On confinement in Yang-Mills theory

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Based mainly on

• [1704.05852] with Masahito Yamazaki

If there is time, also

• [1706.06104] with Hiroyuki Shimizu
Two approaches to gauge theories:

- Lattice gauge theory (I have nothing to say in this talk)
- Sum over perturbation around every classical solution
trans-series expansion \[ g : \text{coupling constant} \]

\[
\sum a_{0,k} g^{2k} + e^{-c_1/g^2} \sum a_{1,k} g^{2k} + e^{-c_2/g^2} \sum a_{2,k} g^{2k} + \cdots
\]

Perturbation around the lowest action

Perturbation around other classical solutions

After resummation, we hopefully get a sensible answer which makes sense for any (possibly large) value of \( g \).
First of all, each term of the trans-series expansion must be well-defined.

But they are not well-defined due to IR divergences!
Example: Instantons on $R^4$.

We want to compute the vacuum energy as a function of the theta angle $\theta$

\[
\exp(-E(\theta) \text{Vol}) = \int [DA] e^{-S}
\]

\[
S = \int d^4x \frac{1}{2g^2} \text{tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{i\theta}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr}(F_{\mu\nu}F_{\rho\sigma})
\]

Remark:
The $E(\theta)$ can have importance for cosmology if we replace

$\theta \rightarrow$ (string) axion, inflaton,…
Introduction

Example: Instantons on $R^4$.

[ ’t Hooft, 1976] [see e.g. Coleman’s book]

Instanton computation of $E(\theta)$

\[
E(\theta) \sim - \int \frac{d\rho}{\rho^5} (\rho \mu)^{b_1} e^{-\frac{8\pi^2}{g^2}} \cos(\theta) + \cdots
\]

\[\rho : \text{instanton size}\]

\[b_1 = \frac{11}{3} N \quad \text{for SU}(N) \quad \mu : \text{RG scale}\]

The integral over the instanton size $\rho$ is ill-defined due to IR divergence at $\rho \to \infty$
Example: Instantons on $R^4$.

Trans-series expansion is **ill-defined** for vacuum energy:

$$E(\theta) \sim \cdots + \infty_{\text{IR}} \cdot e^{-\frac{8\pi^2}{g^2}} \cos(\theta) + \cdots$$

Hand-waving argument:

“The IR divergence is due to strong dynamics in IR and somehow it should be cutoff at the dynamical scale. Hopefully the result would give a qualitatively right answer.”
Introduction

Example: Instantons on $\mathbb{R}^4$.

Trans-series expansion is **ill-defined** for vacuum energy:

$$E(\theta) \sim \cdots + \infty_{\text{IR}} \cdot e^{-\frac{8\pi^2}{g^2}} \cos(\theta) + \cdots$$

Hand-waving argument:

“The IR divergence is due to strong dynamics in IR and somehow should be cutoff at the dynamical scale. Hopefully this will lead to a qualitatively right answer.”
Example: Instantons on $\mathbb{R}^4$.

The correct theta dependence of the vacuum energy

$$E(\theta) \sim \Lambda^4 \theta^2 \quad \text{(in large N limit)}$$

Actually there exists many metastable vacua labeled by integer $\ell \in \mathbb{Z}$ with vacuum energy

$$E_\ell(\theta) \sim \Lambda^4 (\theta + 2\pi \ell)^2 \quad \text{(in large N limit)}$$

Each $E_\ell(\theta)$ is not a $2\pi$ periodic function of $\theta$
Introduction

What is a framework which may have well-defined trans-series expansion without IR divergences?

**Strategy:** [many people in the audience] compactify the space in a way there is no IR divergence.

- We will consider $R_{\text{time}} \times T^3$ with twist

- The running coupling is evaluated at the length scale of $T^3$: small radius $\rightarrow$ weak coupling
Short Summary

4d SU(N) Yang-Mills

\( T^2 \) compactification

\( 2d \, CP^{N-1} \) sigma model

\( T^3 \) compactification twist by 1-form symmetry ('t Hooft magnetic flux)

Confinement at weak coupling

\( S^1 \) with twisted boundary condition

IR divergence is eliminated by twisted boundary condition.
Contents

1. Introduction

2. Twisted compactification of CPN sigma model

3. From SU(N) Yang-Mills to CPN

4. More details of Yang-Mills vacua

5. Confinement v.s. Axial symmetry

6. Summary
Sigma model

Sigma model Lagrangian:

$$\mathcal{L} = h_{ij} \partial_\mu \phi^i \partial^\mu \phi^j \quad (\phi^i: \text{sigma model field})$$

