molecules (or molecular instantons)? Why do we even need a molecular instanton? (Balitsky-Yung in SUSY QM, (86))

Naive calculation of I-anti-I amplitude: meaningless (why?) at $g^2 > 0$. The quasi-zero mode integral is dominated at small-separations where a molecular instanton is meaningless. Continue to $g^2 < 0$, evaluate the integral, and continue back to $g^2 > 0$: two fold-ambiguous!



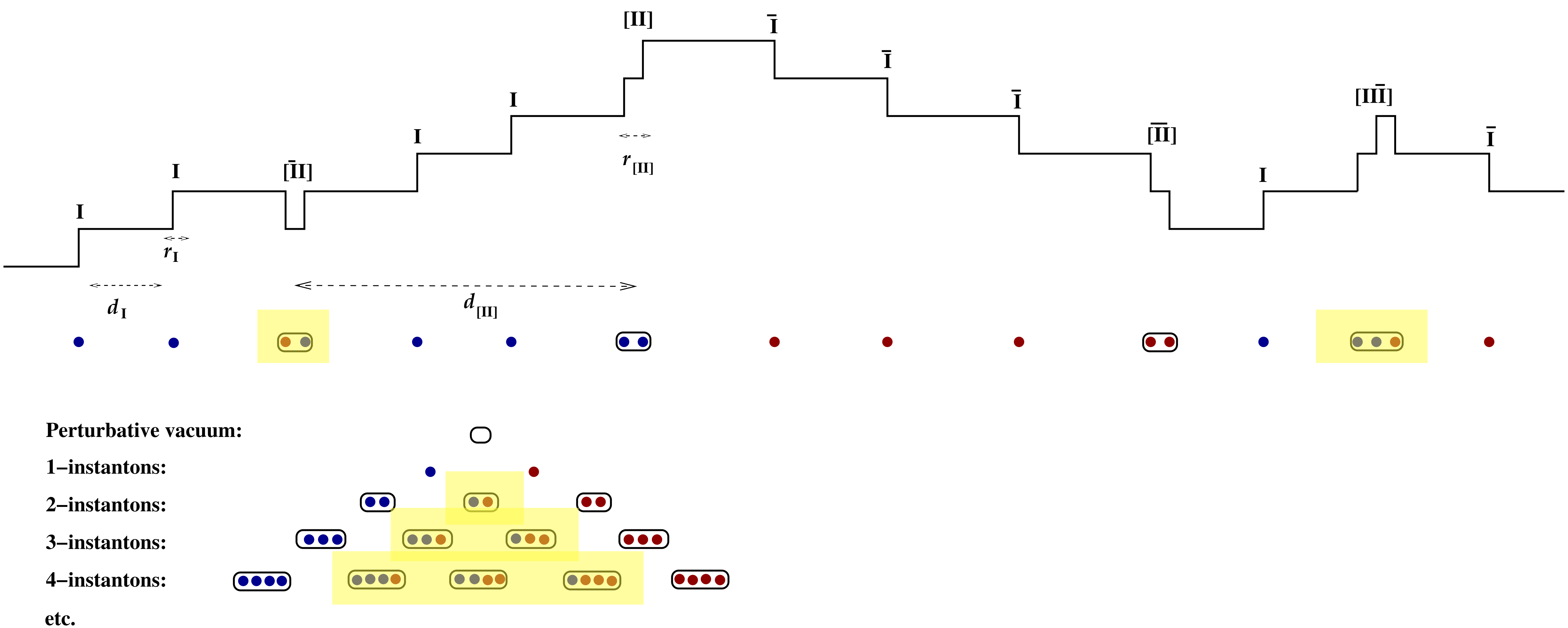
$$[\mathcal{I}\bar{\mathcal{I}}]_{\theta=0^\pm} = \mathrm{Re}\,[\mathcal{I}\bar{\mathcal{I}}] + i\,\mathrm{Im}\,[\mathcal{I}\bar{\mathcal{I}}]_{\theta=0^\pm}$$

Why?: because we are on Stokes line, later....

Remarkable fact: Leading ambiguities cancel. “N.P. CONFLUENCE EQUATION”, elementary incidence of Borel-Ecalle summability which I will return:

$$\mathrm{Im}\, \mathbb{B}_{0,\theta=0^\pm} + \mathrm{Im}\, [\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm} = 0 \,, \qquad \text{up to } O(e^{-4S_I})$$

The ambiguous topological configurations. All are non-BPS quasi-solutions!



Cancellation of ambiguities

path integral version

- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n! \left(1 - \frac{53}{6} \cdot \frac{1}{3} \cdot \frac{1}{n} - \frac{1277}{72} \cdot \frac{1}{3^2} \cdot \frac{1}{n(n-1)} - \ldots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\, E \sim \pi\, e^{-2\frac{1}{6g^2}} \left(1 - \frac{53}{6} \cdot g^2 - \frac{1277}{72} \cdot g^4 - \ldots\right)$$

- The divergent asymptotic part is coded information about the instanton/anti-instanton saddle in the problem (all orders relation)
- Concrete relation even for infinite dimensional (path) integrals.

$$\mathrm{Im}\, \mathbb{B}_{0,\theta=0^\pm} + \mathrm{Im}\, [\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm} = 0\,, \qquad \text{up to } O(e^{-4S_I})$$

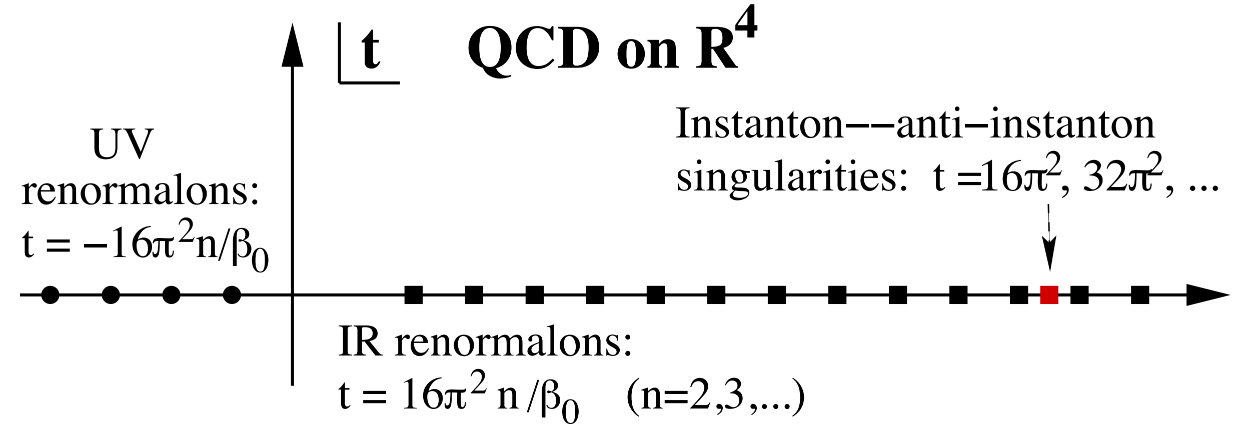
Can this work in QFT? QCD on R4 or NLSM on R2?

't Hooft(79) :**No,** *on R4,* Argyres, MÜ: ***Yes,*** *on R3 x S1,*
F. David(84), Beneke(93) : **No,** *on R2.* Dunne, MÜ: ***Yes,*** *on R1 x S1*

Why doesn't it work, say for YM on R4?

Instanton-anti-instanton contribution, calculated in some way, gives an $\pm i \exp[-2S_I]$.
Lipatov(77): Borel-transform BP(t) has singularities at $t_n = 2n\, g^2\, S_I$. (Modulo the standard IR problems with 2d instantons, also see Bogomolny-Fateyev(77)).

BUT, BP(t) has other (more important) singularities closer to the origin of the Borel-plane. (not due to factorial growth of number of diagrams on R4!)



't Hooft called these IR-renormalon singularituies with the hope/expectation that they would be associated with a saddle point like instantons.
No such configuration is known!!

A real problem in QFT, means pert. theory, as is, ill-defined. How to cure starting from micro-dynamics?

Standard view emanating from late 70s,
e.g. : from **Parisi(78)**

If the theory is renormalizable, the Borel transform has new singularities which cannot be controlled by using semi-classical methods [5-8] .

5 G. 't Hooft, Lectures given at Erice (1977)

6 B. Lautrup, Phys. Lett. 69B (1977) 109

7 G. Parisi, Lectures given at the 1977 Cargèse Summer School

8 P. Olesen, Nordita preprint NBI HE 77.48 (1977)

A perspective that remained with us for 35 years.

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New question: What happens if we can make the most interesting QFTs semi-classically calculable?

Is this even possible?

