Rrenormalization-group Construction
of
the Invariant/Attractive Manifold
for
Regular and Stochastic Dynamics

Teiji Kunihiro (Kyoto)

Resurgence 2017
Kobe, Sep. 6 –Sep.8, 2017

Contents

1. A simple example with secular terms

$$\frac{d^2x}{dt^2} + \epsilon\frac{dx}{dt} + x = 0, \qquad \text{the dumped oscillator!}$$

Exact sol.: $x(t) = A(t)\sin\phi(t), \; A(t) = \bar{A}\exp(-\epsilon t/2), \; \phi(t) = \omega t + \bar{\theta}$

$$\omega \equiv \sqrt{1-\epsilon^2/4}$$

Pert. Sol.:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

$$\ddot{x}_0 + x_0 = 0, \qquad \ddot{x}_{n+1} + x_{n+1} = -\dot{x}_n.$$

$$x_0 = A\sin(t+\theta)$$

$$\mathcal{L}x_1 \equiv \ddot{x}_1 + x_1 = -\dot{x}_0 = -A\cos(t+\theta),$$

$$\mathcal{L} \equiv d^2/dt^2 + 1$$

Zero mode!

secular terms appear.

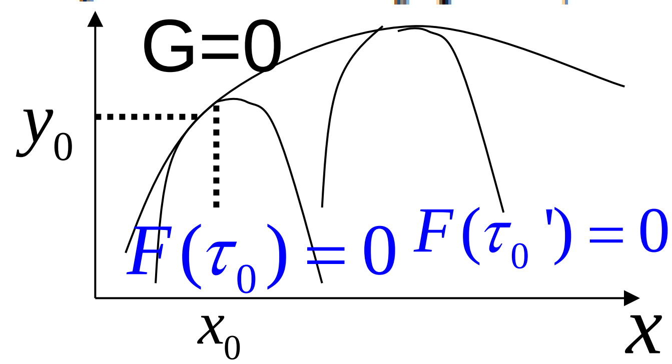
$$\begin{aligned} x(t) &= A\sin(t+\theta) - \epsilon\frac{A}{2}t\sin(t+\theta) + \epsilon^2\frac{A}{8}\{t^2\sin(t+\theta) - t\cos(t+\theta)\} + \ldots\ldots \\ &\simeq A(1-\epsilon/2\cdot t + \epsilon^2/8\cdot t^2)\sin((1-\epsilon^2/8)t + \theta) \\ &\simeq A\exp(-\epsilon t/2)\sin(\sqrt{1-\epsilon^2/4}t + \theta) \end{aligned}$$

The secular terms are renormalized into the slow time-dep. of the amplitude and the phase.

2. Notion of envelops of a family of curves as a key ingredient for the resummation and construction of a global solution.

T.K.Prog. Theor. Phys. ('95)

Let $\{C_\tau\}_\tau$ be a family of curves parametrized by τ in the x-y plane;



$$C_\tau : F(x, y, \tau, \mathbf{C}(\tau)) = 0 \; ?$$

E: The envelope of C_τ $G(x, y) = 0.$

$$F_{\tau_0}(x_0, y_0, \tau_0, C(\tau_0)) \equiv \frac{\partial F(x_0, y_0, \tau_0)}{\partial \tau_0} + \frac{\partial C}{\partial \tau_0} \frac{\partial F(x_0, y_0, \tau_0, C(\tau_0))}{\partial C} = 0.$$

The envelop equation: $dF / d\tau_0 = 0$ ⟺ RG eq.

the solution is inserted to F with the condition

$\tau_0 = x_0$ ⟵ the tangent point

⟹ $G(x, y) = F(x, y, \mathbf{C}(x))$

3.Resummation and extraction of the slow variables and reduced equations

T.K. ('95)

Try to construct a perturbative solution which should be valid around the arbitrary time $t = t_0$.

Perturbative exp:

$$x(t, t_0) = x_0(t, t_0) \ + \ \epsilon x_1(t, t_0) \ + \ \epsilon^2 x_2(t, t_0) \ + \ \ldots,$$

Perturbative equations: $\ddot{x}_0 + x_0 = 0, \quad \ddot{x}_{n+1} + x_{n+1} = -\dot{x}_n.$

Setting the initial value that is yet to be determined and its expantin:

$$x(t_0, t_0) = W(t_0).$$

$$W(t_0) = W_0(t_0) \ + \ \epsilon W_1(t_0) \ + \ \epsilon^2 W_2(t_0) \ + \ \ldots,$$

0-th order solution:

$$x_0(t, t_0) = A(t_0)\sin(t + \theta(t_0)), \quad W_0(t_0) = x_0(t_0, t_0) = A(t_0)\sin(t_0 + \theta(t_0)).$$

1-th order solution:

$$x_1(t, t_0) = -\frac{A}{2}\cdot(t - t_0)\sin(t + \theta), \quad W_1(t_0) = 0$$

The zero mode is added so that the secular term proportional to The zero-th solution vanishes at t=t_0. (A renormalization cond.)

$$x_2(t) = \frac{A}{8}\{(t-t_0)^2 \sin(t+\theta) - (t-t_0)\cos(t+\theta)\}, \qquad W_2(t_0) = 0$$

Secular terms appear again!

Collecting the terms, we have

$$\boxed{\begin{aligned} x(t,t_0) &= A\sin(t+\theta) - \epsilon\frac{A}{2}(t-t_0)\sin(t+\theta) \\ &+\epsilon^2\frac{A}{8}\{(t-t_0)^2\sin(t+\theta) - (t-t_0)\cos(t+\theta)\} \end{aligned}}$$

With I.C.: $W(t_0) = W_0(t_0) = A(t_0)\sin(t_0 + \theta(t_0))$

; parameterized by the functions,

$$A(t_0), \phi(t_0) \equiv t_0 + \theta(t_0)$$

The secular terms invalidate the pert. theroy, like the log-divergence in QFT!

$\{C_{t_0}\}_{t_0}$: $\{x(t,t_0)\}_{t_0}$ $\quad x_E(t) = x(t,t) = W(t).$

$$\frac{dx(t,t_0)}{dt_0} = 0, \quad t_0 = t. \quad \Longrightarrow \quad A(t_0) \text{ and } \theta(t_0)$$

$$x_2(t) = \frac{A}{8}\{(t-t_0)^2 \sin(t+\theta) - (t-t_0)\cos(t+\theta)\}, \qquad W_2(t_0) = 0$$

Secular terms appear again!