Propagator:

$$\langle \phi^i(x) \phi^j(0) \rangle \sim \int \frac{d^2 k}{(2\pi)^2} \frac{e^{ikx}}{k^2}$$

IR divergences analogous to 4d Yang-Mills:

- Log IR divergence of propagator at $k \to 0$
  (The essence of the “no Goldstone boson theorem”)
  \[\text{[Coleman, 1973]}\]
- Instanton integral is also divergent
Twisted compactification

Homogeneous coordinates of $\mathbb{C}P^{N-1}$

\[
[z_1, z_2, \cdots, z_N]
\]

It has global symmetry $Z_N \subset SU(N)$

\[
[z_1, z_2, \cdots, z_N] \rightarrow [e^{2\pi i/N} z_1, e^{4\pi i/N} z_2, \cdots, e^{2N\pi i/N} z_N]
\]

Twisted compactification on $S^1$: boundary condition is twisted by the above $Z_N$ transformation

\[
[\cdots, z_k(x + 2\pi), \cdots] = [\cdots, e^{2\pi ik/N} z_k(x), \cdots]
\]

$x \in S^1$

[Dunne-Unsal,2012]
Twisted compactification

\[ [\cdots, z_k(x + 2\pi), \cdots] = [\cdots, e^{2\pi i k/N} z_k(x), \cdots] \]

\( x \in S^1 \)

This boundary condition \textbf{kills all zero modes} \rightarrow \text{no IR divergence at all.}
Classical vacua

Classical vacua are given by the fixed points of the twisting.

Vacuum: \( \phi = [\cdots, z_k, \cdots] \) independent of \( x \in S^1 \)

\[
[\cdots, z_k, \cdots] = [\cdots, e^{2\pi ik/N} z_k, \cdots]
\]

\( z_k : \text{homogeneous coordinates} \)

\( N \) discrete vacua: fixed points on \( \mathbb{CP}^{N-1} \) by symmetry

\[
P_k = [0, \cdots, 0, 1, 0, \cdots, 0] \quad (k = 1, 2, \cdots, N)
\]
Quantum vacua

Degeneracy of classical vacua is lifted by fractional instantons. [Eto-Fujimori-Isozumi-Nitta-Ohashi-Ohta-Sakai,…]

\[ \mathbb{CP}^{N-1} \]

\[ P_1, P_2, P_3, \ldots, P_N \]

\[ \pm \frac{1}{N} \text{ instanton} \]
Quantum vacua

$|P_k\rangle$ : classical vacuum at the point $P_k$

Due to fractional instantons, the true quantum vacua are

$$|\ell\rangle = \sum_{k=1}^{N} e^{2\pi i k\ell/N} |P_k\rangle \quad \ell \in \mathbb{Z}_N$$

Vacuum energy:

$$E_\ell(\theta) \propto -\cos\left(\frac{\theta + 2\pi \ell}{N}\right)$$

$$\rightarrow (\theta + 2\pi \ell)^2 \quad \text{(in large N limit)}$$

Perfect agreement with large N result!
Contents

1. Introduction

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4. More details of Yang-Mills vacua

5. Confinement v.s. Axial symmetry

6. Summary
Compactification of Yang-Mills

Let’s compactify 4d Yang-Mills on a torus $T^2$ and perform KK reduction.

(Classical) massless modes:
Flat connections $F_{\mu\nu} = 0$ of the gauge field on $T^2$
Compactification of Yang-Mills

**Theorem 1** [Looijenga]

As an algebraic variety, the moduli space of SU(N) flat connections on $T^2$ is given by $CP^{N-1}$

**Theorem 2 (Crude statement)** [Friedman-Morgan-Witten]

Yang-Mills instanton on $T^2 \times \Sigma$ is essentially given by $CP^{N-1}$ instanton on $\Sigma$.  