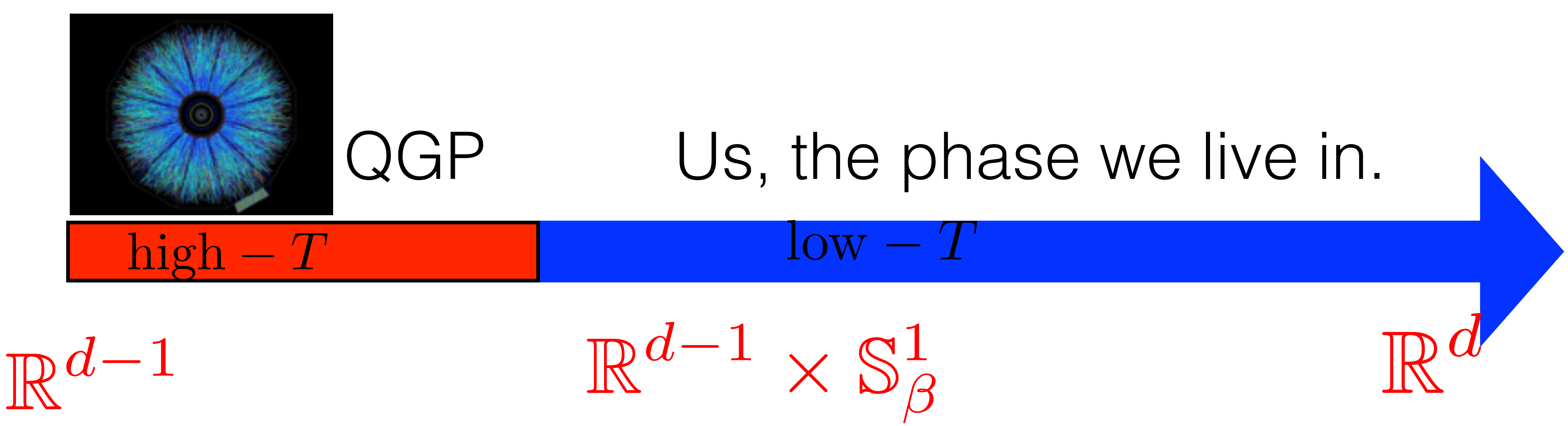
Adiabatic continuity and analyticity for YM? Is it possible?

- We first want a (semi-classically) calculable regime of field theory, say of Yang-Mills or QCD. Of course, everyone want this. But is it possible at all?

- It is NOT known if such a framework exits on R4. In fact, theory becomes strongly coupled at longer distances.

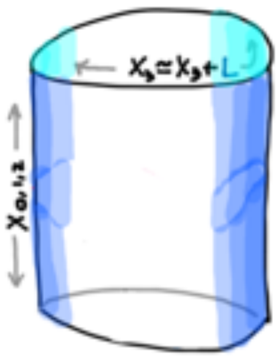
- Supersymmetry does not help for vector-like/chiral theories!

- Consider these theories on four manifold R3 x S1, and study their dynamics as a function of radius. At small-radius, the theory is weakly coupled (great! thanks to asymptotic freedom) at the scale of the radius. But the theory is non-analytic as a function of radius, there is a phase transition. In fact, all QCD-like and chiral theories are under thermal compactification.



Adiabatic continuity and analyticity for non-susy gauge theory? Is it possible?

YM/QCD on R3 x S1:
Idea of adiabatic continuity



Phase transition or
rapid cross-over

high $-T$ low $-T$

$\mathbb{R}^{d-1}$ $\mathbb{R}^{d-1} \times \mathbb{S}^1_\beta$ $\mathbb{R}^d$

$$V_{1-\text{loop}}[\Omega] = (-1)\frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\,|\text{tr}\,\Omega^n|^2 \qquad \Omega = e^{\int_{S^1} A}$$

Gross, Pisarski, Yaffe 1980

We want continuity:

$\mathbb{R}^{d-1} \times \mathbb{S}^1_L$

Prevent phase-transition by using circle compactification
(pbc for fermions) or double-trace deformation!

Adiabatic continuity vs. volume independence.

Adiabatic continuity in non-susy theories is a spin-off of a brilliant idea by Eguchi and Kawai (82)

What does EK say? It says something far more stronger than continuity, it implies volume independence, observable being independent of compactification radius at large-N.

But it was tricky to achieve EK, original proposal failed.

We lived in the mirage that it worked for 25 years with TEK and QEK: QEK and old-TEK also shown to fail many years later.

TEK is brilliant, why did it fail (put a classical twisted b.c. in both directions, alter classical action. Happy right? No.)

Because in this game, your enemy (thermal or quantum fluctuations) is always O(N^2), yes, it is quantum loop effect but it is O(N^2), and your weapons in old-TEK is O(N).

Large *N* volume independence

“Eguchi-Kawai reduction” or “large-N reduction”

$SU(N)$ gauge theory on toroidal compactifications of $\mathbb{R}^4$

down to four-manifold $\mathbb{R}^{4-d} \times (S^1)^d$

No volume dependence in leading large *N* behavior of topologically trivial single-trace observables (or their connected correlators)

provided

no spontaneous breaking of center symmetry or translation invariance. (i.e, no phase transition as the radius is reduced.)

Proof: Comparison of large *N* loop equations (also called Migdal-Makeenko or Schwinger-Dyson equation (Eguchi-Kawai 82) or N=∞ classical dynamics (Yaffe 82)

Center-stabilized (deformed) YM

$$S^{\mathrm{YM}^*} = S^{\mathrm{YM}} + \int_{R^3 \times S^1} P[\Omega(\mathbf{x})] \qquad P[\Omega] = A \frac{2}{\pi^2 L^4} \sum_{n=1}^{\lfloor N/2 \rfloor} \frac{1}{n^4} \left| \mathrm{tr}\,(\Omega^n) \right|^2$$

- A double-trace deformation prevents center-breaking.

- This is a large deformation of the action/Hamiltonian, not a small perturbation. We are changing the action with something as large as action itself.

- For any sane person, this should ring alarm bells because, naively, we are dramatically altering the theory. But there is something deeper here!

- Indeed, deformation is O(N^2). But after it does its job of stabilization, its effect on the dynamics is N-suppressed (Yaffe, MU, 2008).

Such deformations were also considered in Ogilvie, Myers, Schaden,Pisarski for other purposes

Why double-trace deformation is important? (despite the fact that many folks expressed that they do not like it, or they despise it.)

- The answer is very simple. Because it is very difficult to make a "zombie" live and kicking again. (And believe me, I am not expert on anything else but zombies.)
- Talk by Gonzalez-Arroyo: Birth, death and rebirth of large-N volume independence

DYM-Loop equations

$$\langle \delta S \cdot \delta W[C] \rangle + \langle \delta^2 W[C] \rangle = 0\,.$$

$$\tfrac{1}{2}|C| \langle W[C] \rangle = \sum_{\ell \subset C} \sum_{p | l \subset \partial p} \frac{\widetilde{\beta}}{4} \left[\langle W[(\overline{\partial p})C] \rangle - \langle W[(\partial p)C] \rangle \right] + \sum_{\mathrm{self-intersections}} \mp \langle W[C'] \rangle \langle W[C''] \rangle\,.$$

SD-loop equations for original YM, for W(C) Wilson loop observable

$$\langle \delta(\Delta S) \cdot \delta W[C] \rangle = \sum_{k \neq 0} b_k[C] \, \langle W[\Omega^k C] \, W[\Omega^{-k}] \rangle\,,$$

$$\langle W[\Omega^k C] W[\Omega^{-k}] \rangle = \langle W[\Omega^k C] \rangle \langle W[\Omega^{-k}] \rangle + O(1/N^2)\,.$$

$$\langle \delta(\Delta S) \cdot \delta W[C] \rangle = O(1/N^2)\,.$$

The effect of deformation

Factorization, thanks to unbroken center!

N-suppressed effect of deformation, thanks to unbroken center!

- Like a good Samaritan, it does the good deed, and you do not even know it existed (as Gabriele Veneziano insightfully put it in 2009.)

Preserving center with adjoint fermions

Kovtun, Unsal, Yaffe,07

$N_f \geq 1$ massless adjoint rep. fermions.
periodic boundary conditions ➡ stabilized center symmetry

$N_f = 1$ N=1 SYM. One loop perturbative potential is zero. (To all loop order it is zero.) I realized that this was not an "ordinary zero". It was

1-1=0

$$V[\Omega] = (1-1)\frac{1}{L^4}\sum_{n=1}^{\infty}|\mathrm{tr}\Omega^n|^2$$

Plus one is coming from periodic adjoint fermions, nothing to do with supersymmetry. This meant, we have a friend on our side, as powerful as the gauge fluctuations, and trying to undo the harm. After some thinking, I came up with the idea that

2-1=1>0! (or VI works!)

Digression: Newer ideas about VI

Whenever VI works, there is always a deep reason behind it, and implications that follow with it.

I want to point out two line of works. Due to time constraints, I cannot expand on neither ideas, but I think they are important.

By Cherman: Emergent fermionic symmetries at large N (sourced by the question: how does QCD(Adj) avoids Hagedorn transition which would normally forces existence of phase transition?)