Collecting the terms, we have

$$\boxed{\begin{aligned} x(t, t_0) = & A \sin(t+\theta) - \epsilon \frac{A}{2}(t-t_0)\sin(t+\theta) \\ & + \epsilon^2 \frac{A}{8}\{(t-t_0)^2 \sin(t+\theta) - (t-t_0)\cos(t+\theta)\} \end{aligned}}$$

With I.C.: $W(t_0) = W_0(t_0) = A(t_0)\sin(t_0 + \theta(t_0))$

; parameterized by the functions,

$A(t_0), \phi(t_0) \equiv t_0 + \theta(t_0)$

Let us try to construct the envelope function of the set of locally divergent functions, Parameterized by t_0 !

$\{C_{t_0}\}_{t_0}$: $\{x(t, t_0)\}_{t_0}$ $\qquad x_E(t) = x(t, t) = W(t).$

$\frac{dx(t, t_0)}{dt_0} = 0,$ $t_0 = t.$ $\Longrightarrow$ $A(t_0)$ and $\theta(t_0)$

$$\frac{dA}{dt_0} + \epsilon A = 0, \quad \frac{d\theta}{dt_0} + \frac{\epsilon^2}{8} = 0, \quad A(t_0) = \bar{A}\mathrm{e}^{-\epsilon t_0/2}, \quad \theta(t_0) = -\frac{\epsilon^2}{8}t_0 + \bar{\theta},$$

Extracted the (slow) collective variables as the would-be integral constants, and the governing reduced equations, i.e.,
The amplitude and phase equations, separately!

The global solution as the `initial value':

$$x_{_E}(t) = x(t,t) = W_0(t) = \bar{A}\exp(-\frac{\epsilon}{2}t)\sin((1-\frac{\epsilon^2}{8})t + \bar{\theta}),$$

$$\sqrt{1-\epsilon^2/4} = 1 - \epsilon^2/8 + O(\epsilon^4)$$

The envelop function $x_{_E}(t) = W_0(t)$ an approximate but

global solution in contrast to the pertubative solutions

which have secular terms and valid only in
local domains.

Notice also the resummed nature!

Geometrical structure of the reduction of dynamics

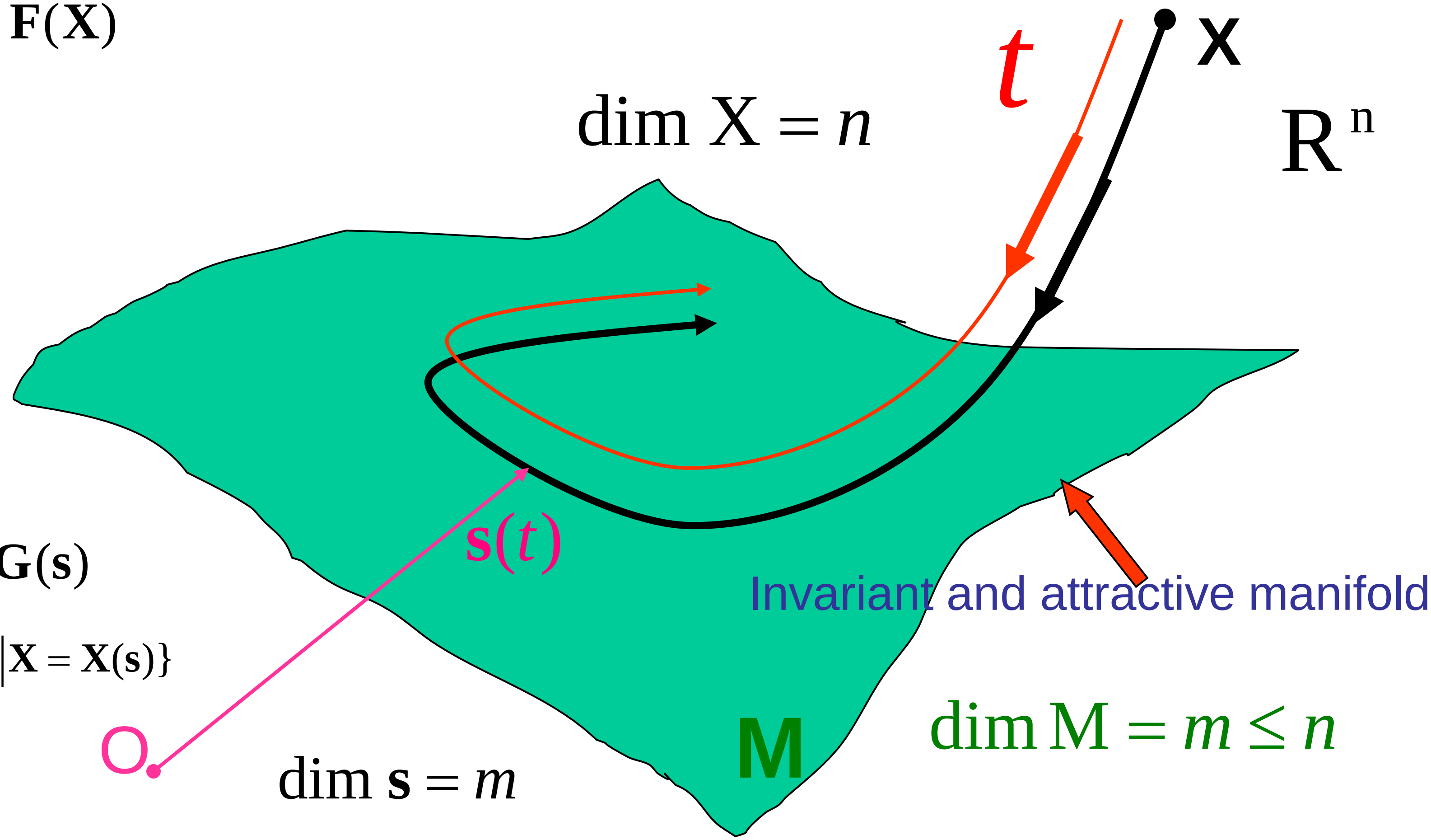
Geometrical image of reduction of dynamical systems

(including Hydrodynamic limit of Boltzmann equation)

n-dimensional dynamical system:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

$$\begin{cases} \dfrac{d\mathbf{s}}{dt} = \mathbf{G}(\mathbf{s}) \\ \mathrm{M}=\{\mathbf{X} \mid \mathbf{X} = \mathbf{X}(\mathbf{s})\} \end{cases}$$



eg.