(For more precise statements see their paper.)
1-form $\mathbb{Z}_N$ symmetry

The $\mathbb{Z}_N$ global symmetry of $\mathbb{C}P^{N-1}$

$$[z_1, z_2, \cdots, z_N] \rightarrow \left[e^{2\pi i/N} z_1, e^{4\pi i/N} z_2, \cdots, e^{2N\pi i/N} z_N\right]$$

It turns out that this is realized by dimensional reduction of 1-form center symmetry of SU($N$) Yang-Mills

$$U \rightarrow e^{2\pi i/N} U$$

$U$ : Wilson line

$e^{2\pi i/N}$ : in the center $\mathbb{Z}_N$ of SU($N$)

A little more details are discussed later.
Correspondence in 4d/2d

4d SU(N) Yang-Mills

\( T^2 \) compactification

2d \( CP^{N-1} \) sigma model

Correspondence on

- Instantons
- \( Z_N \) symmetry

These properties are enough to guarantee the argument of the next slide:
Quantum vacua of Yang-Mills

Quantum vacua: \(|\ell\rangle = \sum_{k=1}^{N} e^{2\pi i k \ell/N} |P_k\rangle \quad \ell \in \mathbb{Z}_N\)

\[ E_\ell(\theta) \propto -\cos\left(\frac{\theta + 2\pi \ell}{N}\right) \]
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Reminder of Situation

4d SU(N) Yang-Mills

$T^2$ compactification

2d $CP^{N-1}$ sigma model

$T^3$ compactification

Confinement at weak coupling

$S^1$ with twisted boundary condition

Yang-Mills theory is put on $R_{\text{time}} \times T^3$
Yang-Mills flat connections

What are the states \(|P_k\rangle\) in Yang-Mills in the box \(T^3\)?

Lowest energy states

\[
F_{\mu\nu} = 0
\]

Flat connections on \(T^3 = S^1_A \times S^1_B \times S^1_C\)

Characterized by \(N \times N\) matrices

\[
(U_A, U_B, U_C) \quad : \text{Wilson lines}
\]

\[
U_{A,B,C} = \exp i \int_{S_{A,B,C}} A_\mu dx^\mu
\]
$Z_N$ 1-form center symmetry

• 1-form symmetry acts on line operators like Wilson lines (ordinary “0-form” symmetry acts on local operators)

• center symmetry is, roughly, the symmetry

$$U \rightarrow e^{2\pi i/N} U$$

$U$ : Wilson line

$e^{2\pi i/N}$ : in the center $Z_N$ of SU(N)
How can we implement twisted compactification by the 1-form center symmetry? [’t Hooft, 1979]

I’m going to give a rough explanation. Please don’t care subtle details (because I don’t remember).
Twist boundary condition by 1-form symmetry

\[ U_B(x_C + 2\pi) = e^{2\pi i/N} U_B(x_C) \]
Twist

\[ U_B(2\pi) = U_C U_B(0) U_C^{-1} \quad \text{(if } F_{\mu\nu} = 0 \text{)} \]
Twist

- $U_B(2\pi) = e^{2\pi i/N}U_B(0)$
- $U_B(2\pi) = U_CU_B(0)U_C^{-1}$

Combining them and defining $U_B := U_B(0)$, we get

\[ U_CU_B = e^{2\pi i/N} U_BU_C \]

\[ UAUA = UAU_B \]
\[ UAU_C = UCU_A \]

(The $U_A$ have trivial commutation relations in our twist.)
Classical vacua

Classical vacua are solutions of the commutation relations up to gauge transformations:

\[
U_C U_B = e^{2\pi i/N} U_B U_C
\]

\[
U_A U_B = U_B U_A
\]

\[
U_A U_C = U_C U_A
\]

**Algebra:** \( U_C \) may be regarded as “lowering operator”

If \( U_B \vec{v} = e^{i\alpha} \vec{v} \) then \( U_B(U_C \vec{v}) = e^{i\alpha - 2\pi i/N} (U_C \vec{v}) \)
Classical vacua

Solutions up to gauge transformation

\[ U_B = \text{diag}(1, e^{2\pi i/N}, e^{4\pi i/N}, \ldots) \]

\[ U_C = (\delta_{i+1,j}) \]

\[ U_A = e^{2\pi i k/N}, \quad k = 1, 2, \ldots, N \]

It turns out that \( N \) classical vacua are

\[ U_A = e^{2\pi i k/N}, \quad |P_k\rangle \]
Confinement

**Definition:**

confinement $\Longleftrightarrow$ center symmetry unbroken

How about in our case?