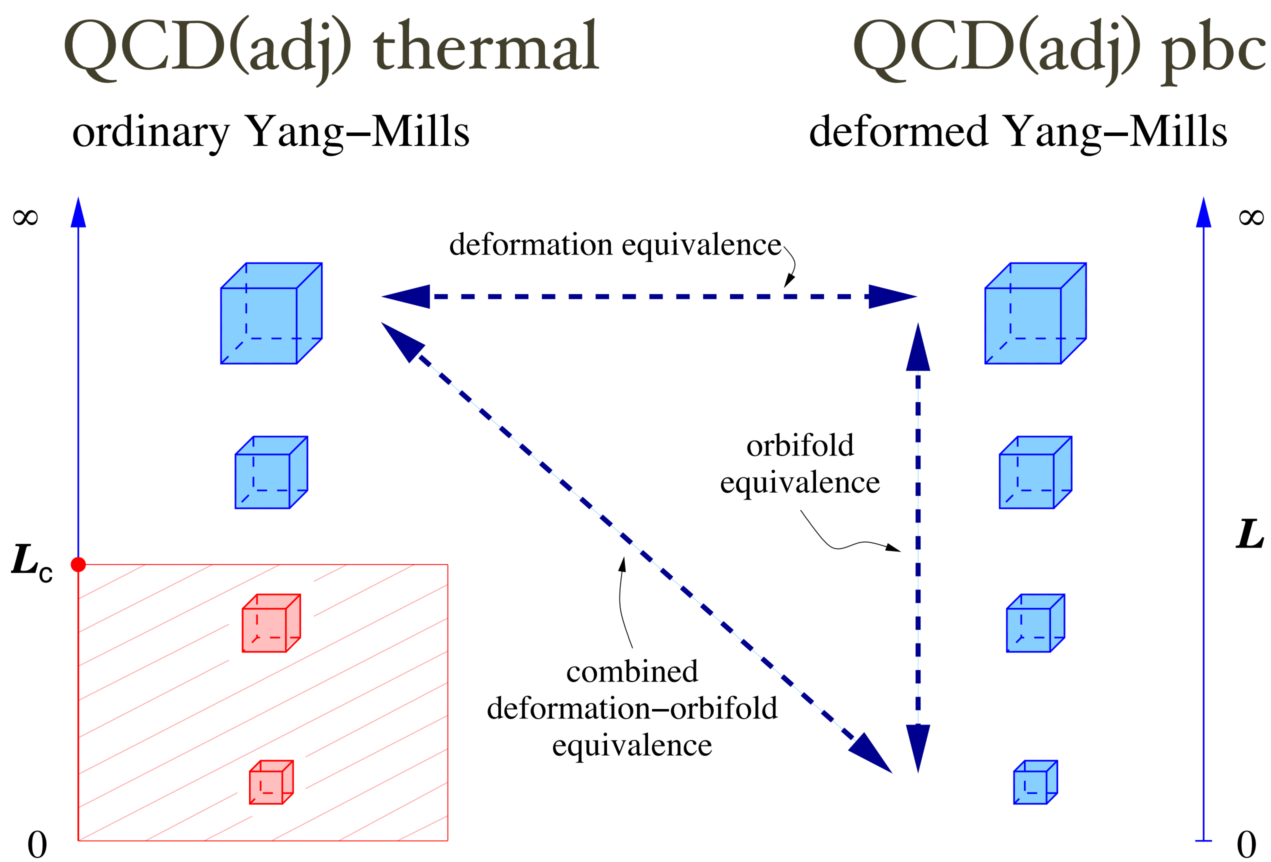
By Sulejmanpasic: VI in theories with global symmetries, new graded partitions functions.

In both, the common denominator that makes things works is an extreme spectral cancellation conspiracy in the appropriate twisted partition function.

I also think that the reason why new-TEK (Gonzalez-Arroyo, Okawa) works ought to be extremely important and related!

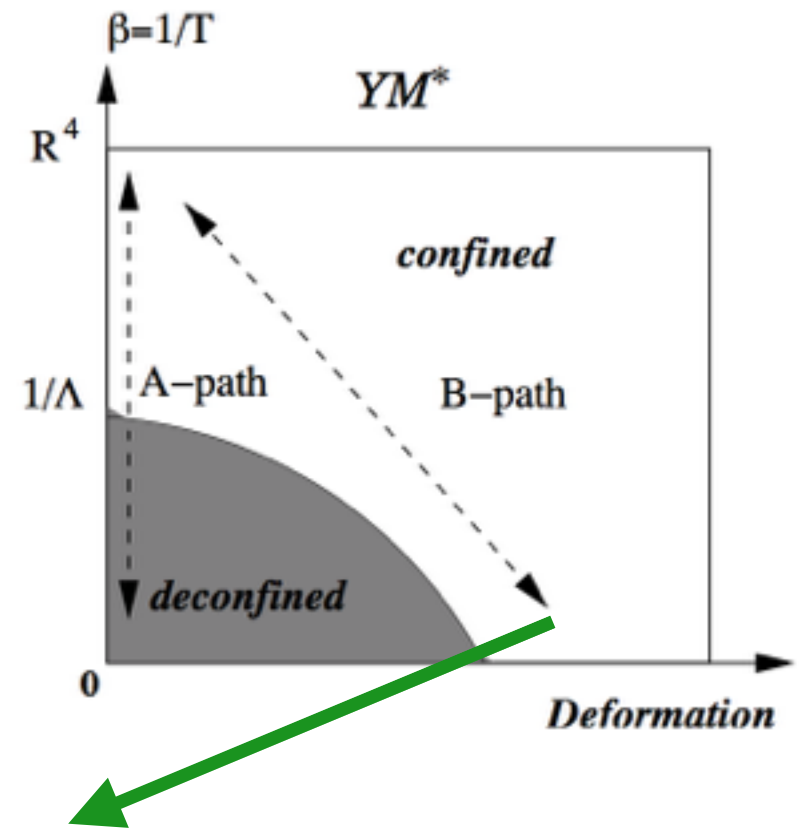
Physical implication side of VI is never sufficiently explored. Potentially fruitful.

Large-N:exact volume independence



• (w/ Yaffe 2008)

Finite-N: analiticity or continuity



We can now do reliable semi-classics here, and it is continuously connected to YM on R4.

By using twisted partition function $Z(L) = \mathrm{tr} e^{-LH}(-1)^F$ in QCD with adjoint fermions, one can also achieve unbroken center symmetry and a calculable regime in small-L.

Twisted partition function is the supersymmetric Witten index for N=1 SYM compactified on a circle.

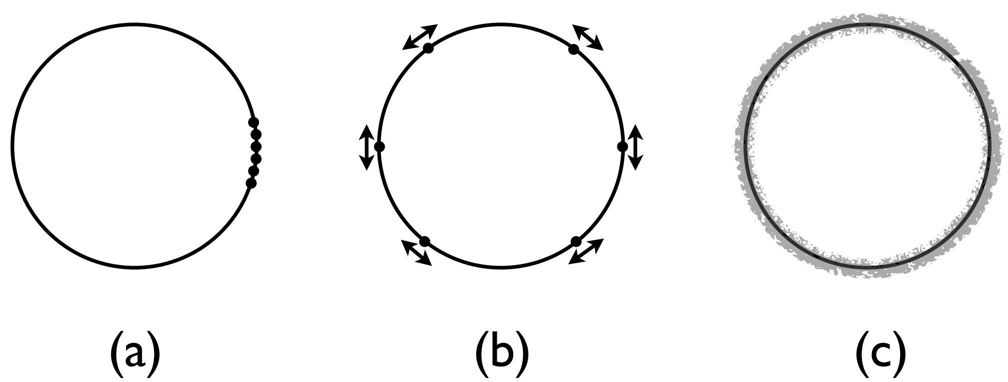
Abelianization and abelian duality

$SU(N) \to U(1)^{N-1}$

Similar to Polyakov model in 3d and SW in 4d, dynamics abelianize.

Three types of holonomy

Crucial difference of (a) and (b): **van Baal, Kraan** in YM theory on R_3 x S_1



center broken

center-stable weak coupling

center-stable strong coupling

$$L = \frac{1}{4} F_{\mu\nu}^2 \longleftrightarrow \frac{1}{2} (\partial_\mu \sigma)^2$$

Gapless to all orders in perturbation theory.
How about NP-effects?

Topological configurations, 1-defects

1-defects, Monopole-instantons: Associated with the N-nodes of the affine Dynkin diagram of SU(N) algebra. The Nth type corresponds to the affine root and is present only because the theory is *locally 4d*! [**van Baal, Kraan, (97/98), Lee-Yi, Lee-Lu (97)**]

$$\mathcal{M}_k \sim e^{-S_k} e^{-\alpha_k \cdot b + i\alpha_k \sigma}, \qquad k = 1, \ldots, N$$

$$S_k = \frac{8\pi^2}{g^2 N} = \frac{S_I}{N}$$

Action 1/N of the 4d instanton, keep this in mind!

Proliferation of monopole-instantons generates a non-perturbative mass gap for gauge fluctuations, similar to 3d Polyakov model (Polyakov, 77). It is first generalization thereof to local 4d theory!

$$m_g \sim (LN)^{-1} e^{-S_k/2} \sim \Lambda(\Lambda L N)^{5/6}$$

Sounds like happy ending..... But actually more like new beginnings. See below.

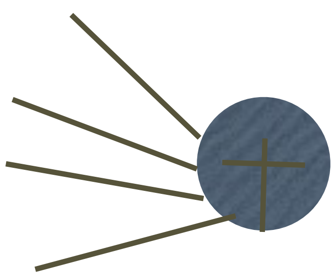
In theories with adjoint fermions?