$\mathbf{X} = f(\mathbf{r}, \mathbf{p})$; **distribution function in the phase space (infinite dimensions)**

$\mathbf{s} = \{u^{\mu}, T, n\}$; **the hydrodinamic quantities or conserved quantities for 1st-order eq.**

Perturbative Approach:

For dynamical systems:

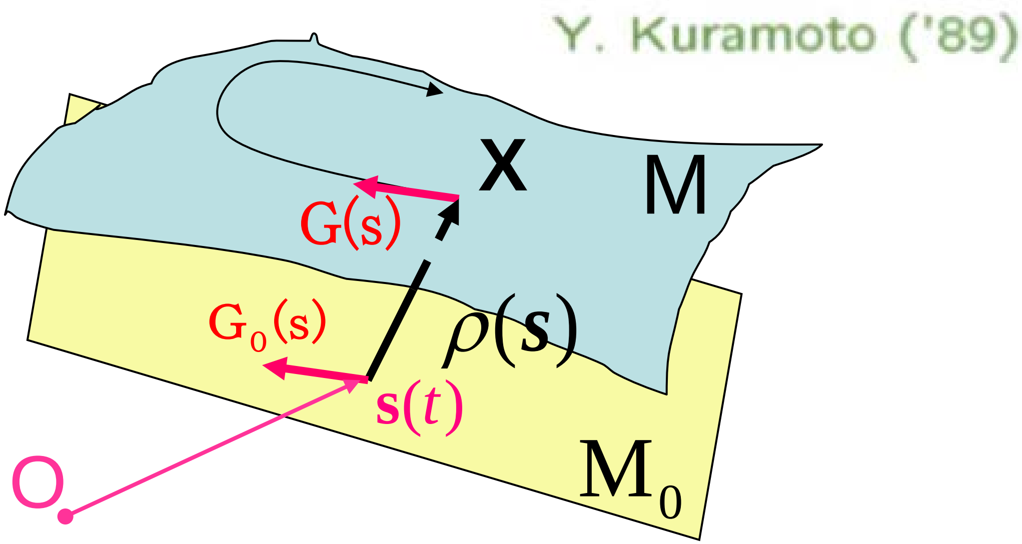
$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{F}(\boldsymbol{X}, t),$$

$$\downarrow$$

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{G}(\boldsymbol{s}), \quad \text{, reduced dynamics on M}$$

$$\boldsymbol{X} = \boldsymbol{R}(\boldsymbol{s}); \quad \text{, representation of M}$$

Y.Kuramoto('89)



Perturbative reduction of dynamics

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{F}(\boldsymbol{X}, t),$$

$$\downarrow$$

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{G}_0(\boldsymbol{s}) + \gamma(\boldsymbol{s}), \quad \text{, reduced dynamics on M}$$

$$\boldsymbol{X} = \boldsymbol{R}_0(\boldsymbol{s}) + \rho(\boldsymbol{s}); \quad \text{, the invariant manifold M}$$

Geometrical image of perturbative reduction of dynamics

Krylov-Bogoliubov-Mitropolosky

Reduction of dynamics and invariant/attractive manifold

The reduction problem of some class of evolution equations may be formulated as a construction of asymptotic invariant/attractive manifold with a possible space-time coarse-graining.

We adopt the perturbative RG method (Chen et al, (1995)*; T.K. (1995)) to construct the attractive/invariant manifolds and the reduced equations for the would-be zero modes as the slow/collective variables defined on the manifold.

* L.Y.Chen, N. Goldenfeld and Y.Oono, PRL.72('95),376:
Discovery that allowing secular terms appear, and then applying RG-like eq gives `all' basic equations of exiting asymptotic methods

A foundation of the RG method a la non-pert. RG.

T.K. PRD57(1998);
S.Ei, K.Fujii and TK (2000)

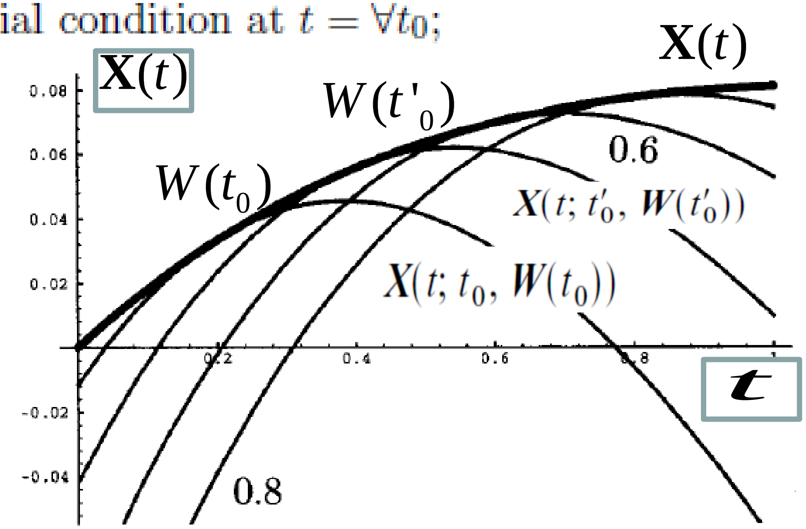
Let us take the following n-dimensional equation;

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{X}, t), \tag{B·11}$$

where n may be infinity. Let $\boldsymbol{X}(t) = \boldsymbol{W}(t)$ be an yet unknown exact solution to Eq.(B·11), and we try to solve the equation with the initial condition at $t = \forall t_0$;

$$\boldsymbol{X}(t = t_0) = \boldsymbol{W}(t_0).$$

Then, the solution may be written as $\boldsymbol{X}(t; t_0, \boldsymbol{W}(t_0))$.



$$\boldsymbol{X}(t; t_0, \boldsymbol{W}(t_0)) = \boldsymbol{X}(t; t_0', \boldsymbol{W}(t_0')).$$

Taking the limit $t_0' \to t_0$, we have

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t_0} = \frac{\partial \boldsymbol{X}}{\partial t_0} + \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{W}}\frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}t_0} = \boldsymbol{0}.$$

Pert. Theory:

$\boldsymbol{X}(t; t_0, \boldsymbol{W}(t_0))$ and $\boldsymbol{X}(t; t_0', \boldsymbol{W}(t_0'))$ may be valid only for $t \sim t_0$ and $t \sim t_0'$,

$$t_0 < t < t_0' \ (\text{or } t_0' < t < t_0)$$

$$\left.\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t_0}\right|_{t_0=t} = \left.\frac{\partial \boldsymbol{X}}{\partial t_0}\right|_{t_0=t} + \left.\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{W}}\frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}t_0}\right|_{t_0=t} = \boldsymbol{0},$$ with $t_0 = t$

RG equation!

