On confinement in Yang-Mills theory

Kazuya Yonekura,
Kavli IPMU

Based mainly on

- [1704.05852] with Masahito Yamazaki

If there is time, also

- [1706.06104] with Hiroyuki Shimizu

Introduction

Two approaches to gauge theories:

- Lattice gauge theory (I have nothing to say in this talk)

- Sum over perturbation around every classical solution

Introduction

<u>**trans-series expansion**</u> g : coupling constant

$$\sum a_{0,k} g^{2k} + e^{-c_1/g^2} \sum a_{1,k} g^{2k} + e^{-c_2/g^2} \sum a_{2,k} g^{2k} + \cdots$$

Perturbation around the lowest action

Perturbation around other classical solutions

After resummation, we hopefully get a sensible answer which makes sense for any (possibly large) value of g .

Introduction

First of all, each term of the trans-series expansion must be well-defined.

But they are not well-defined due to IR divergences!

Introduction

Example: Instantons on R^4.

We want to compute the vacuum energy as a function of the theta angle θ

$$\exp(-E(\theta)\mathrm{Vol}) = \int [DA] e^{-S}$$

$$S = \int d^4x \frac{1}{2g^2} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{i\theta}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

Remark:

The $E(\theta)$ can have importance for cosmology if we replace

$\theta \to$ (string) axion, inflaton,…

Introduction

Example: Instantons on R^4.

['t Hooft,1976]
[see e.g. Coleman's book]

Instanton computation of $E(\theta)$

$$E(\theta) \sim -\int \frac{d\rho}{\rho^5}(\rho\mu)^{b_1} e^{-\frac{8\pi^2}{g^2}} \cos(\theta) + \cdots$$

ρ : instanton size

$b_1 = \frac{11}{3}N$ for SU(N) μ : RG scale

The integral over the instanton size ρ is ill-defined due to IR divergence at $\rho \to \infty$

Introduction

Example: Instantons on R^4.

Trans-series expansion is ill-defined for vacuum energy:

$$E(\theta) \sim \cdots + \infty_{\mathrm{IR}} \cdot e^{-\frac{8\pi^2}{g^2}} \cos(\theta) + \cdots$$

Hand-waving argument:

> “The IR divergence is due to strong dynamics in IR and somehow it should be cutoff at the dynamical scale. Hopefully the result would give a qualitatively right answer.”

Introduction

Example: Instantons on R^4.

Trans-series expansion is ill-defined for vacuum energy:

$$E(\theta) \sim \cdots + \infty_{\mathrm{IR}} \cdot e^{-\frac{8\pi^2}{g^2}} \cos(\theta) + \cdots$$

Hand-waving argument:

“The IR divergence is d in IR and
someh ical scale.
Hopefu ld give a qualitatively right answer.”

WRONG!!!

Introduction

Example: Instantons on R^4.

The correct theta dependence of the vacuum energy

[Witten,1980,1998]

$$E(\theta) \sim \Lambda^4 \theta^2 \quad \text{(in large N limit)}$$

Actually there exists many metastable vacua labeled by integer $\ell \in \mathbb{Z}$ with vacuum energy

$$E_\ell(\theta) \sim \Lambda^4 (\theta + 2\pi\ell)^2 \quad \text{(in large N limit)}$$