(Part of) relevant gauge invariant operators:

$$\text{tr}(U_B) = \text{tr}(U_C) = 0$$

$$\text{tr}(U_A) = Ne^{2\pi ik/N} \neq 0$$

The center symmetry is broken by the nonzero vacuum expectation value of $\text{tr}(U_A)$ at each classical vacua $|P_k\rangle$
Confinement

Quantum vacua: \[ |\ell\rangle = \sum_{k=1}^{N} e^{2\pi ik\ell/N} |P_k\rangle \quad \ell \in \mathbb{Z}_N \]

\[ \langle \ell| \text{tr}(U_A) |\ell\rangle = 0 \quad : \text{confinement} \]
Remarks

The center symmetry restoration itself is not surprising because all the spatial directions are compactified.

The points are:

• We realized it in completely weakly coupled regime

• Our results are expected to be continued to large volume (assuming resurgence and mass gap)

• I will later discuss an example of center symmetry breaking in the presence of fermions.
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6. Summary
Let’s include fermions $\lambda$ in the adjoint representation.

Just for simplicity, in this talk I discuss the case of a single adjoint fermion

$\rightarrow$ $\text{N}=1$ Super-Yang-Mills

Axial symmetry

$Z_{2N} : \lambda \rightarrow e^{2\pi i/2N} \lambda$
Axial current

Naive axial current \( J^\mu \sim \bar{\lambda} \gamma_5 \gamma^\mu \lambda \)
\[
\partial_\mu J^\mu = N \cdot \frac{1}{16\pi^2} \text{tr} F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}
\]

Conserved axial current

\[
\partial_\mu \tilde{J}^\mu = 0
\]
\[
\tilde{J}^\mu = J^\mu - N \cdot \text{CS}^\mu
\]

\( \text{CS}^\mu : \text{Chern-Simons} \)
\[
\partial_\mu (\text{CS}^\mu) = \frac{1}{16\pi^2} \text{tr} F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}
\]
Axial charge

Axial charge: \[ Q = \int d^3 x \tilde{J}^0 \]

\[ = \int d^3 x J^0 - N \int \text{CS} \]

\[ +1 \frac{1}{N} \text{ instanton} \]

\[ = \int \text{CS} = \frac{k}{N} \mod 1 \]

\[ = \int \text{CS} = \frac{k + 1}{N} \mod 1 \]
Axial charge

\[ Q \text{ : defined only modulo N (due to Chern-Simons)} \]

\[ e^{2\pi i Q/N} \text{ : well-defined charge for discrete } Z_{2N} \]

\[ |P_k\rangle \text{ are eigenstates of the axial charge } e^{2\pi i Q/N} \]

\[ e^{2\pi i Q/N} |P_k\rangle = e^{2\pi i k/N} |P_k\rangle \]
Axial v.s. center symmetry

\[ |P_k\rangle : \text{eigenstates of axial charge} \]

\[ |\ell\rangle = \sum_{k=1}^{N} e^{2\pi i k \ell/N} |P_k\rangle : \text{eigenstates of center symmetry (confinement)} \]

In Hilbert space, there is no simultaneous eigenstate of the axial symmetry and the center symmetry! One of them (or both of them) is always spontaneously broken.
Thermal phase transition

Finite temperature scenario

\[ T : \text{temperature} \]

\[ T_{\text{deconfine}} \leq T_{\text{chiral}} \]

The equality is possible only for 1st order transition

\[ Z_{\text{center}} \text{ broken} \]

\[ T_{\text{deconfine}} \]

\[ Z_{\text{axial}} \text{ broken} \]

\[ T_{\text{chiral}} \]

[Komargodski-Sulejmanpasic-Unsal]
[Shimizu-Yonekura]
Remarks

• More deep reason behind it is a mixed anomaly between axial and center symmetry
  
  [Gaiotto-Kapustin-Seiberg-Willett]

• It is also possible to constrain phase transitions with fermions in the fundamental representation (massless QCD!).
  
  [Shimizu-Yonekura]

• For details, please see our paper
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Summary

A setup which is free from IR divergences reproduce qualitative features of confinement in Yang-Mills even at weak coupling regime.

Vacuum structure has rich phenomena such as

• Nontrivial $\theta$ angle dependence of vacuum energy
• Relation between confinement and axial symmetry