Let $\boldsymbol{X}(t;t_0)$ is an approximate solution to Eq.(B·11) around $t \sim t_0$;

$$\frac{\mathrm{d}\boldsymbol{X}(t;t_0)}{\mathrm{d}t} \simeq \boldsymbol{F}(\boldsymbol{X}(t;t_0),t).$$

Then, we have

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{W}(t)}{\mathrm{d}t} &= \left.\frac{\partial \boldsymbol{X}(t;t_0)}{\partial t}\right|_{t_0=t} + \left.\frac{\partial \boldsymbol{X}(t;t_0)}{\partial t_0}\right|_{t_0=t} \\
&= \left.\frac{\partial \boldsymbol{X}(t;t_0)}{\partial t}\right|_{t_0=t} \\
&\simeq \boldsymbol{F}(\boldsymbol{X}(t;t_0),\, t)|_{t_0=t}, \\
&= \boldsymbol{F}(\boldsymbol{W}(t),t),
\end{aligned}$$

showing that our envelope function satisfies the original equation (B.11) in the global domain uniformly.

The correspondence to the renormalization-group theory

S.Ei, K.Fujii and TK, Ann. Phys. (2000)

arbitrary initial time t_0

RG/Flow equation in QFT (Wilson, Wegner):

$$\Gamma(\phi, \boldsymbol{g}(\Lambda), \Lambda) = \Gamma(\phi, \boldsymbol{g}(\Lambda'), \Lambda').$$

If we take the limit $\Lambda' \to \Lambda$, we have

integral const's

$$\frac{d\Gamma(\phi, \boldsymbol{g}(\Lambda), \Lambda)}{d\Lambda} = 0,$$

which is the RG/flow equation.

Equations for the would-be integ. constants, i.e., the infrared/slow variables

$$\frac{\partial \Gamma}{\partial \boldsymbol{g}} \cdot \frac{d\boldsymbol{g}}{d\Lambda} = -\frac{\partial \Gamma}{\partial \Lambda}.$$

The yet unknown function $\mathbf{g}$ is solved exactly and inserted into Γ, which then becomes valid in a global domain of the energy scale.

How to deal with zero modes which appear in the pert. Theory?

A linear eq. with zero modes:

: A is an nxn Hermite matrix with

$$A\boldsymbol{x} = \boldsymbol{b} \quad (*) \qquad \dim[\ker A] = m > 0$$

$$\boldsymbol{x}_k^{(0)} \in \ker A \qquad (k = 1,\, 2,\, \ldots m < n)$$

If $(\boldsymbol{x}_k^{(0)}, \boldsymbol{b}) = 0 \qquad \forall k.$ (solubility cond.)

Then, $$\boldsymbol{x} = A^{-1}\boldsymbol{b} + \sum_{k=1}^{m} c_k \boldsymbol{x}_k^{(0)}$$

Otherwise, no solution to (*).

(**Fredholm alternation th.**)

See for resummation methods based on the solubility cond,C.M.Bender and S.A. Orszag,``Advanced Mathematical Methods For Scientists and Engineers" (McGraw-Hil,1978)

Such problems with a linear op. having zero modes is ubiqutous in mathematical sciences.

How about the time-dependent case?

Appearance of secular terms!

Generic example with zero modes

S.Ei, K. Fujii & T.K.('00)

$$\partial_t \mathbf{u} = A\mathbf{u} + \epsilon \mathbf{F}(\mathbf{u}), \quad |\epsilon| < 1$$

$$\mathbf{u}(t;t_0) = \mathbf{u}_0(t;t_0) + \epsilon \mathbf{u}_1(t;t_0) + \epsilon^2 \mathbf{u}_2(t;t_0) + \cdots$$

$$\begin{aligned} \mathbf{W}(t_0) &= \mathbf{W}_0(t_0) + \epsilon \mathbf{W}_1(t_0) + \epsilon^2 \mathbf{W}_2(t_0) + \cdots, \\ &= \mathbf{W}_0(t_0) + \boldsymbol{\rho}(t_0), \end{aligned}$$

$$\begin{aligned} (\partial_t - A)\mathbf{u}_0 &= 0, \\ (\partial_t - A)\mathbf{u}_1 &= \mathbf{F}(\mathbf{u}_0), \\ (\partial_t - A)\mathbf{u}_2 &= \mathbf{F}'(\mathbf{u}_0)\mathbf{u}_1, \end{aligned}$$

$$(\mathbf{F}'(\mathbf{u}_0)\mathbf{u}_1)_i = \sum_{j=1}^{n} \left\{ \partial(F'(\mathbf{u}_0))_i / \partial(u_0)_j \right\} (u_1)_j$$

When A has semi-simple zero eigenvalues.

$$A\mathrm{U}_i = 0, \quad (i = 1, 2, \ldots, m).$$

We suppose that other eigenvalues have negative real parts;

$$A\mathrm{U}_\alpha = \lambda_\alpha \mathrm{U}_\alpha, \quad (\alpha = m+1, m+2, \cdots, n),$$

where $\mathrm{Re}\lambda_\alpha < 0$. One may assume without loss of generality that U_i's and U_α's are linearly independent.

The adjoint operator $A^\dagger$ has the same eigenvalues as A has;

$$A^\dagger \tilde{\mathrm{U}}_i = 0, \quad (i = 1, 2, \ldots, m),$$
$$A^\dagger \tilde{\mathrm{U}}_\alpha = \lambda_\alpha^* \tilde{\mathrm{U}}_\alpha, \quad (\alpha = m+1, m+2, \cdots, n).$$

Def. P **the projection onto the kernel** $\ker A$

$$P + Q = 1$$

An asymptotic analysis

Since we are interested in the asymptotic state as $t \to \infty$, we may assume that the lowest-order initial value belongs to $\ker A$:

$$\mathbf{W}_0(t_0) = \sum_{i=1}^{m} C_i(t_0)\mathbf{U}_i = \mathbf{W}_0[\mathbf{C}]. \quad \Longleftrightarrow \quad \mathrm{M}_0$$

$$\mathbf{u}_0(t;t_0) = \mathrm{e}^{(t-t_0)A}\mathbf{W}_0(t_0) = \textstyle\sum_{i=1}^{m} C_i(t_0)\mathbf{U}_i.$$

Parameterized with m variables, Instead of n!

$$\mathbf{C} = {}^t(C_1, C_2, \cdots, C_m)$$

$$\mathbf{u}_1(t;t_0) = \mathrm{e}^{(t-t_0)A}[\mathbf{W}_1(t_0) + A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0))] \\ +(t-t_0)P\mathbf{F}(\mathbf{W}_0(t_0)) - A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0)).$$

The would-be rapidly changing terms can be eliminated by the choice;

$$\mathbf{W}_1(t_0) = -A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0)), \qquad P\mathbf{W}_1(t_0) = 0$$

Then, the secular term appears only the P space;

$$\mathbf{u}_1(t;t_0) = (t-t_0)P\mathbf{F} - A^{-1}Q\mathbf{F}$$

a deformation of the manifold ρ

Deformed (invariant) slow manifold: $M_1 = \{\mathbf{u} | \mathbf{u} = \mathbf{W}_0 - \epsilon A^{-1} Q \mathbf{F}(\mathbf{W}_0)\}$

$$\mathbf{u}(t; t_0) = \mathbf{W}_0 + \epsilon\{(t - t_0) P\mathbf{F} - A^{-1} Q\mathbf{F}\}$$

A set of locally divergent functions parameterized by t_0 !