Each $E_\ell(\theta)$ is not a 2π periodic function of θ

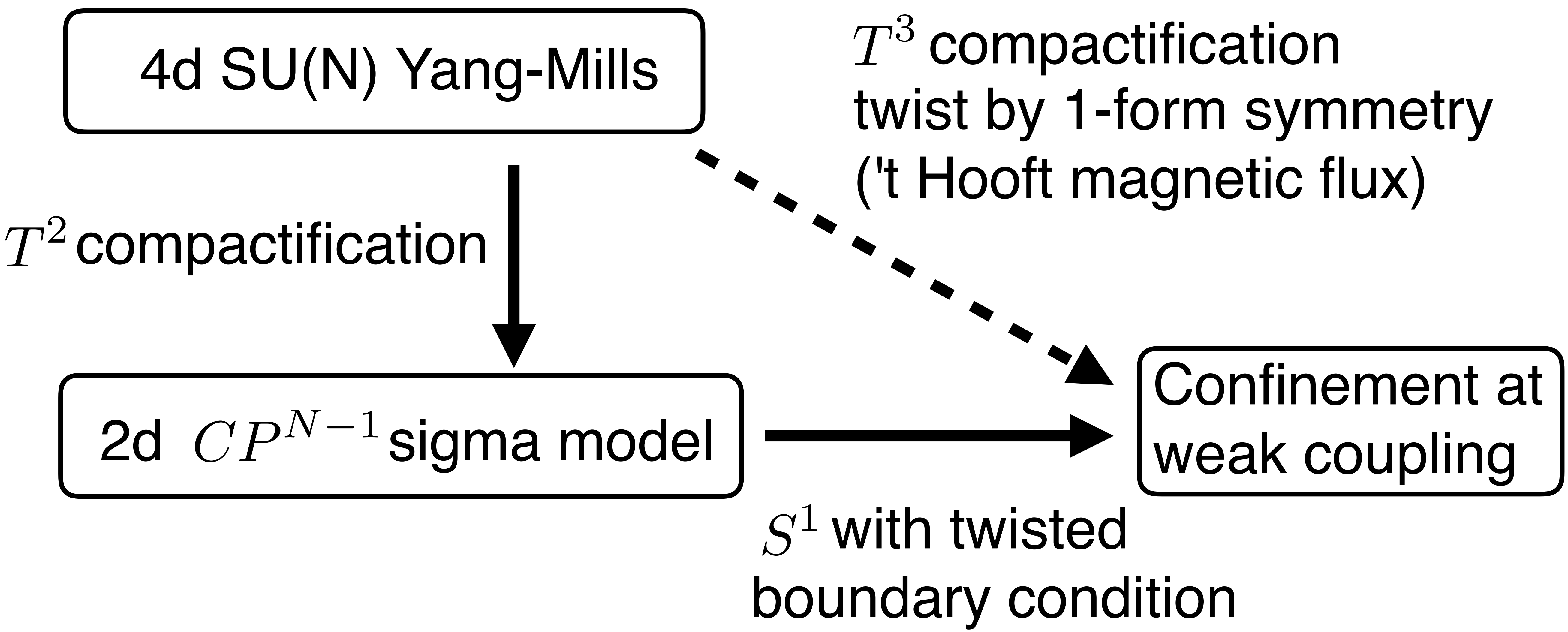
Introduction

What is a framework which may have well-defined trans-series expansion without IR divergences?

Strategy: [many people in the audience]
compactify the space in a way there is no IR divergence.

- We will consider $R_{\rm time} \times T^3$ with twist
- The running coupling is evaluated at the length scale of T^3 : small radius → weak coupling

Short Summary



IR divergence is eliminated by twisted boundary condition.

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Sigma model

Sigma model Lagrangian:

$$\mathcal{L} = h_{ij}\partial_\mu \phi^i \partial^\mu \phi^j \qquad (\ \phi^i\text{: sigma model field})$$

Propagator:

$$\langle \phi^i(x)\phi^j(0)\rangle \sim \int \frac{d^2k}{(2\pi)^2}\frac{e^{ikx}}{k^2}$$

<u>IR divergences analogous to 4d Yang-Mills:</u>

- Log IR divergence of propagator at $k \to 0$
  (The essence of the “no Goldstone boson theorem”)
  [Coleman,1973]

- Instanton integral is also divergent

Twisted compactification

Homogeneous coordinates of CP^{N-1}

$$[z_1, z_2, \cdots, z_N]$$

It has global symmetry $Z_N \subset SU(N)$

$$[z_1, z_2, \cdots, z_N] \to [e^{2\pi i/N} z_1, e^{4\pi i/N} z_2, \cdots, e^{2N\pi i/N} z_N]$$