Theories with massless fermions: take SU(2) QCD(adj)

$$S = \int_{R^3 \times S^1} \frac{1}{g^2} \mathrm{tr} \left[\frac{1}{4} F_{MN}^2 + i \bar{\psi}^I \bar{\sigma}^M D_M \psi_I \right]$$

monopole operators have fermionic zero modes.

$$e^{-S_0} e^{i\sigma} \qquad \underbrace{\psi \ldots \psi}_{\text{fermion zero modes}}$$



Hence, unlike Polyakov mechanism, monopoles can no longer induce mass gap or confinement, instead a photon-fermion interaction Affleck-Harvey-Witten(82).
This is viewed as death of Polyakov mechanism in theories with fermions.
AHW proved gaplessness in Polyakov model with Dirac adjoint fermions in 82 on R3. What happens on R3 x S1?

Is there a gap or not? If so, there must be something new with respect to AHW? How? First, let us count the zero modes.

Index theorems

Journal of Functional Analysis **177**, 203–218 (2000)
doi:10.1006/jfan.2000.3648, available online at http://www.idealibrary.com on IDEAL®

An L^2-Index Theorem for Dirac Operators on $S^1 \times \mathbb{R}^3$

Tom M. W. Nye and Michael A. Singer

APPENDIX A. ADIABATIC LIMITS OF η-INVARIANTS

$$\begin{aligned} \operatorname{ind}\,(D^+_{\mathbb{A}}) &= \int_X \mathrm{ch}(\mathbb{E}) + \frac{1}{\mu_0}\sum_\mu \epsilon_\mu c_1(E_\mu)[S^2_\infty] \\ &= \int_X \mathrm{ch}(\mathbb{E}) - \frac{1}{2}\overline{\eta}_{\mathrm{lim}} \end{aligned} \tag{22}$$

Very important theorem! Importance of it is not yet sufficiently appreciated in literature.

index theorems

Atiyah-M.I.Singer 1975

Callias 1978 → E. Weinberg 1980

Nye-A.M.Singer, 2000 → Poppitz, MU 2008: The one relevant for us!

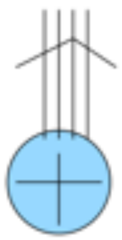
Topological excitations in QCD(adj), SU(2), Nf=2

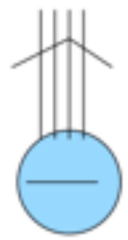
MÜ 2007

$$\left(\int_{S^2} F,\ \int_{R^3\times S^1} F\tilde{F}\right)$$

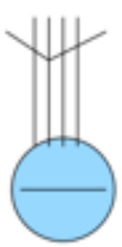
index theorems
Callias 1978
E. Weinberg 1980
Nye-A.M.Singer, 2000
Poppitz, MU 2008
Atiyah-M.I.Singer 1975

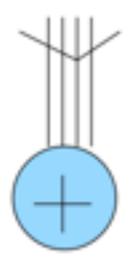
Monopole-instantons


BPS
(1, 1/2)


KK
(–1, 1/2)

$$e^{-S_0} e^{i\sigma} \det_{I,J} \psi^I \psi^J,$$


$\overline{\text{BPS}}$
(–1, –1/2)

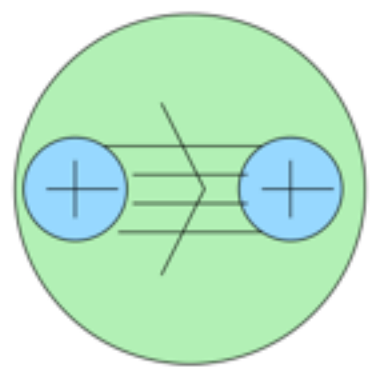

$\overline{\text{KK}}$
(1, –1/2)

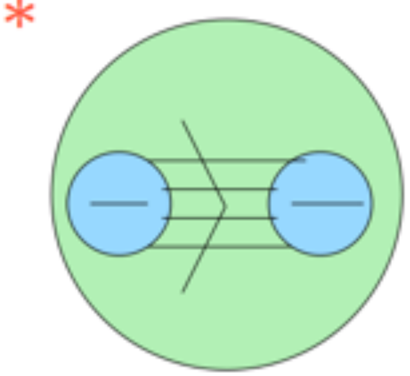
$$e^{-S_0} e^{i\sigma} \det_{I,J} \bar{\psi}^I \bar{\psi}^J$$

Magnetic Bions

Mass gap for gauge fluctuations!

$(\mathbb{Z}_2)_*$


(2,0)


(–2, 0)

No net topological charge.

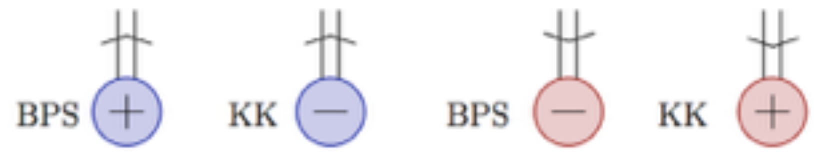
$$e^{-2S_0}\left(e^{2i\sigma} + e^{-2i\sigma}\right)$$

Discrete shift symmetry : $\sigma \longrightarrow \sigma + \pi \qquad \psi^I \longrightarrow e^{i\frac{2\pi}{8}} \psi^I$

Crucial earlier work: van Baal, Kraan 97/98 and Lee, Lu, Yi, 97/98

Topological objects: Coupling to low energy fields

$$(Q_M, Q_{top}) = \left(\int_{S_2} B \cdot d\Sigma, \int_{R^3 \times S_1} F\tilde{F}\right)$$



$e^{\mp b} e^{\pm i\sigma} (\lambda\lambda)$ $\quad$ $e^{\mp b} e^{\mp i\sigma} (\bar{\lambda}\bar{\lambda})$

monopoles
mass gap for fermions

$(\lambda\lambda)(\lambda\lambda)$ $\quad$ $(\bar{\lambda}\bar{\lambda})(\bar{\lambda}\bar{\lambda})$

instantons
Anomaly

Take-home page

e^{-2b} $\quad$ e^{2b}

Neutral bions
Center stabilizing NP-potential!

$$b = \frac{4\pi}{g^2}\Delta\theta$$

$e^{2i\sigma}$ $\quad$ $e^{-2i\sigma}$

Magnetic bions
Mass gap for gauge fluctuations

Pretty complete dynamical description of the theory. Summary: Confinement with discrete chiral symmetry breaking, N-vacua.

Mixed anomalies (center vs. discrete chiral)

Can symmetries make their own (independent) decisions on how to be realized?

Once center is unbroken, it actually has dramatic implications. To see this, we need to resort to ’t Hooft anomalies of discrete symmetries (Komargodski et.al.)

Idea: Gauge Z_N center. Then, one is dealing with PSU(N) gauge theory. Then, it is possible to show that topological term is modified.

$$\Delta Q_5 = T(\mathrm{adj}) \times \frac{1}{8\pi^2}\int \mathrm{Tr}(F \wedge F) \in 2N\mathbb{Z}$$ Global (ABJ) anomaly.

$$\frac{1}{8\pi^2}\int\left(\mathrm{Tr}F' \wedge F' - \frac{1}{N}\mathrm{Tr}F' \wedge \mathrm{Tr}F'\right) \in \frac{1}{N}\mathbb{Z},$$

$$\Delta Q_5 \in 2N \times \frac{1}{N}\mathbb{Z} = 2\mathbb{Z}$$

Once center gauge, chiral sym. is gone! This is mixed discrete ’t Hooft anomaly. Means, at least either one or the other must always be broken. Can never have them both restored. (theorem) This is sometimes called **persistent order.**
In QCD(adj) with circle compactification, center is never broken, thus, the discrete chiral symmetry was actually destined to be broken. In this sense, adiabatic continuity is a stronge version of persistent order. (Komargodski, Sulejmanpasic, MU. 2017)

Topological molecules: 2-defects

2-defects are universal, dictated by Cartan matrix of Lie algebra: Charged and neutral bions

- *Magnetic bions:* For each pair (i,j) such that $(\alpha_i, \alpha_j) < 0$, there exists a magnetic bion $[\mathcal{M}_i \overline{\mathcal{M}}_j]$ with magnetic and topological charges

$$(\mu, \nu) = \left(\alpha_i^\vee - \alpha_j^\vee \,,\, \nu^{(i)} - \nu^{(j)}\right), \quad (0,\, 2/N) \tag{5.1}$$

associated with an operator in the effective action proportional to

$$\mathcal{B}_{ij} \sim e^{-S_i(\varphi) - S_j(\varphi)} e^{-2\pi i \sigma(\alpha_i^\vee - \alpha_j^\vee)}, \tag{5.2}$$

- *Neutral bions:* For each i there exists a bion $[\mathcal{M}_i \overline{\mathcal{M}}_i]$ with magnetic and topological charges

$$(\mu, \nu) = (0, 0), \tag{5.3}$$

associated with an operator proportional to

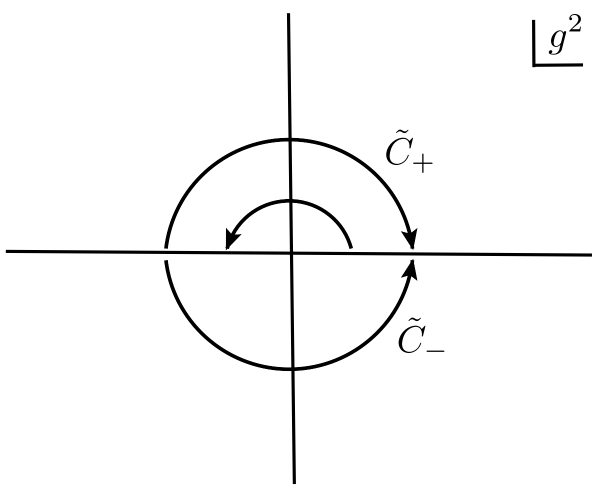
$$\mathcal{B}_{ii} \sim e^{-2S_i(\varphi)}. \tag{5.4}$$