The RG/E equation $\partial \mathbf{u} / \partial t_0 \big|_{t_0 = t} = \mathbf{0}$ gives the envelope, which is globally valid:

$$\dot{\mathbf{W}}_0(t) = \epsilon P\mathbf{F}(\mathbf{W}_0(t)),$$

which is reduced to an m-dimensional coupled equation,

$$\dot{C}_i(t) = \epsilon\langle \tilde{\mathbf{U}}_i, \mathbf{F}(\mathbf{W}_0[C])\rangle, \quad (i = 1, 2, \cdots, m).$$

The global solution (the invariant manifod):

$$\mathbf{u}(t) = \mathbf{u}(t; t_0 = t) = \sum_{i=1}^{m} C_i(t)\mathbf{U}_i - \epsilon A^{-1} Q\mathbf{F}(\mathbf{W}_0[C]).$$

We have derived the invariant manifold and the **slow dynamics** on the manifold by the RG method.

The RG/E equation $\left.\partial \mathbf{u} / \partial t_0\right|_{t_0=t} = \mathbf{0}$

gives the envelope, which is globally valid:

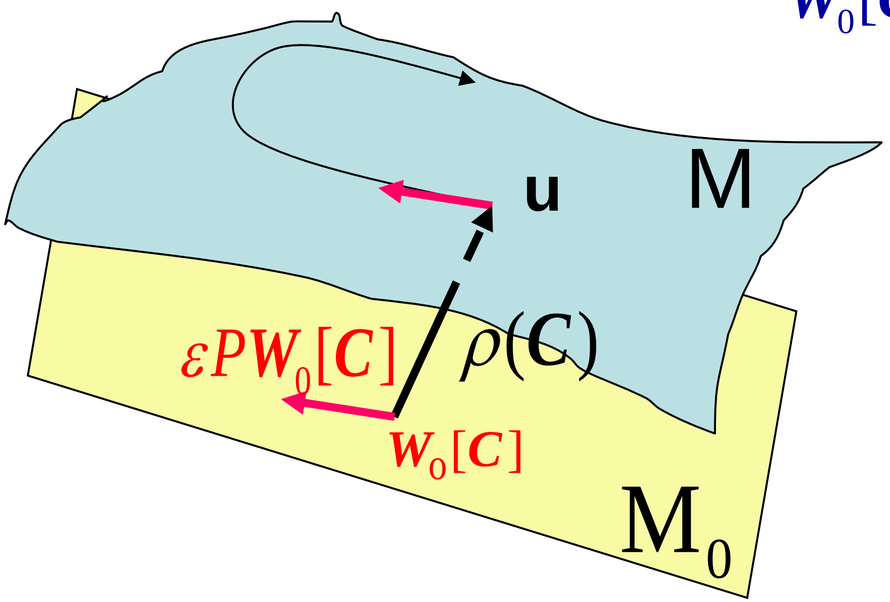
$$\dot{\mathbf{W}}_0(t) = \epsilon P \mathbf{F}(\mathbf{W}_0(t)),$$

which is reduced to an m-dimensional coupled equation,

$$\dot{C}_i(t) = \epsilon \langle \tilde{\mathbf{U}}_i, \mathbf{F}(\mathbf{W}_0[C]) \rangle, \quad (i = 1, 2, \cdots, m).$$

The global solution (the invariant manifod):

$$\mathbf{u}(t) = \mathbf{u}(t; t_0 = t) = \underbrace{\sum_{i=1}^{m} C_i(t) \mathbf{U}_i}_{W_0[C]} \underbrace{- \epsilon A^{-1} Q \mathbf{F}(\mathbf{W}_0[C])}_{\rho(C)}.$$



c.f. Polchinski theorem
in renormalization theory
In QFT.

Remarks

We have derived the invariant manifold and the **slow dynamics** on the manifold by the RG method.

Extension;

a) A is not semi-simple. b) Higher orders.

S. Ei, K. Fujii and T.K. , Ann.Phys 280.('00), 236

c) PDE; Brusselator; Layered pulse dynamics for TDGL and Non-lin Schroedinger

TK, Jpn.J.Ind.Appl.Math.,14(1997); S. Ei, K. Fujii and T.K. ('00).

d) Derivation of kinetic eq. including **Langevin to Fokker-Planck** and further adiabatic elimination of fast variables.

Y. Hatta and T.K. Ann. Phys. 298 (2002), 24

e) Reduction of difference equations

J. Matsukidaiara and T.K. PRE57 (1998),4817

f) Incorporation of excited modes for constructing inv. manifold and derivation of **(non-)relativistic diss. Fluid dynamics**

K. Tsumura, Y. Kikuchi and T.K. Physica D336 (2016),1 and many References cited therein.

e) **Application to QM**

T.K. PRD57 (1998), 2035; RG construction of WF of AHO.

T.K. (2009) (unpublished); t-dep. Perturbation th.

An example of the nonlinear eq.
--- Limit cycle in van der Pol eq. ---

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}.$$

Let $\tilde{x}(t;\, t_0)$ be a local solution around $t \sim \forall t_0$

Expand: $$\tilde{x}(t;\, t_0) = \tilde{x}_0(t;\, t_0) + \epsilon\, \tilde{x}_1(t;\, t_0) + \epsilon^2\, \tilde{x}_2(t;\, t_0) + \cdots .$$

with 'Initial Cond.'.

$$W(t_0) \equiv \tilde{x}(t_0;\, t_0) = x(t_0)$$

;supposed to be an exact sol.,

which is also expanded as

$$W(t_0) = W_0(t_0) + \epsilon\, W_1(t_0) + \epsilon^2\, W_2(t_0) + \cdots .$$

$O(\varepsilon^0)$ $$\mathcal{L}\tilde{x}_0 \equiv \left[\frac{d^2}{dt^2} + 1\right]\tilde{x}_0 = 0, \qquad \tilde{x}_0(t;\, t_0) = A(t_0)\, \cos(t + \theta(t_0)).$$

$O(\varepsilon^1)$ $$\mathcal{L}\tilde{x}_1 = -A\left(1 - \frac{A^2}{4}\right)\sin\phi(t) + \frac{A^3}{4}\sin 3\,\phi(t),$$

$$\phi(t) = t + \theta_0(t_0).$$

$$\tilde{x}_1(t;t_0) = (t-t_0)\,\frac{A}{2}\left(1-\frac{A^2}{4}\right)\cos\phi(t) - \frac{A^3}{32}\sin 3\,\phi(t).$$

$$\left.\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t_0}\right|_{t_0=t} = 0,$$

$$\tilde{x} = \tilde{x}_0 + \epsilon\,\tilde{x}_1$$

$$\dot{A} = \epsilon\,\frac{A}{2}\left(1-\frac{A^2}{4}\right),$$

$$\dot{\phi} = 1.$$

Limit cycle with a radius 2!

$$A(t) = \frac{A_0}{\sqrt{e^{-\varepsilon t} + (\frac{A_0}{2})^2(1-e^{-\varepsilon t})}}$$

Extracted the asymptotic dynamics that is a slow dynamics.