<u>Twisted compactification on S^1:</u>
boundary condition is twisted by the above Z_N transformation

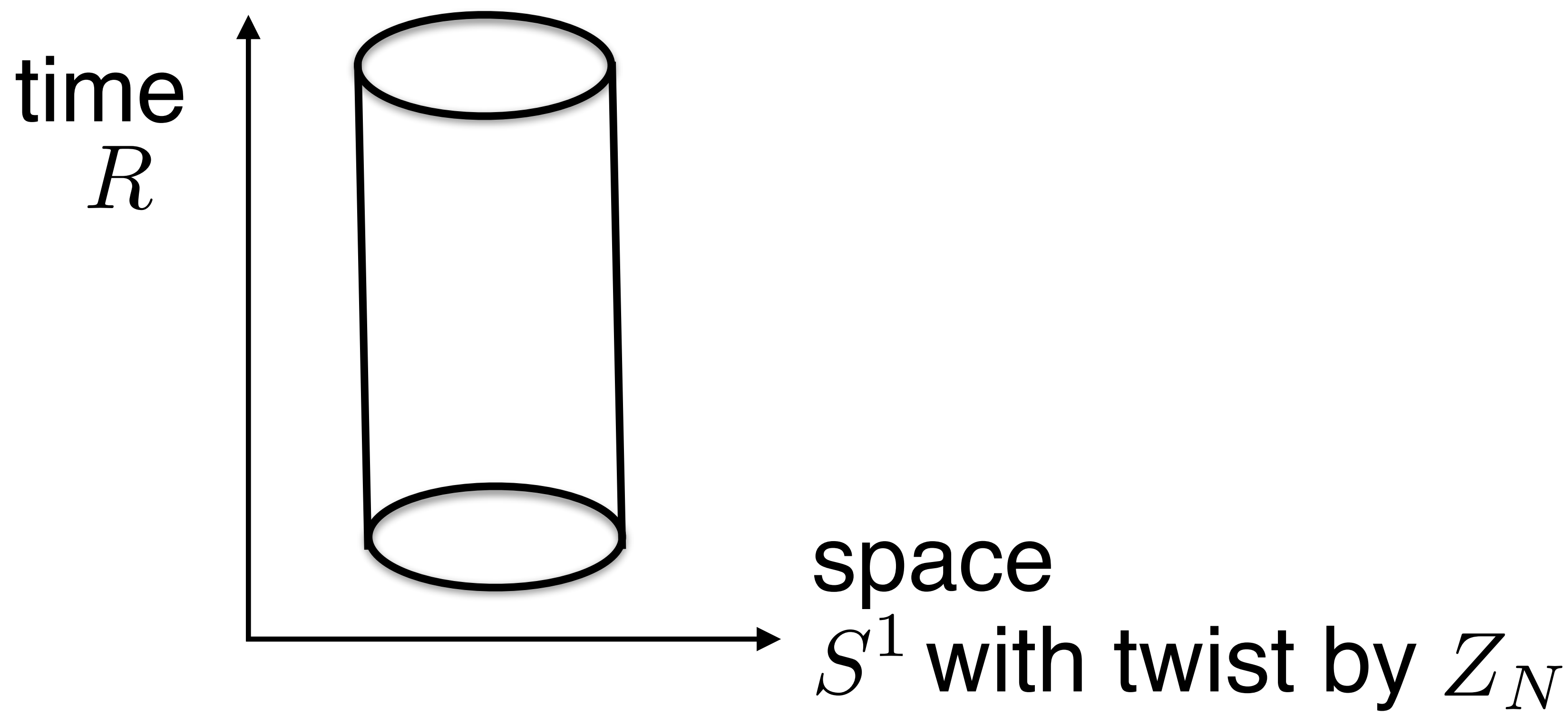
$$[\cdots, z_k(x+2\pi), \cdots] = [\cdots, e^{2\pi i k/N} z_k(x), \cdots]$$
$$x \in S^1$$

[Dunne-Unsal,2012]

Twisted compactification

$$[\cdots, z_k(x+2\pi), \cdots] = [\cdots, e^{2\pi i k/N} z_k(x), \cdots]$$

$$x \in S^1$$



This boundary condition kills all zero modes
→no IR divergence at all.