Magnetic bion: mass gap for gauge fluctuations, MÜ 2007

Neutral bion generates a center-stabilizing potential:
Poppitz-MÜ 2011, Poppitz-Schäfer-MÜ, Argyres-MÜ 2012

Neutral bion and non-perturbative ambiguity in semi-classical expansion

Naive calculation of neutral bion amplitude, as you may guess as per QM example, meaningless at $g^2 > 0$. The quasi-zero mode integral is dominated at small-separations where a molecular event is meaningless. Continue to $g^2 < 0$, evaluate the integral there, and continue back to $g^2 > 0$. Result is two fold-ambiguous!



$$\mathcal{B}_{ii,\theta=0^\pm} = [\mathcal{M}_i \overline{\mathcal{M}}_i]_{\theta=0^\pm} \sim e^{-\frac{16\pi^2}{g^2 N}} \pm i e^{-\frac{16\pi^2}{g^2 N}}$$

for deformed YM

As it stands, this is a **disaster!** Semi-classical expansion at second order is void of meaning? This is a general statement valid for many QFTs admitting semi-classical approximation. e.g. the Polyakov model.

In QFT literature, people rarely discussed second or higher order effects in semi-classics, most likely, they thought no new phenomena would occur, and they would only calculate exponentially small subleading effects. The truth is far more subtler!

NP ambiguity in semi-classical expansion:
Disaster or blessing in disguise?

$$\mathcal{B}_{ii,\theta=0^\pm} = [\mathcal{M}_i \overline{\mathcal{M}}_i]_{\theta=0^\pm}$$
$$\sim e^{-\frac{16\pi^2}{g^2 N}} \pm i e^{-\frac{16\pi^2}{g^2 N}}$$

for deformed YM

$$\mathcal{B}_{ii} = [\mathcal{M}_i \overline{\mathcal{M}}_i]$$
$$\sim e^{-\frac{16\pi^2}{g^2 N}}$$

for N=1 SYM and QCD(adj)

NP-ambiguity in PT   Ambiguity in neutal-bions amplitude

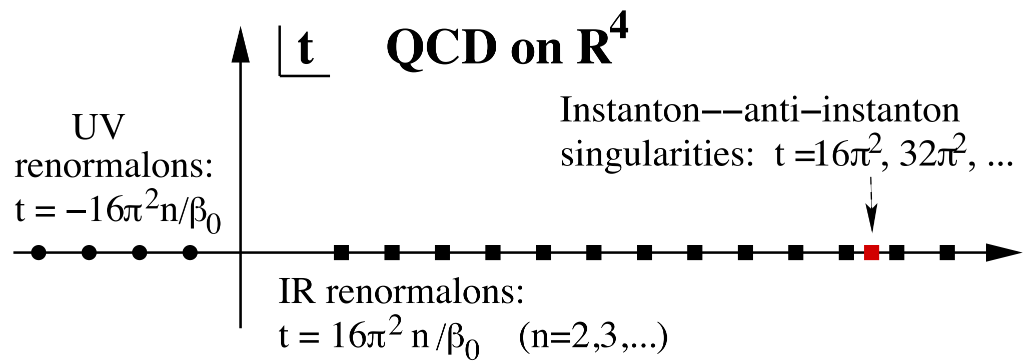
$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Im}[\mathcal{B}_{ii}]_\pm \, , \qquad (\text{up to } e^{-4S_0}) \qquad \mathrm{YM, CP(N-1)}$$
$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Im}[\mathcal{B}_{ij}\overline{\mathcal{B}}_{ij}]_\pm \, , \qquad (\text{up to } e^{-6S_0}) \qquad \mathrm{QCD(adj)}$$

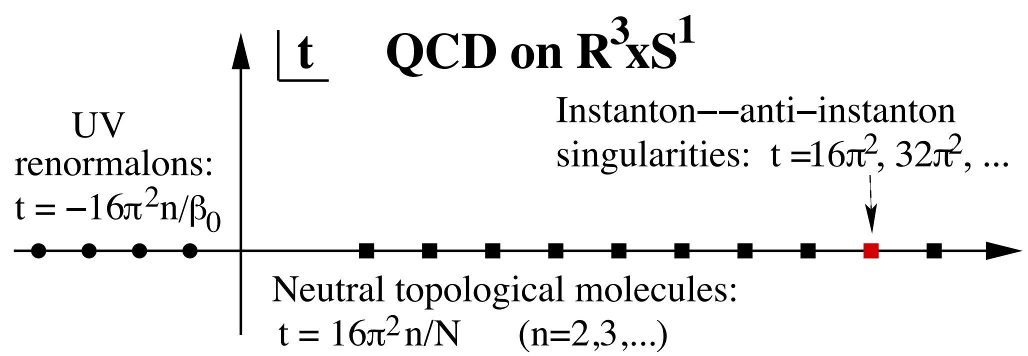
$$\mathrm{Im}[\mathcal{B}_{ii}]_\pm = \mathrm{Im}[\mathcal{M}_i \overline{\mathcal{M}}_i]_\pm \qquad \mathrm{Im}[\mathcal{B}_{ij}\overline{\mathcal{B}}_{ij}]_\pm = \mathrm{Im}[\mathcal{M}_i \overline{\mathcal{M}}_j \mathcal{M}_j \overline{\mathcal{M}}_i]_\pm$$

The ambiguities at order exp[-2S$_I$/N] cancel and
QFT is well-defined up to the ambiguities of order exp[-4S$_I$/N] !
Ambiguities in the IR-renormalon territory as per 't Hooft, David, Beneke,....

Semi-classical renormalons as neutral bions

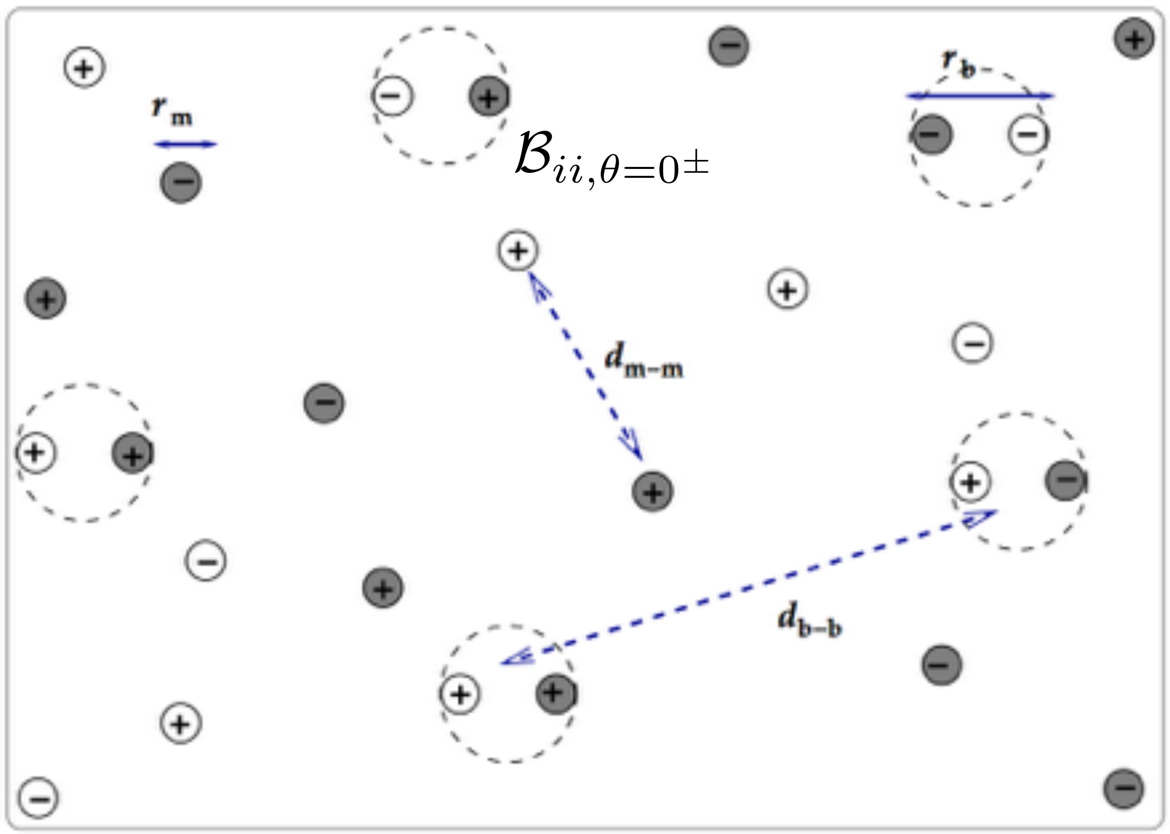
Claim (with Argyres in 4d) and (with Dunne in 2d): Neutral bions and neutral topological molecules are semi-classical realization of ‘t Hooft’s elusive renormalons, and it is possible to make sense out of combined perturbative semi-classical expansion. We showed this only at leading (but most important) order for 2d sigma models, but it is conjectural in 4d.