An approximate but globaly valid sol. as the envelope;

$$x_{\mathrm{E}}(t) \equiv \tilde{x}(t;\,t) = W(t) = A(t)\cos(t+\theta_0) - \epsilon\,\frac{A^3(t)}{32}\sin(3\,t+3\,\theta_0),$$

The deformation of
The unpert. sol. like
the Pauli term in
QED.

The secular terms are renormalized
Into the would-be integral
const.

A typical linear equation with zero modes apeears in perturbation theory in QM.

Time-indep. Case: $\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle \qquad \hat{H} = \hat{H}_0 + \lambda \hat{H}_1$

$$\hat{H}_0|\varphi_n^{(0)}\rangle = E_n^{(0)}|\varphi_n^{(0)}\rangle, \qquad \langle\varphi_{n'}^{(0)}|\varphi_n^{(0)}\rangle = \delta_{n'n}.$$

Pert. Exp.

$$|\varphi_n\rangle = |\varphi_n^{(0)}\rangle + \lambda|\varphi_n^{(1)}\rangle + \lambda^2|\varphi_n^{(2)}\rangle + \dots,$$

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots.$$

$\mathcal{O}(\lambda^1)$ $\quad \hat{L}_0|\varphi_n^{(1)}\rangle = \hat{H}_1|\varphi_n^{(0)}\rangle - E_n^{(1)}|\varphi_n^{(0)}\rangle = |b^{(1)}\rangle$

$L_0 = E_n^{(0)} - \hat{H}_0$ has zero modes!

Solubility cond.:

$$0 = \langle\varphi_n^{(0)}|b^{(1)}\rangle \Rightarrow E_n^{(1)} = \langle\varphi_n^{(0)}|\hat{H}_1|\varphi_n^{(0)}\rangle$$

$$|\varphi_n^{(1)}\rangle = \hat{L}_0^{-1}|b^{(1)}\rangle + c_{nn}^{(1)}|\varphi_n^{(0)}\rangle$$

And so on.

How about the time-dep. case? $\quad (i\partial_t - H_0)\psi = \lambda V\psi,$

Appearance of secular terms, not so familiar fact.

Time-dep. perturbation theory in QM revisited

T.K. unpublished (2009, 2016)

- Some ambiguous part of the pert. Theory in QM a la Dirac has been long noted.

See for example, P.W. Langhoff, S.T.Epstein and M. Karplus, RMP44(1972), 602.

H. Ezawa, `Quantum Mechanics I' (Iwanami syoten, 1978)

The problem is how to deal with the initial state due to the appearance of the secular terms. We shall show that the RG method nicely resolve the difficulty.

$$(i\partial_t - H_0)\psi = \lambda V\psi,$$ with the `initial cond.' $\psi(t=t_0;t_0) = \Psi(t_0)$

$$H_0|n\rangle = \epsilon_n|n\rangle \qquad \langle n|m\rangle = \delta_{nm}$$

Conversion to an integral equation:

$$\psi(t;t_0) = \mathrm{e}^{-iH_0(t-t_0)}\Psi(t_0) + \lambda\mathrm{e}^{-iH_0t}\int_{t_0}^{t} ds\mathrm{e}^{iH_0s}(-iV)\psi(s;t_0)$$

Pert. Exp.

$$\begin{aligned}\psi(t;t_0) &= \psi_0(t;t_0) + \lambda\psi_1(t;t_0) + \lambda^2\psi_2(t;t_0) + \cdots,\\ \Psi(t_0) &= \Psi_0(t_0) + \lambda\Psi_1(t_0) + \lambda^2\Psi_2(t_0) + \cdots\end{aligned}$$

0-th order

$$\psi_0(t;t_0) = \mathrm{e}^{-iH_0(t-t_0)}\Psi_0(t_0) = c(t_0)\mathrm{e}^{-i\epsilon_n t}|n\rangle$$

We assume that the 0-th initial state is an unpert. eigen state.

Def. Projection operators

$$P \equiv |n\rangle\langle n|, \quad Q = 1 - P$$

1st-order sol.

$$\psi_1(t;t_0) = c(t_0)\mathrm{e}^{-i\epsilon_n t}\{(-iV_{nn})\underline{(t-t_0)} + \frac{Q}{\epsilon_n - H_0}V\}|n\rangle$$

The integral const.

secular term!

We can proceed to higher order straightforwardly.

Collecting them up to the 2nd order,

$$\psi(t;t_0) \simeq \psi_0(t;t_0) + \lambda\psi_1(t;t_0) + \lambda^2\psi_2(t;t_0)$$

Now applying **the RG/Envelope equation**,

$$\partial\psi/\partial t_0|_{t_0=t} = 0$$

we have

$$\dot{c} = -i\Delta\epsilon_n c(t),$$ and thus, $$c(t) = c_0\mathrm{e}^{-i\Delta\epsilon_n t}$$

with $$\Delta\epsilon_n \equiv \lambda\langle n|V|n\rangle + \lambda^2\langle n|V\frac{Q}{\epsilon_n - H_0}V|n\rangle$$

The energy correction to be obtained in the time-indep. perturbation theory!

The resummed wave function as the initial value reads

$$\begin{aligned}\psi_n(t) &= \psi(t;t) = \Psi(t) = c_0\epsilon^{-iE_n t}\Big[1 + \lambda\frac{Q}{\epsilon_n - H_0}V \\ &\quad + \lambda^2\{\frac{Q}{\epsilon_n - H_0}V\frac{Q}{\epsilon_n - H_0}V - \frac{Q}{(\epsilon_n - H_0)^2}VPV\}\Big]|n\rangle, \\ E_n &= \epsilon_n + \Delta\epsilon_n\end{aligned}$$

Both the energy and wave function are consistent with the time-independent Rayleigh-Schroedinger perturbation theory.

We note that the secular terms are renormalized into the energy as the time-dep. phase.

The method can be successfully applied to
the (quasi) degenerate initial state,
leading to the effective Hamiltonian acting the degenerate subspace.

