Reconstruction of the tunneling amplitude from the perturbation series

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- H.S. and H. Yasuta, “Observing quantum tunneling in perturbation series,” Phys. Lett. B **400**, 341 (1997) [hep-th/9612165].
- H.S. and H. Yasuta, “Quantum bubble nucleation beyond WKB: Resummation of vacuum bubble diagrams,” Phys. Rev. D **57**, 2500 (1998) [hep-th/9704105].

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- The *delta expansion* (Seznec-Zinn-Justin (1979), Kleinert (1992), Bender-Duncan-Jones (1993), and many others) applied to the anharmonic oscillator:

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- The real answer E_0 should not depend on Ω, but the finite order approximation $E_0^{(N)}$ *does*.
- So, fix Ω order by order.

Various criteria to fix Ω order by order

- Principle of minimal sensitivity (Stevenson (1981)):

$$\frac{\partial E_0^{(N)}}{\partial \Omega} = 0.$$

- Fastest apparent convergence (Duncan-Jones (1992)):

$$E_0^{(N)} - E_0^{(N-1)} = 0.$$

- Scaled delta expansion (Guida-Konishi-H.S. (1994)):

$$\Omega = \omega C N^{\gamma}.$$

The delta expansion can be constructed from PT

- Note that the delta expansion can be constructed by simply making the substitutions,
$$\omega^2 \to \Omega^2 + \delta(\omega^2 - \Omega^2), \qquad g \to \delta g,$$
in the conventional perturbation series,
$$E_0 \sim \omega \sum_{n=0}^{\infty} c_n \left(\frac{g}{\omega^3}\right)^n,$$
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- We find
$$E_0^{(N)} = \omega \sum_{n=0}^{N} c_n \left(\frac{g}{\omega^3}\right)^n \left(\frac{\omega}{\Omega}\right)^{3n-1} \sum_{k=0}^{N-n} \left(1 - \frac{\omega^2}{\Omega^2}\right)^k \frac{\Gamma(3n/2 + k - 1/2)}{\Gamma(3n/2 - 1/2)\Gamma(k+1)}.$$

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- Perturbation series to order N suffices!

Conventional perturbation series

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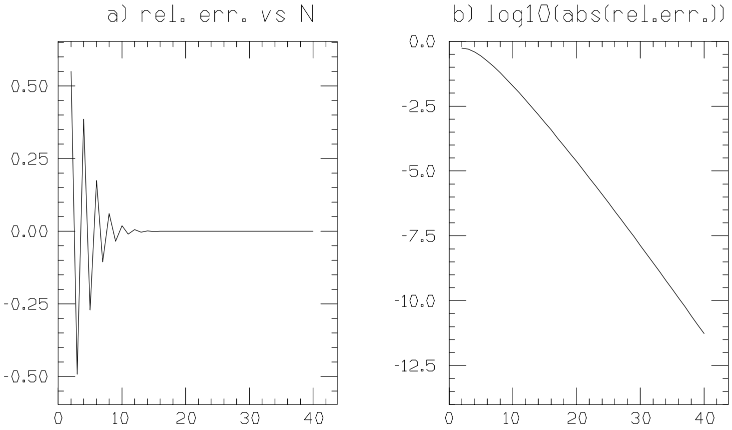
$$E_0 \sim \omega \sum_{n=0}^{\infty} c_n \left(\frac{g}{\omega^3}\right)^n,$$

is terribly diverging as (here, I used the method of Bender-Wu (1969))

$$c_0 = \frac{1}{2}, \quad c_1 = \frac{3}{16}, \quad c_2 = -\frac{21}{128}, \quad c_3 = \frac{333}{1024}, \quad c_4 = -\frac{30885}{32768}, \quad c_5 = \frac{916731}{262144},$$
$$c_6 = -\frac{65518401}{4194304}, \quad c_7 = \frac{2723294673}{33554432}, \quad c_8 = -\frac{1030495099053}{2147483648}, \quad c_9 = \frac{54626982511455}{17179869184},$$
$$c_{10} = -\frac{6417007431590595}{274877906944}, \quad c_{11} = \frac{413837985580636167}{2199023255552}, \quad c_{12} = -\frac{116344863173284543665}{70368744177664},$$
$$c_{13} = \frac{8855406003085477228503}{562949953421312}, \quad c_{14} = -\frac{1451836748576538293163705}{9007199254740992},$$
$$c_{15} = \frac{127561682802713500067360049}{72057594037927936}, \quad c_{16} = -\frac{191385927852560927887828084605}{9223372036854775808},$$
$$c_{17} = \frac{19080610783320698048964226601511}{73786976294838206464}, \quad c_{18} = -\frac{4031194983593309788607032686292335}{1180591620717411303424},$$
$$c_{19} = \frac{449820604540765836160529697491458635}{9444732965739290427392},$$
$$c_{20} = -\frac{211491057584560795425148309663914344715}{302231454903657293676544}, \ldots$$

The (scaled) delta expansion converges very quickly!