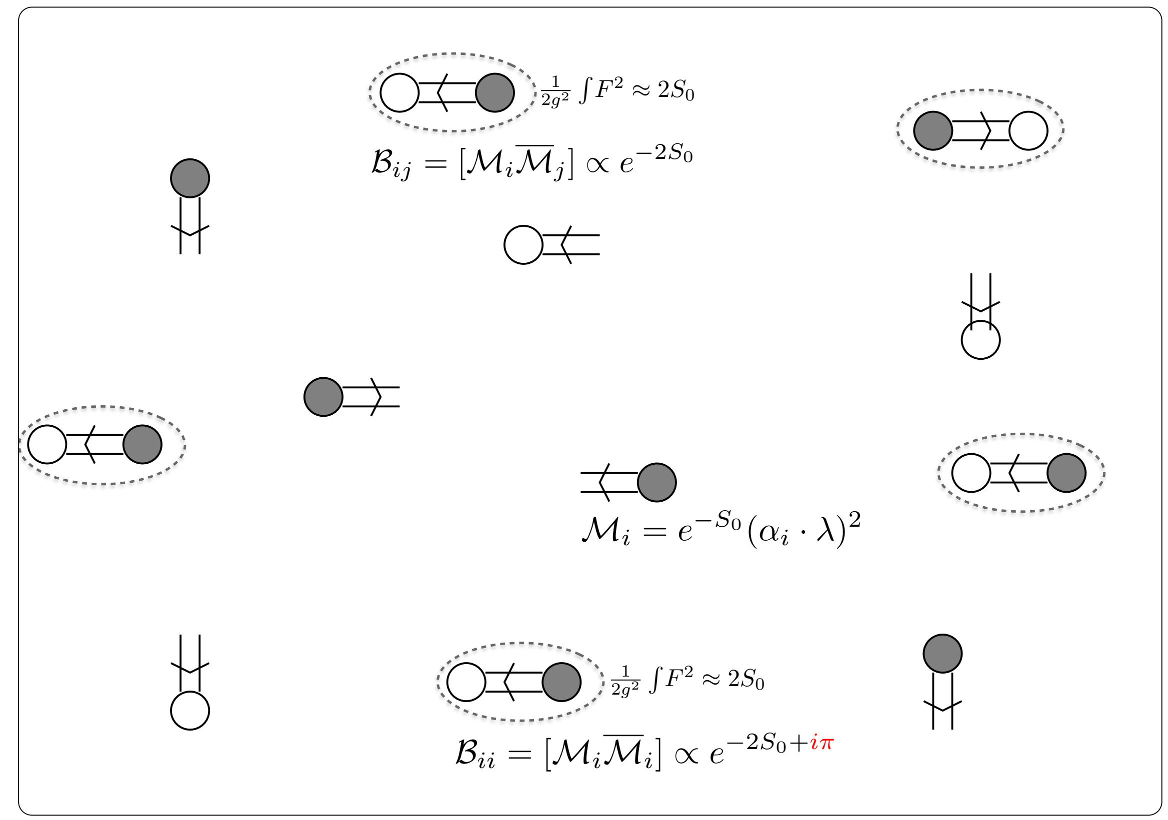


Our work is the concrete realization of link between two different deep ideas, and two man in the picture. monopole-instantons vs. renormalons!

More than three decades ago, ‘t Hooft gave a famous set of (brilliant) lectures(79): *Can we make sense out of QCD?* He was thinking a non-perturbative continuum formulation. It seem plausible to me that, we can do so, at least, in the semi-classical regime of QFT. (Comment on Sulejmanpasic-Anber) deformed YM, Euclidean vacuum



N=1 SYM, Euclidean vacuum



Relation to R4?

$$\langle F^2 \rangle_{0^\pm} \propto \mathcal{M}_i + [\mathcal{M}_i \bar{\mathcal{M}}_j] + [\mathcal{M}_i \bar{\mathcal{M}}_i]_{0^\pm} + \dots$$

Ambiguity in condensate sourced by neutral bion.

$$\langle F^2 \rangle \propto 0 \times n_{\mathcal{M}_i} + (n_{\mathcal{B}_{ij}} + e^{i\pi} n_{\mathcal{B}_{ii}}) = 0\,.$$

Condensate vanishes, due to a hidden topological angle. (related to stationary phase associated with thimbles). First micro-realization of a negative contribution to condensate!

Graded Resurgence triangle for dYM

The structure of dYM and many QFTs

pert. theory around center symmetric vacuum

monopole x (pert. fluctuations)

bions x (pert. fluctuations)

$$
\begin{array}{ccccccccc}
 & & & & f_{(0,0)} & & & & \\
 & & & e^{-\frac{A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(1,1)} & & e^{-\frac{A}{\lambda}-i\frac{\widetilde{\Theta}_k}{N}} f_{(1,-1)} & & & \\
 & & e^{-\frac{2A}{\lambda}+2i\frac{\widetilde{\Theta}_k}{N}} f_{(2,2)} & & e^{-\frac{2A}{\lambda}} f_{(2,0)} & & e^{-\frac{2A}{\lambda}-2i\frac{\widetilde{\Theta}_k}{N}} f_{(2,-2)} & & \\
 & e^{-\frac{3A}{\lambda}+3i\frac{\widetilde{\Theta}_k}{N}} f_{(3,3)} & & e^{-\frac{3A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(3,1)} & & e^{-\frac{3A}{\lambda}-i\frac{\widetilde{\Theta}_k}{N}} f_{(3,-1)} & & e^{-\frac{3A}{\lambda}-3i\frac{\widetilde{\Theta}_k}{N}} f_{(3,-3)} & \\
e^{-\frac{4A}{\lambda}+4i\frac{\widetilde{\Theta}_k}{N}} f_{(4,4)} & & e^{-\frac{4A}{\lambda}+2i\frac{\widetilde{\Theta}_k}{N}} f_{(4,2)} & & e^{-\frac{4A}{\lambda}} f_{(4,0)} & & e^{-\frac{4A}{\lambda}-2i\frac{\widetilde{\Theta}_k}{N}} f_{(4,-2)} & & e^{-\frac{4A}{\lambda}-4i\frac{\widetilde{\Theta}_k}{N}} f_{(4,-4)} \\
 & \iddots & & & \vdots & & & \ddots &
\end{array}
$$

No two column can mix with each other in the sense of cancellation of ambiguities.

N.P. confluence equations

In order QFT to have a meaningful semi-classical continuum definition, a set of perturbative--non-perturbative confluence equations must hold. Examples are

$$0 = \mathrm{Im}\Big(\mathbb{B}_{[0,0],\theta=0^\pm} + \mathbb{B}_{[2,0],\theta=0^\pm}[\mathcal{B}_{ii}]_{\theta=0^\pm} + \mathbb{B}_{[4,0],\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_{\theta=0^\pm} + \mathbb{B}_{[6,0]\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{jk}\mathcal{B}_{ki}]_{\theta=0^\pm} + \ldots\Big)$$

Meaning, order by order hierarchical confluence equations:

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Re}\mathbb{B}_{[2,0]}\mathrm{Im}[\mathcal{B}_{ii}]_\pm\ , \qquad (\text{up to } e^{-4S_0})$$

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Re}\mathbb{B}_{[2,0]}\mathrm{Im}[\mathcal{B}_{ii}]_\pm + \mathrm{Im}\mathbb{B}_{[2,0]\pm}\mathrm{Re}[\mathcal{B}_{ii}] + \mathrm{Re}\mathbb{B}_{[4,0]}\mathrm{Im}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_\pm \qquad (\text{up to } e^{-6S_0})$$

$$0 = \ldots$$

Similar to The reality of resurgent transseries by Schiappa, and Aniceto, 2013

Decoding late terms in pert. theory.

$$\mathrm{Disc}\,\mathbb{B}_{[0,0]} = -2\pi i \lambda^{-r_2} P_{[2,0]} e^{-2A/\lambda} + \mathcal{O}(e^{-4A/\lambda}), \tag{1}$$

Using dispersion relation, we obtain

$$\begin{aligned} a_{[0,0],q} &= \sum_{q'=0}^{\infty} a_{[2,0],q'} \frac{\Gamma(q+r_2-q')}{(2A)^{q+r_2-q'}} + O\left(\left(\frac{1}{4A}\right)^q\right) \\ &= \frac{\Gamma(q+r_2-q')}{(2A)^{q+r_2}} \left[a_{[2,0],0} + \frac{2A}{(q+r_2-1)} a_{[2,0],1} + \frac{(2A)^2}{(q+r_2-1)(q+r_2-2)} a_{[2,0],2} + \dots \right] \\ &+ O\left(\left(\frac{1}{4A}\right)^q\right) \end{aligned} \tag{2}$$

Late terms in pert.exp. around the pert. vac.

Early terms in pert.exp. around neutral bion= 1/q corrections:

Neutral bion action

Exponentially suppressed corrections: Bion-bion etc. terms.

Resurgence theory in path integrals

Key step is in the analytic continuation of paths in field space (cf. Pham, and recent papers by Witten), to make sense of steepest descent and Stokes phenomenon in path integrals. (We actually use this implicitly, but need to make it more systematic.)

cf. a recent talk by Kontsevich "Resurgence from the path integral perspective", Perimeter Institute, August, 2012.