Classical vacua

Classical vacua are given by the fixed points of the twisting.

Vacuum: $\phi = [\cdots, z_k, \cdots]$ independent of $x \in S^1$

$$\downarrow$$

$$[\cdots, z_k, \cdots] = [\cdots, e^{2\pi i k/N} z_k, \cdots]$$

$\downarrow$ z_k : homogeneous coordinates

$\boxed{\text{}}$

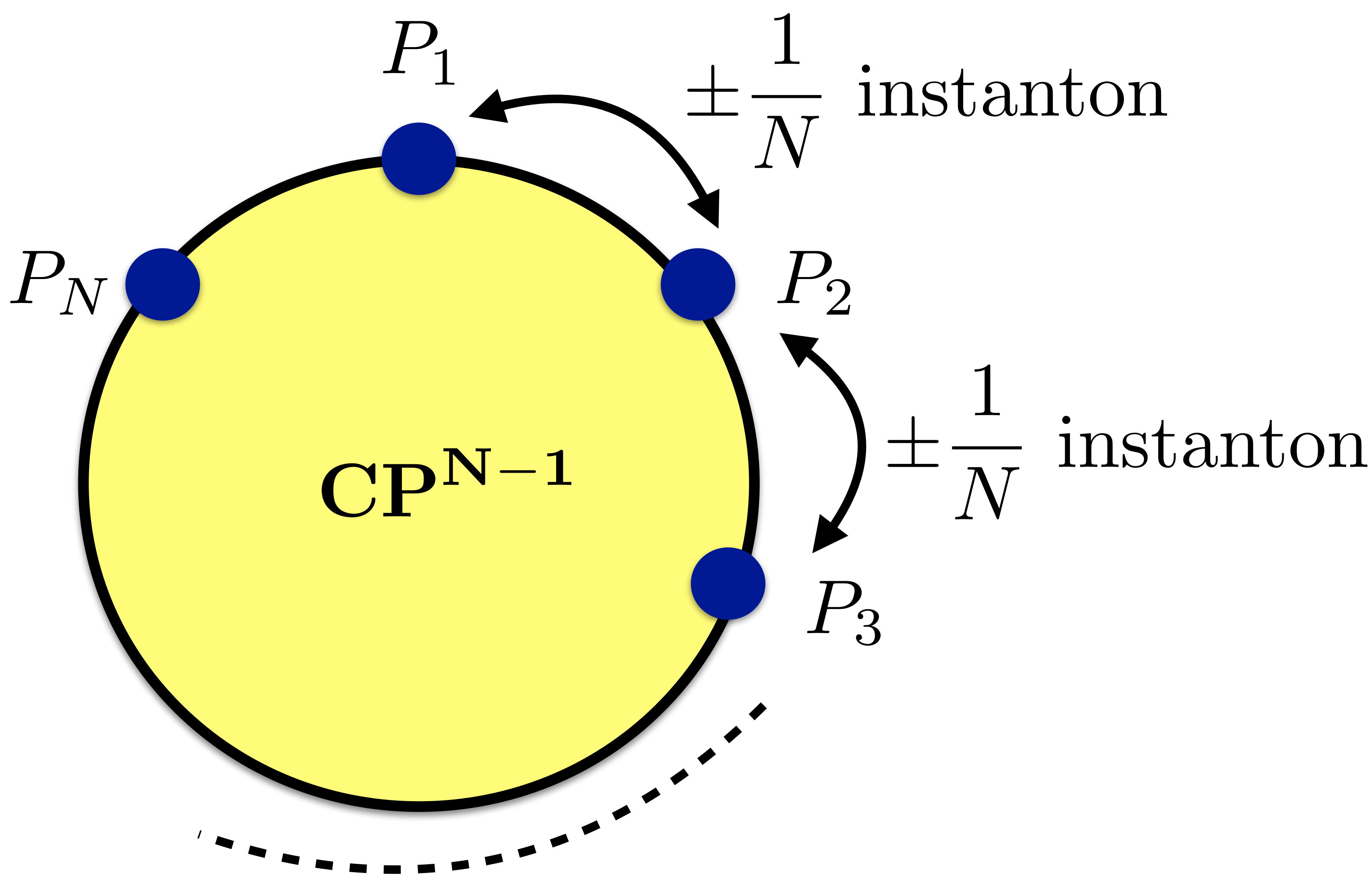
$\mathbf{N}$ discrete vacua: fixed points on $\mathbf{CP^{N-1}}$ by symmetry