- The relative error for $g/\omega^3 = 4.0$, $\gamma = 0.35$ (Guida-Konishi-H.S. (1994)): Similar behavior is observed for other criteria for Ω.



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$$\Omega = \omega C N^{\gamma},$$

and

$$\frac{1}{3} < \gamma < \frac{1}{2}, \qquad C > 0,$$

or

$$\gamma = \frac{1}{3}, \qquad C \geq \alpha_c g^{1/3}, \qquad \alpha_c \simeq 0.5708751028937741,$$

then $E_K^{(N)}$ (K: energy level) converges to the exact value E_K.

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- One can show that the error $|E_K - E_K^{(N)}| \to 0$ as $N \to 0$ by using an exact expression for the energy (Loeffel-Martin-Simon-Wightman (1968)):

$$E_K = c_0 \omega + \frac{g}{\pi} \int_{-\infty}^{0} dg' \, \frac{\operatorname{Im} E_K(g')}{g'(g' - g)}.$$

The order dependent mapping method (Seznec-Zinn-Justin (1979), Le Guillou-Zinn-Justin (1983))

- Introduce a conformal mapping in the coupling constant place:

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- The convergence proof applies: $\rho = gC'/N^{\gamma'}$ with

$$1 < \gamma' < \frac{3}{2}, \quad C' > 0, \qquad \text{or} \qquad \gamma' = 1, \quad C' \leq 1/\alpha_c^3.$$

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- Convergence can be proven for $g > g_0 \simeq 0.1$ ($g_0^{-2/3}$ is the radius of convergence of the strong coupling expansion due to the Bender-Wu singularity).
- It thus cannot replace the WKB/bounce calculus for weak couplings $g \ll 1$:

$$\mathrm{Im}\, E_0 \sim -\sqrt{\frac{8}{\pi g}} \exp\left(-\frac{4}{3g}\right) \left(1 - \frac{95}{96}g + O(g^2)\right).$$

Tunneling vs perturbation theory?

- The WKB/bounce formula,

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- If so, we may take "perturbation theory monism"（摂動論一元論）at least in this system; PT should saturate.

Tunneling vs perturbation theory?

- In fact, the WKB/bounce formula for $g \ll 1$ tells us the large order behavior of PT:

$$E_0 \sim \sum_{n=0}^{\infty} c_n g^n, \quad c_n \overset{n\to\infty}{\sim} -\sqrt{\frac{6}{\pi^3}}\left(\frac{3}{4}\right)^n \Gamma(n+1/2).$$

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- The saturation of PT suggests:

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- We have to resolve the mixing between the interaction among bounces and PT around bounces; ex. 2 bounces (Bogomolny, Zinn-Justin)

$$\mathrm{Im}\, E_0|_{2\text{ bounces}} \sim \pm\frac{6}{g}\exp\left(-\frac{8}{3g}\right).$$

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- $\Leftarrow$ Resurgence!?

Im E_0 from c_n

- The Borel transform

$$B(z) \equiv \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n$$

possess a fractional branch point

$$B(z) \sim -\frac{2\sqrt{2}}{\pi}(z_0 - z)^{-1/2} + \cdots , \quad z_0 = \frac{4}{3}$$

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- The Borel integral along the positive real axis,

$$E_0 = \frac{1}{g}\int_0^{\infty} dz\, e^{-z/g} B(z),$$

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- Here, we implicitly assumed that all singularities on the Borel transform are on the real positive axis.

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- To avoid this analytic continuation, we consider a conformal mapping on the Borel z plane (Loeffel (1976)):

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- The singularity z_0 is mapped to $\lambda = 1$ and the radius of convergence of

$$B(z) = \sum_{k=0}^{\infty} d_k \lambda^n, \quad d_k = \sum_{n=0}^{k} (-1)^{k-n} \frac{\Gamma(k+n)}{(k-n)!\Gamma(2n)} (4z_0)^n \frac{c_n}{n!},$$

is now 1.