Work in progress: Basar, Dunne, MU,
Also, recent works by Y. Tanizaki, T. Kanazawa, 2014, w/ Cherman, 2014
T. Misumi et.al. , 2015, to appear.

Lefschetz thimbles vs. resurgence, and puzzles.

- Both heavily depend on the behavior of the theory upon analytic continuation, where asymptotic expansions are consistent with analytic continuation properties. (both taking into account Stokes phenomena)

- Witten shows that if gradient flow equations (GFE) in field theory are elliptic, nice properties of finite dimensional case carry over to infinite dimensions. (determination of the Stokes multipliers of the thimbles.)

- Based on our communications with him, he thinks that for parabolic gradient flow equations (GFE), there is no such simplicity.

- Almost all interesting QFTs have parabolic GFE, including QM in configuration space. CS and QM in phase space has elliptic GFE, and are exceptional cases.

- All QM and QFT examples we studied to date (non-linear sigma models, QM, non-abelian gauge theories) have parabolic GFE, and resurgence seems to be working fine!

Picard-Lefschetz equations for YM theory

Reminder: If S(A) is Chern-Simons functional in **3d,** the flow equations are **4d** instanton equations. This is an infinite dimensional version of real Morse theory (in field space.) Crucial in Floer homology.

If S(A) is a complex Chern-Simons functional, Picard-Lefschetz equations gives a complex generalization of 4d instanton equation:

$$\mathcal{F}_{\mu\nu} + e^{-i\theta}(\star\overline{\mathcal{F}})_{\mu\nu} = 0$$

Equation appeared first in other contexts
MU, 06 in lattice-susy, hep-th/0603046
Kapustin-Witten 06 Geometric Langland, hep-th/0604151
(same derivation!)

In our case:

$$\frac{d\mathcal{A}^\mu}{dt} = -e^{-i\theta}\frac{\partial \bar{S}}{\partial \bar{\mathcal{A}}_\mu} \qquad \mathcal{A}_\mu \in SL(N,\mathbb{C})$$

$$\frac{d\mathcal{A}^\mu}{dt} = -e^{-i\theta}\bar{\mathcal{D}}_\nu \bar{\mathcal{F}}^{\nu\mu}$$

Fixed points of flow are monopole-instantons, bions, etc. Im(S) conserved over thimble, and has physical consequences.

Attach a down-ward flow manifold to each one of the critical point.
Guess: In the semi-classical regime, this provides a (homology) cycle/Lefschetz thimble decomposition of the space of fields. The integrations over the homology cycles are finite by construction. There is a possibility that this provides a NP-definition, at least in the EFT sense where short distance is integrated out.

Why do we think that resurgence triangle may be a complete NP description in semi-classical domain?

- Because every non-perturbative quantity we calculated so far is qualitatively consistent with lattice field theory or exact large-N results or mixed 't Hooft anomalies. No single exception.

- Take Berry-Holws multi-dimensional generalization of hyperasymptotics/resurgence. The structure that we found in QM with Borel singularities on positive and negative real axis is compatible with their finite dimensional studies.

- Witten et.al. claim to give as a non-working example, Liouville theory, for which GFE is parabolic. But some of us think that (including et.al part) the evidence provided in their section 5 is in favor of complete semi-classical decomposition of exact known results.

Why do we think that resurgence triangle may be a complete NP description in semi-classical domain?

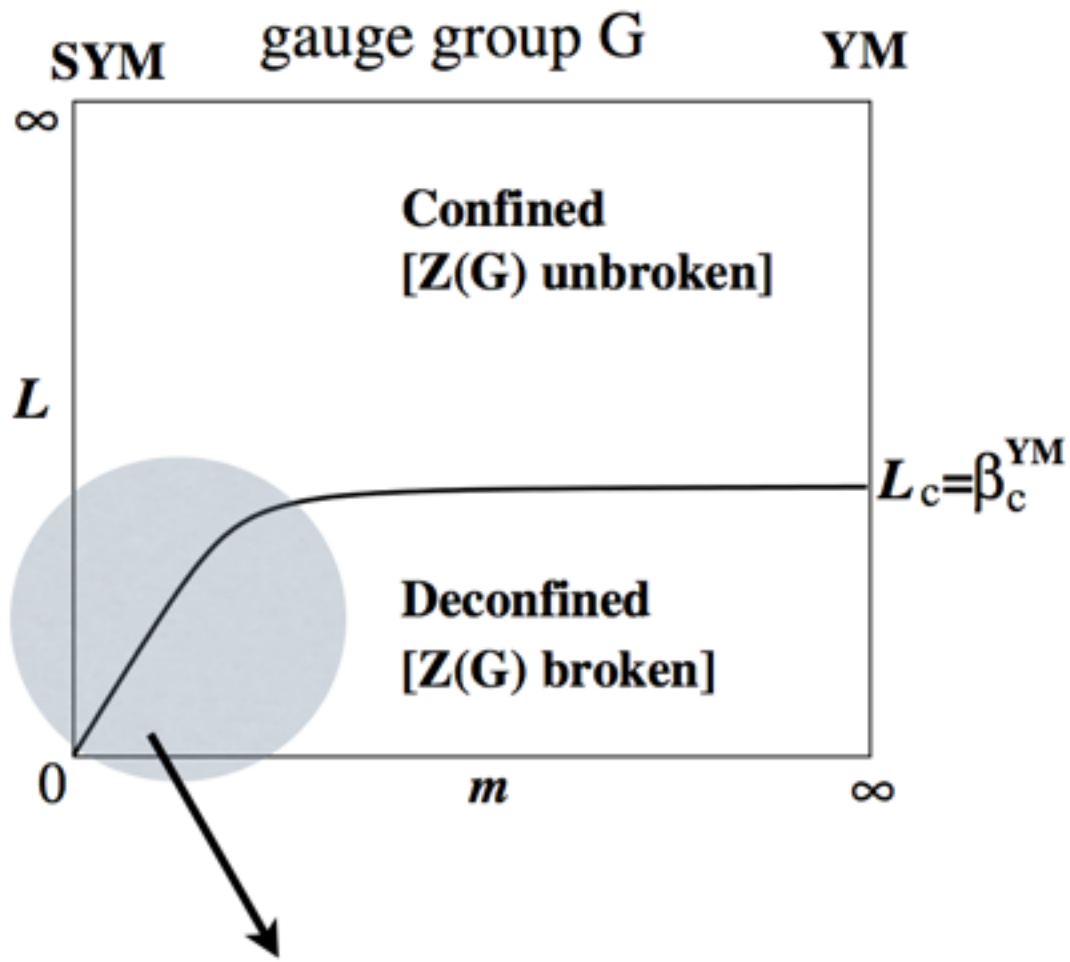
- Consider an extremely non-trivial non-perturbative phenomenon. Center symmetry changing deconfinement phase transition in Yang-Mills theory, as well as SYM theory on R3 x S1.

- This transition proved to be impossible to study by using continuum methods (except for models) and is a manifestly non-perturbative phenomenon. To date, no reliable semi-classical method existed.

- Lattice simulations can be used to see this phase transition, and provide its detailed description.

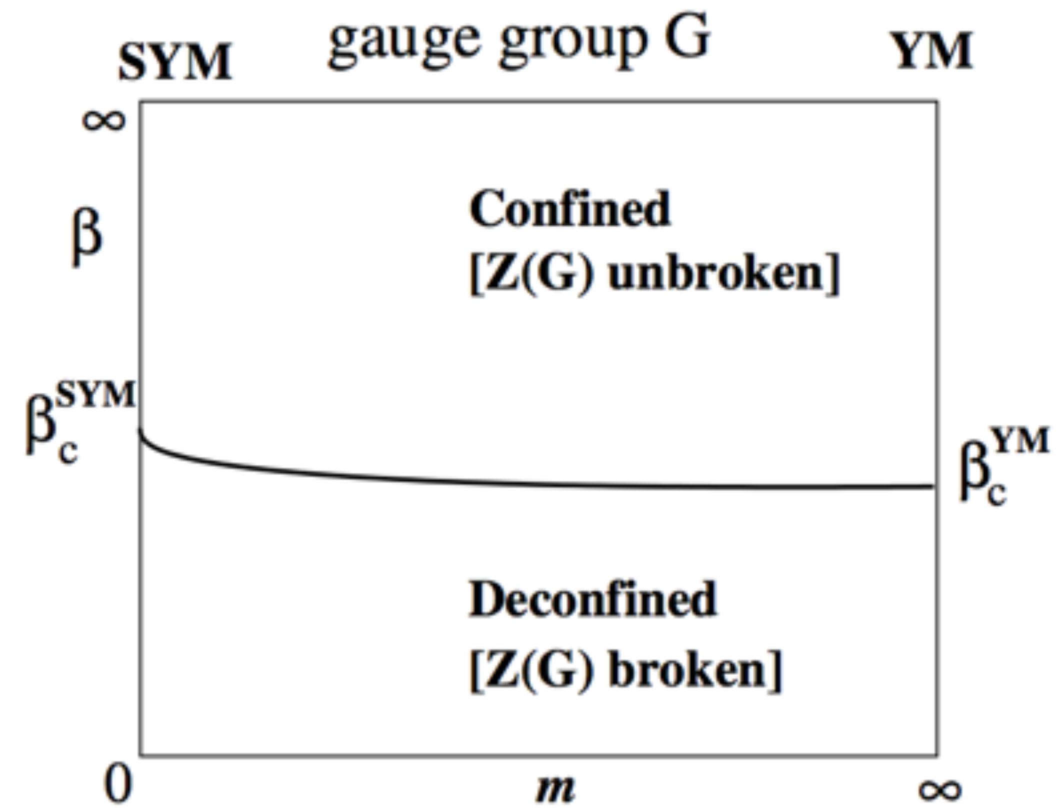
Yang-Mills theory with gauge group G and one Weyl fermion mass m, phase diagram in L-m plane

Spatial compactification



NP-Calculable,
in g^2 and $\exp[-A/g^2]$.
NP-effects
under control !