$$P_k = [0, \cdots, 0, \overset{k}{1}, 0, \cdots, 0] \qquad (k = 1, 2, \cdots, N)$$

Quantum vacua

Degeneracy of classical vacua is lifted by fractional instantons.

[Eto-Fujimori-Isozumi-Nitta-Ohashi-Ohta-Sakai,…]



Quantum vacua

$|P_k\rangle$: classical vacuum at the point P_k

Due to fractional instantons, the true quantum vacua are

$$|\ell\rangle = \sum_{k=1}^{N} e^{2\pi i k\ell/N} |P_k\rangle \qquad \ell \in \mathbb{Z}_N$$

Vacuum energy:

$$E_\ell(\theta) \propto -\cos(\frac{\theta + 2\pi\ell}{N})$$

$$\to (\theta + 2\pi\ell)^2 \text{ (in large N limit)}$$

Perfect agreement with large N result!

Contents

Compactification of Yang-Mills

Let’s compacfity 4d Yang-Mills on a torus T^2
and perform KK reduction.

(Classical) massless modes:
Flat connections $F_{\mu\nu} = 0$ of the gauge field on T^2

Compactification of Yang-Mills

Theorem 1 [Looijenga]

As an algebraic variety, the moduli space of SU(N) flat connections on T^2 is given by CP^{N-1}

Theorem 2 (Crude statement) [Friedman-Morgan-Witten]

Yang-Mills instanton on $T^2 \times \Sigma$ is essentially given by CP^{N-1} instanton on Σ . (Σ : Riemann surface)

(For more precise statements see their paper.)

1-form Z_N symmetry

The Z_N global symmetry of CP^{N-1}

$$[z_1, z_2, \cdots, z_N] \to [e^{2\pi i/N} z_1, e^{4\pi i/N} z_2, \cdots, e^{2N\pi i/N} z_N]$$

It turns out that this is realized by dimensional reduction of 1-form center symmetry of SU(N) Yang-Mills

$$U \to e^{2\pi i/N} U$$

U : Wilson line

$e^{2\pi i/N}$: in the center Z_N of SU(N)

A little more details are discussed later.