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- The radius of convergence of the sum

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- Parametrizing $\lambda = e^{i\theta}$, we arrive at

$$\mathrm{Im}\, E_0 = \frac{z_0}{g} \int_0^{\infty} d\theta \, \exp\left(-\frac{z_0}{g} \frac{1}{\cos^2 \theta/2}\right) \frac{\sin \theta/2}{\cos^3 \theta/2} \sum_{k=0}^{N} d_k \sin k\theta,$$

  $z_0 = 4/3$ is the bounce action.

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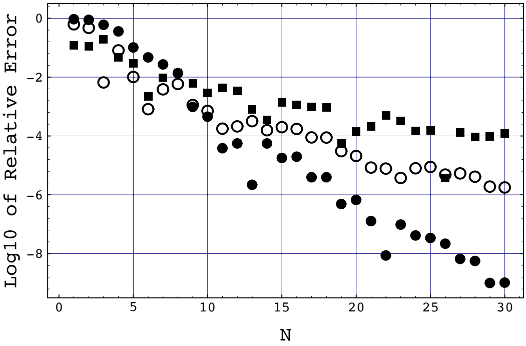
  $z_0=4/3$ is the bounce action.
- This is our formula.

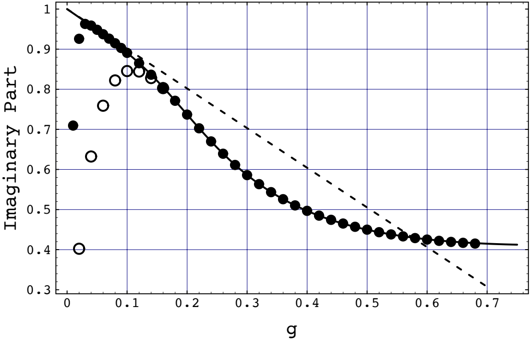
The order N vs the logarithm of the relative error

- $\bullet\ g = 0.08$, $\circ\ g = 0.3$, $\square\ g = 0.6$.



g vs Im E_0 normalized by the leading bounce calculus

- ○ $N = 5$, • $N = 15$, the solid line; exact value, the broken line; the two-loop bounce.



Extension to the field theory

- D-dimensional $O(N)$ symmetric $\lambda\phi^4$-theory:

$$H = \int d^{D-1}x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{1}{4!}(\phi^2)^2\right], \qquad g > 0.$$

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- The leading bounce calculus gives rise to (setting $m = 1$)

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- Corresponding to this imaginary part, the Borel transform

$$B(z) = \sum_{n=0}^{\infty}\frac{c_n}{\Gamma(n+(D+N)/2)}z^n, \qquad \mathcal{E} \sim \sum_{n=0}^{\infty} c_n g^n,$$

develops the square-root branch point,

$$B(z) = -\frac{1}{\sqrt{\pi}}A_N C_{D,N}\frac{S_0^{1/2}}{(2\pi)^{(D+N-1)/2}}(S_0 - z)^{-1/2} + \cdots.$$

Extension to the field theory

- Assuming that the branch point is the singularity closest to the origin, using the conformal mapping trick, we arrive at

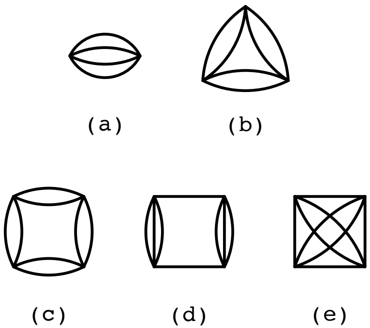
$$\operatorname{Im}\mathcal{E} = \left(\frac{S_0}{g}\right)^{(D+N)/2} \int_0^\pi d\theta \, \exp\left(-\frac{S_0}{g}\frac{1}{\cos^2\theta/2}\right) \times \frac{\sin\theta/2}{\cos^{D+N+1}\theta/2} \sum_{k=0}^{N} d_k \sin k\theta,$$

$$d_k = \sum_{n=0}^{k} (-1)^{k-n} \frac{\Gamma(k+n)(4S_0)^n}{(k-n)!\Gamma(2n)\Gamma(n+(D+N)/2)} c_n,$$

where

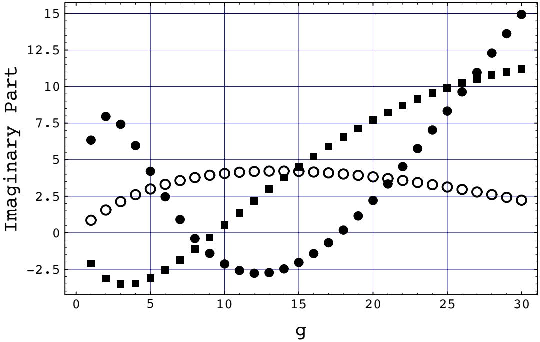
$$S_0 = \begin{cases} 8, & D=1, \\ 35.10269, & D=2, \\ 113.38351, & D=3. \end{cases}$$

- We calculated vacuum bubble diagrams to five loops.