T.K. , in preparation.

From Langevin to Fokker-Planck equation

Y. Hatta and TK Ann. Phys. 298 (2002), 24

Langevin eq. with multiplicative noise:

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{h}(\boldsymbol{u}) + \hat{g}(\boldsymbol{u})\boldsymbol{R} \quad \text{--- (1)} \qquad \boldsymbol{u} = {}^t(u_1, u_2, \ldots, u_n),\ \boldsymbol{h} = {}^t(h_1, h_2, \ldots, h_n)$$

$(\hat{g}(\boldsymbol{u})\boldsymbol{R})_i = \sum_j g_{ij} R_j \quad R_i\ (i = 1, 2, \ldots, n)$; stochastic variable with $\langle \boldsymbol{R}(t) \rangle = 0$,

Let $f(\boldsymbol{u}, t)$ be the distribution function of the dynamical variable u, which satisfies the continuity equation with $\boldsymbol{v} = d\boldsymbol{u}/dt$;.

$$\frac{\partial f(\boldsymbol{u}, t)}{\partial t} + \nabla_u \cdot (\boldsymbol{v} f(\boldsymbol{u}, t)) = 0,$$

Because of (1), we have $\frac{\partial f}{\partial t} = -\nabla_u \cdot [(\boldsymbol{h} + \hat{g}\boldsymbol{R}) f] \equiv L(s) f$ --- (2)

$$L(s) = -\nabla_u \cdot (\boldsymbol{h}(\boldsymbol{u}) + \hat{g}\boldsymbol{R}(s))$$

We apply **t-dep. perturbation theory familiar in QM** to derive Fokker-Planck equation from (2). $L = L_0 + L_1,$ with $L_0 = -\nabla_u \cdot \boldsymbol{h},\ L_1 = -\nabla_u \hat{g} \boldsymbol{R}.$

We take the interaction picture of L_1.

c.f. Path-integral approach, J. Zinn-Justin,``QFT and Critical Phenomena” (1989)

The formal solution to (2) with the initial cond. at $t = t_0$ reads

$$\tilde{f}(\boldsymbol{u}, t; t_0) = T \exp\left[\int_{t_0}^{t} ds\, L(s)\right] \underline{\tilde{f}(\boldsymbol{u}, t_0; t_0)}$$

R.Kubo JMP 4 (1963)

The initial distribution fun. $\tilde{f}(\boldsymbol{u}, t_0; t_0)$,which is to be specified later, will be found to play the essential role.

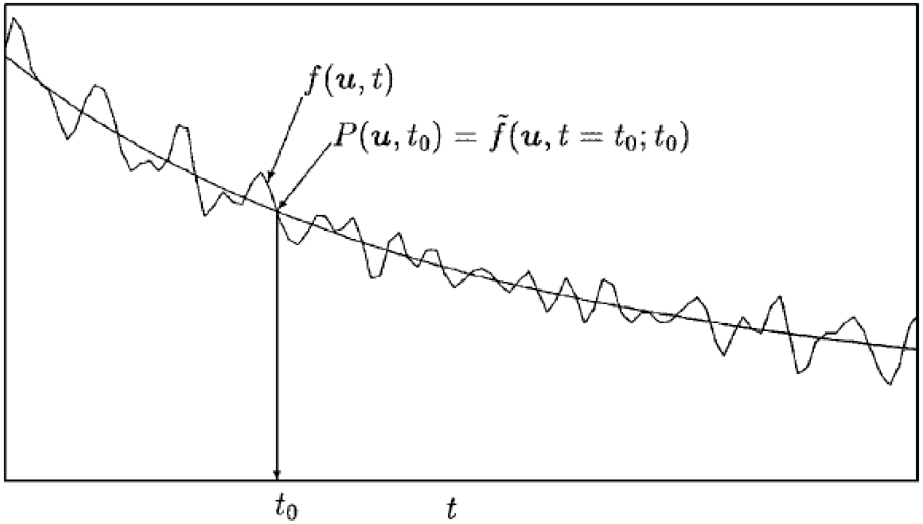
Def. Unpert. Evolution op. $U_0(t, t_0) = T \exp\left[\int_{t_0}^{t} ds\, L_0(s)\right] \quad L_0 = -\nabla_u \cdot \boldsymbol{h},$

Def. $\rho_1(\mathbf{u}, t; t_0)$ by $\tilde{f}(\boldsymbol{u}, t; t_0) = U_0(t, t_0)\rho_1(\boldsymbol{u}, t; t_0)$

Def. The averaged distribution function $\tilde{P}(\boldsymbol{u}, t; t_0) \equiv \left\langle \tilde{f}(\mathbf{u}, t; t_0) \right\rangle_{\mathbf{R}}$

We require that the bare distribution coincide the averaged one at $t = t_0$

$$\tilde{f}(\boldsymbol{u}, t = t_0; t_0) = P(\boldsymbol{u}, t_0)$$
$$= \rho_1(\boldsymbol{u}, t = t_0, t_0)$$



$$\rho_1(\boldsymbol{u}, t; t_0) = T \exp\left[\int_{t_0}^{t} ds\, \mathcal{L}_1(s; t_0)\right]\rho_1(\boldsymbol{u}, t_0; t_0),$$

$$\mathcal{L}_1(t; t_0) = U_0^{-1}(t, t_0) L_1(t) U_0(t, t_0)$$

$$\tilde{P}(\boldsymbol{u}, t; t_0) = \langle U_0(t, t_0)\rho_1(\boldsymbol{u}, t; t_0)\rangle \equiv U_0(t, t_0) S(t; t_0) P(\boldsymbol{u}, t_0),$$

$$S(t; t_0) \equiv \left\langle T \exp\left[\int_{t_0}^{t} ds\, \mathcal{L}_1(s; t_0)\right]\right\rangle.$$

Evaluation of $S(t; t_0)$ by a naiive perturbation exp, which is to be resummed by the RG equation.

$$S(t; t_0) = 1 + T\int_{t_0}^{t} ds \langle \mathcal{L}_1(s)\rangle + \frac{1}{2} T \int_{t_0}^{t} ds_1 \int_{t_0}^{t} ds_2 \langle \mathcal{L}_1(s_1)\mathcal{L}_1(s_2)\rangle + \cdots$$

$$= 1 + \frac{1}{2} T \int_{t_0}^{t} ds_1 \int_{t_0}^{t} ds_2 \Gamma(s_1, s_2) + \cdots \qquad \Gamma(s_1, s_2) \equiv \langle \mathcal{L}_1(s_1)\mathcal{L}_1(s_2)\rangle$$

Assumption: the stationary noise and the microscopic time-reversal invariance.