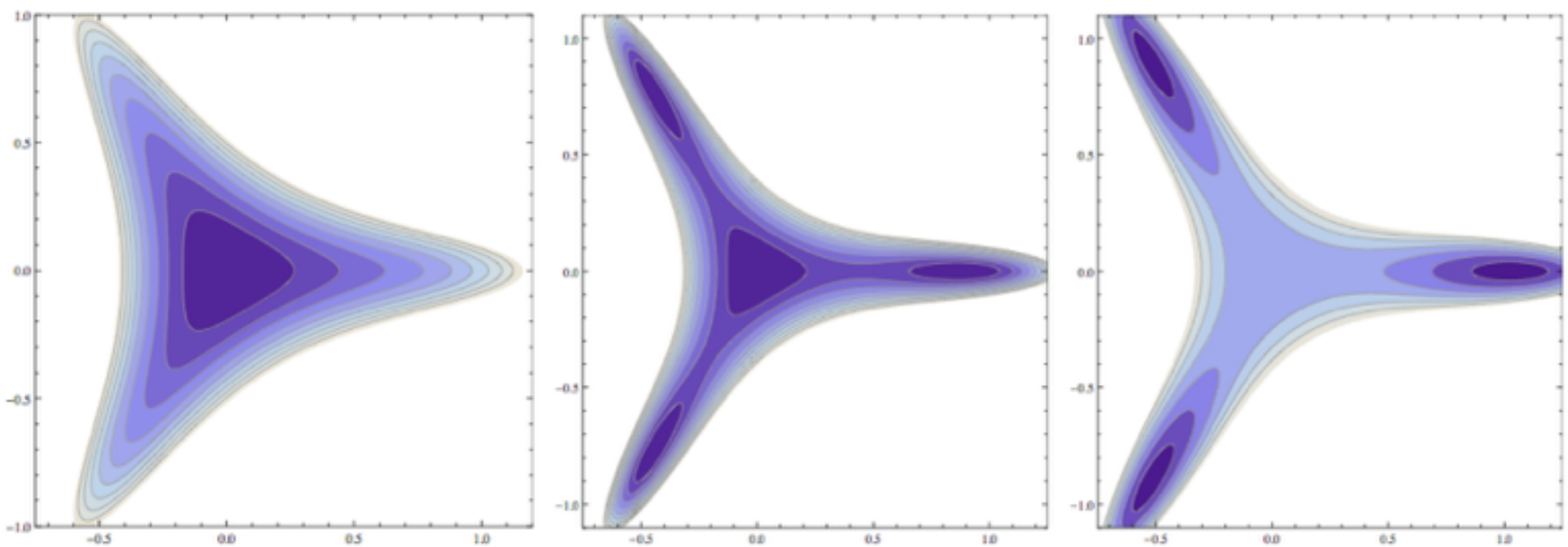
Thermal compactification



w/ Poppitz-Schäfer, 2012

SU(3): Extrama of the NP-potential for trace of Wilson line, large-L to small-L.

Not a model; no tuned parameters, result of justified semi-classics.



Confined

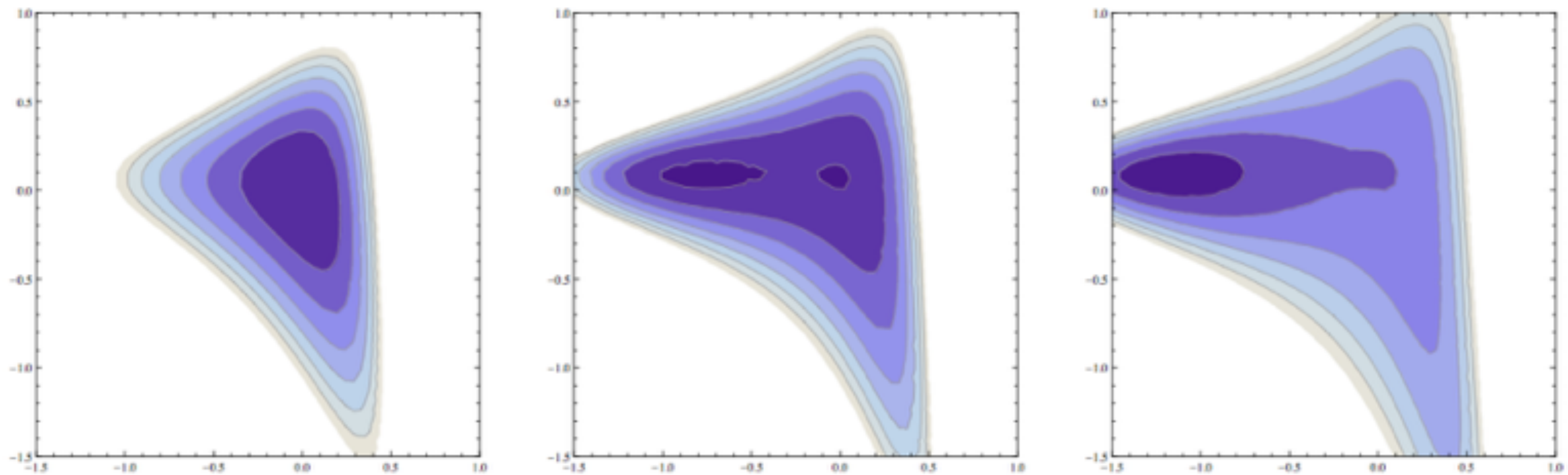
between the limits of metastability

Deconfined

Due to competition between different order terms in transseries expansion of free energy!

G$_2$: Extrama of the NP-potential for trace of Wilson line, large-L to small-L

G_2: First order transition without change of symmetry.



Both qualitatively and quantitavely (numerical ratio of jump) identical to LGT result for YM.

LGT
[1] K. Holland, P. Minkowski, M. Pepe and U. J. Wiese,
[2] M. Pepe and U. -J. Wiese,
[3] J. Greensite, K. Langfeld, S. Olejnik, H. Reinhardt and T. Tok,
[4] G. Cossu, M. D'Elia, A. Di Giacomo, B. Lucini and C. Pica,

Due to competition between different order terms in transseries expansion of free energy!

$$f_{(0,0)}$$

$$e^{-\frac{A}{\lambda}+i\frac{\tilde{\Theta}_k}{N}} f_{(1,1)} \qquad e^{-\frac{A}{\lambda}-i\frac{\tilde{\Theta}_k}{N}} f_{(1,-1)}$$

$$e^{-\frac{2A}{\lambda}+2i\frac{\tilde{\Theta}_k}{N}} f_{(2,2)} \qquad e^{-\frac{2A}{\lambda}} f_{(2,0)} \qquad e^{-\frac{2A}{\lambda}-2i\frac{\tilde{\Theta}_k}{N}} f_{(2,-2)}$$

$$e^{-\frac{3A}{\lambda}+3i\frac{\tilde{\Theta}_k}{N}} f_{(3,3)} \qquad e^{-\frac{3A}{\lambda}+i\frac{\tilde{\Theta}_k}{N}} f_{(3,1)} \qquad e^{-\frac{3A}{\lambda}-i\frac{\tilde{\Theta}_k}{N}} f_{(3,-1)} \qquad e^{-\frac{3A}{\lambda}-3i\frac{\tilde{\Theta}_k}{N}} f_{(3,-3)}$$

$$e^{-\frac{4A}{\lambda}+4i\frac{\tilde{\Theta}_k}{N}} f_{(4,4)} \qquad e^{-\frac{4A}{\lambda}+2i\frac{\tilde{\Theta}_k}{N}} f_{(4,2)} \qquad e^{-\frac{4A}{\lambda}} f_{(4,0)} \qquad e^{-\frac{4A}{\lambda}-2i\frac{\tilde{\Theta}_k}{N}} f_{(4,-2)} \qquad e^{-\frac{4A}{\lambda}-4i\frac{\tilde{\Theta}_k}{N}} f_{(4,-4)}$$

$$\iddots \qquad \vdots \qquad \ddots$$

Perturbative potential for Polyakov loop: O(m^2)
Monopole-induced potential: m^1 exp[-A/g^2]
Neutral bion induced potential: exp[-2A/g^2]

Conclusions

Continuity and resurgence theory can be used in combination to provide a possibly non-perturbative continuum definition of asymptotically free theories, and more general QFTs. All must be (and is so far) with 't Hooft anomalies.

In simple cases, Lefschetz thimbles is geometrization of resurgence. There is a possibility that this may be true for non-trivial path integrals. Regardless, resurgent trans-series seems to be capturing both perturbative and non-perturbative properties correctly.

The construction will have practical utility and region of overlap with lattice field theory. One can check predictions of the formalism numerically.