Result for $D = 2$ and $N = 1$

- g vs $\mathrm{Im}\,\mathcal{E}_0$ normalized by the leading bounce calculus.
- $\circ$ $N = 2$, ■ $N = 3$, $\bullet$ $N = 4$.



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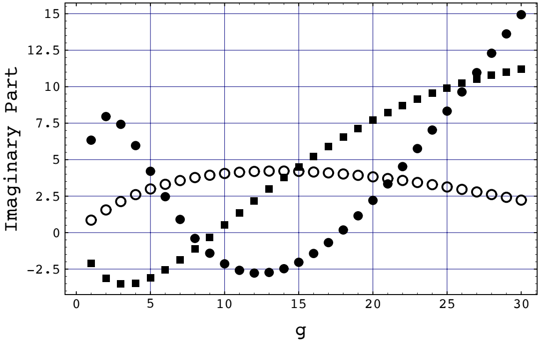

Gaussian propagator model (H. Yasuta, Phys. Lett. B **418**, 145 (1998) [hep-th/9707161])

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- The formula

$$\operatorname{Im} \mathcal{E} = \left(\frac{A_0}{g} \right)^{(D+1)/2} \int_0^\pi d\theta \, \exp \left[- \frac{A_0}{g} \frac{1}{\cos^2(\theta/2)} \right] \times \frac{\sin(\theta/2)}{\cos^{D+2}(\theta/2)} \sum_{k=0}^{N} d_k \sin(k\theta),$$

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Result for $D = 1$

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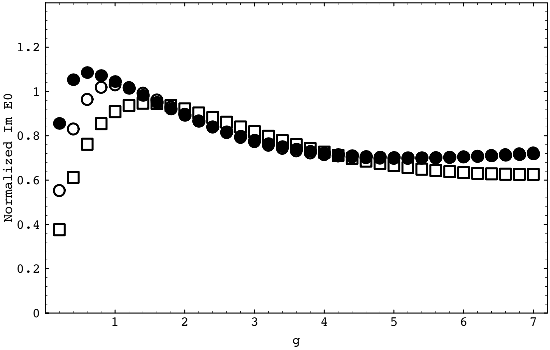

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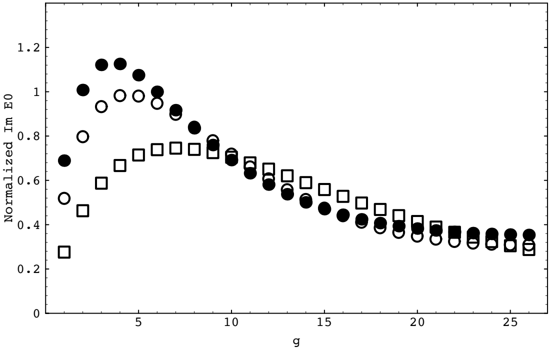

Result for $D = 3$

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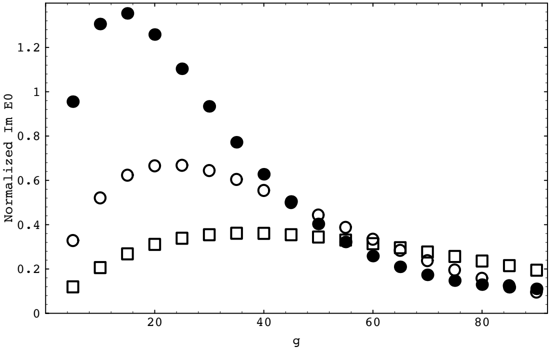

Result for $D = 4$

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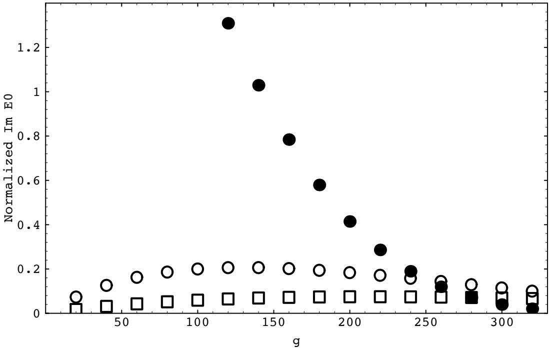

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- Presumably, it is now the time to reflect the logic, especially on the effect of the renormalization. . .