$$\tilde{P}(\boldsymbol{u}, t; t_0) = U(t; t_0)[1 + (t - t_0) G(t - t_0)] P(\boldsymbol{u}, t_0)$$

$$G(t) = \int_0^t ds\, \Gamma(s)$$

Now we apply the RG/Envelope equation as

$$\partial \tilde{P}(\boldsymbol{u}, t; t_0)/\partial t_0|_{t_0=t} = 0,$$

$$\partial_{t_0} U_0(t, t_0)|_{t_0=t} P(\boldsymbol{u}, t) + \partial_t P(\boldsymbol{u}, t) - G(0) P(\boldsymbol{u}, t) = 0$$

Recalling tha $\partial_{t_0} U_0(t, t_0)|_{t_0=t} = -L_0 = \nabla_u \cdot \boldsymbol{h}$, we arrive at the Foller-Planck eq.;

$$\partial_t P(\boldsymbol{u}, t) = -\nabla_u \cdot \boldsymbol{h} P(\boldsymbol{u}, t) + G(0) P(\boldsymbol{u}, t)$$

Here the explicit form of G(0) depends on the properties of the noise.

Simple example: Gaussian noise $\langle R_i(t) R_j(t') \rangle = 2\delta_{ij} D_i \delta(t - t')$

$$\Gamma(s) = U_0^{-1} \partial_i g_{ij} \partial_k g_{kl} 2 D_j \delta_{jl} \delta(s),$$ with $\partial_i = \partial/\partial u_i$.

Thus

$$G(t) \equiv \int_0^t ds\, \Gamma(s) = \frac{1}{2} U_0^{-1} \partial_i g_{ij} \partial_k g_{kl} 2 D_j \delta_{jl} = G(0).$$

t-independent.

$\theta(0) = 1/2$

We have the familiar form of the Fokker-Planck equation for the multiplicative Gaussian noise as,

$$\partial_t P(\boldsymbol{u}, t) = -\nabla_u \cdot \boldsymbol{h} P(\boldsymbol{u}, t) + D_j \partial_i g_{ij} \partial_k g_{kj} P(\boldsymbol{u}, t)$$

Brief Summary and concluding remarks

(1) The notion of the RG plays an important role in the reduction theory of dynamics.
(2) In the classical theory of envelops gives a nice intuitive understanding of the perturbative RG reduction.
(3) The RG method provides a very powerful tool for the reduction of dynamics and the explicit construction of the invariant/attractive manifold.
(4) The RG method gives a natural framework for the resummation method for the time-dep. Perturbation theory in Quantum Mechanics.
(5) We have seen also that the method can be also applied to a reduction of the stochastic equations.
(6) There are and will be many subjects to which the RG method can be nicely applied.

Back Up slides

Soliton-Soliton interaction in the KdV equation

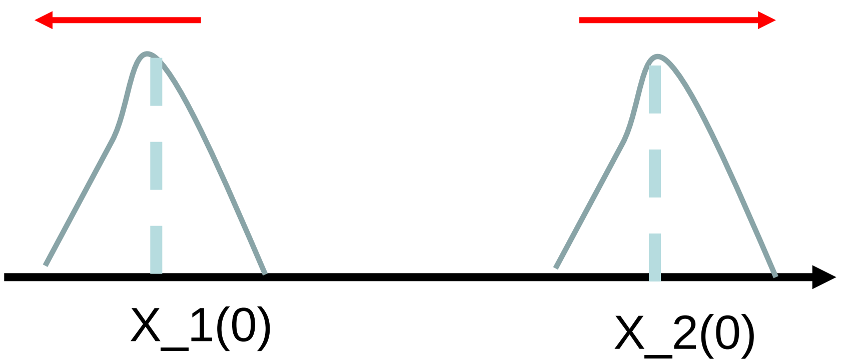
S.Ei, K.Fujii and TK (2000)

The KdV eq.: $\partial_t u + 6u\,\partial_x u + \partial_x^3 u = 0$

one-soliton sol.: $u(x,t) = \frac{c}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{c}(x-ct)}{2}\right] \equiv \varphi(x-ct;\,c)$

Remark: (a)Translational invariance.
(b)The velocity c is arbitrary.

When two one-soitons are located at x_1(0) and x_2(0) at t=0, what would happen for t>0?



A variational approach with an ansatz:
S.Ei and T. Ohta,PRE50 (1994),4672

$$u(x,t) = \varphi(x-ct-x_1;\,c+\dot{x}_1) + \varphi(x-ct-x_2;\,c+\dot{x}_2) + b(x-ct,\,t)$$

The RG method gives the foundation and more.

$t = t,\ z = x - ct$

$$\partial_t u + F[u] = 0,$$

$$F[u] = -c\,\partial_z u + 6u\,\partial_z u + \partial_z^3 u$$

One-soliton is a fixed `point' : $F[\varphi(z-b;\,c)] = 0,$

$$u(z,\,t) = \varphi(z - z_1(t_0);\,c) + \varphi(z - z_2(t_0);\,c) + v(z,\,t)$$

For the case of either $z \sim z_1$ or $z \sim z_2$

$$\partial_t v + F'[\varphi^{(1)} + \varphi^{(2)}]\,v + 6\partial_z(\varphi^{(1)}\varphi^{(2)}) + O(|v|^2) = 0.$$

$$\partial_t v = A^{(1)} v - 6\delta\partial_z(\varphi^{(1)} g) + O(\delta^2 + |v|^2),$$

$$A^{(1)} = -F'[\varphi^{(1)}] = c\,\partial_z - \partial_z^3 - 6(\partial_z \varphi^{(1)} + \varphi^{(1)}\,\partial_z)$$

A has a Jordan cell $AU_1 = 0, \qquad AU_2 = U_1,$

where $U_1 = \partial_z \varphi, \qquad U_2 = -\partial_c \varphi$

$$\begin{cases} \dot{z}_1 = -C_1 - 28ce^{-\sqrt{c}\,(z_2 - z_1)}, \\ \dot{C}_1 = 16c^{5/2}e^{-\sqrt{c}\,(z_2 - z_1)}, \end{cases}$$

Similar equation to z_2.

Not only the positions of the one-solitons but also the velocity of the Soliton acquire time-dependence, though quite small.

Eliminating the velocity correction C_i, we have equations only for the positions;

$$\dot{z}_1 \simeq -16c^{5/2}e^{-\sqrt{c}\,(z_2 - z_1)}$$

$$\ddot{z}_2 \simeq 16c^{5/2}e^{-\sqrt{c}\,(z_2 - z_1)},$$

which is exactly what Ei-Ohta obtained starting from a plausible ansatz.

S.Ei and T. Ohta,PRE50 (1994),4672