Correspondence in 4d/2d

4d SU(N) Yang-Mills

T^2 compactification

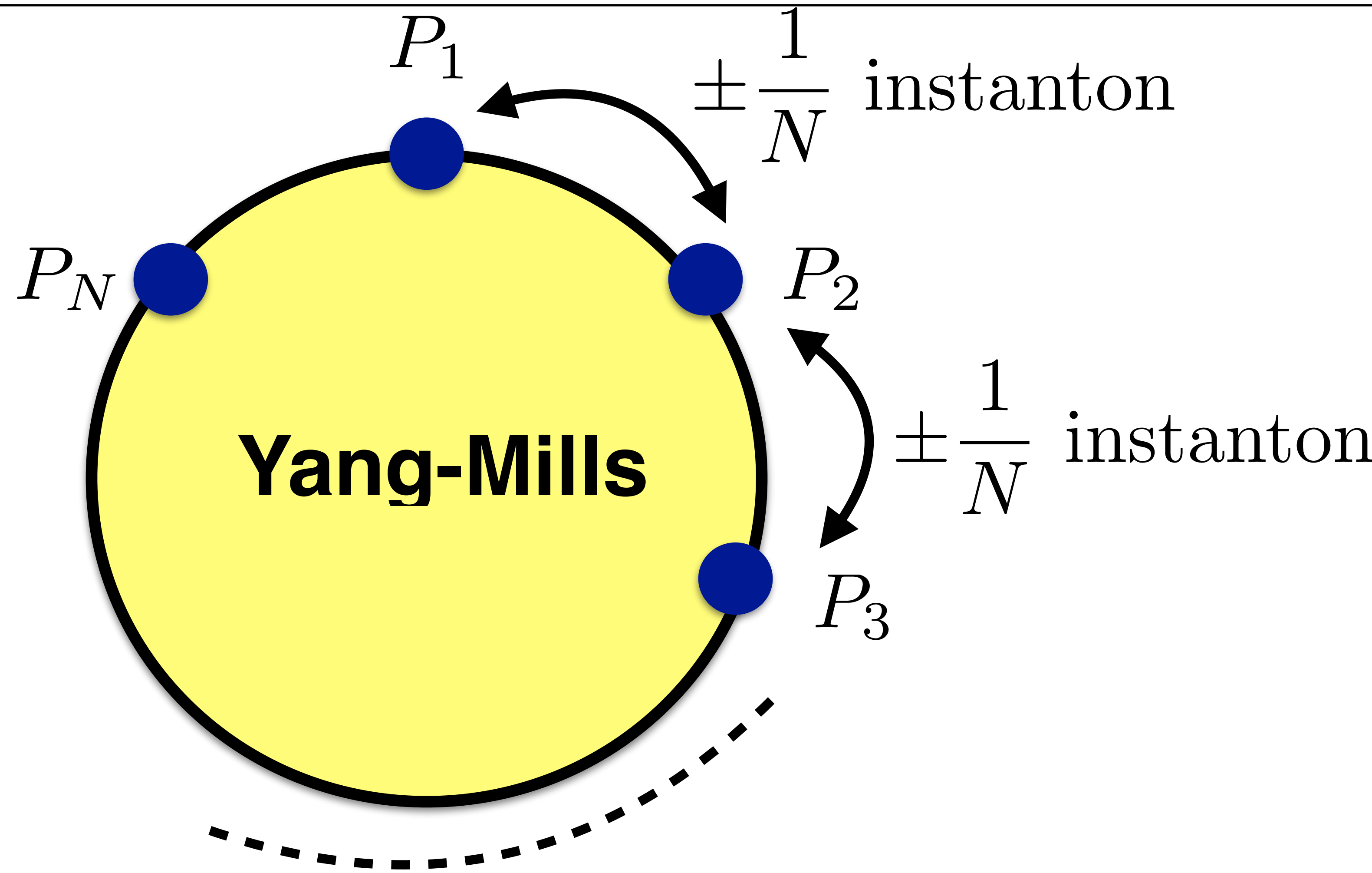
2d CP^{N-1} sigma model

Correspondence on

- Instantons
- Z_N symmetry

These properties are enough to guarantee the argument of the next slide:

Quantum vacua of Yang-Mills



Quantum vacua: $|\ell\rangle = \sum_{k=1}^{N} e^{2\pi i k\ell/N}|P_k\rangle \qquad \ell \in \mathbb{Z}_N$

$$E_\ell(\theta) \propto -\cos\left(\frac{\theta+2\pi\ell}{N}\right)$$

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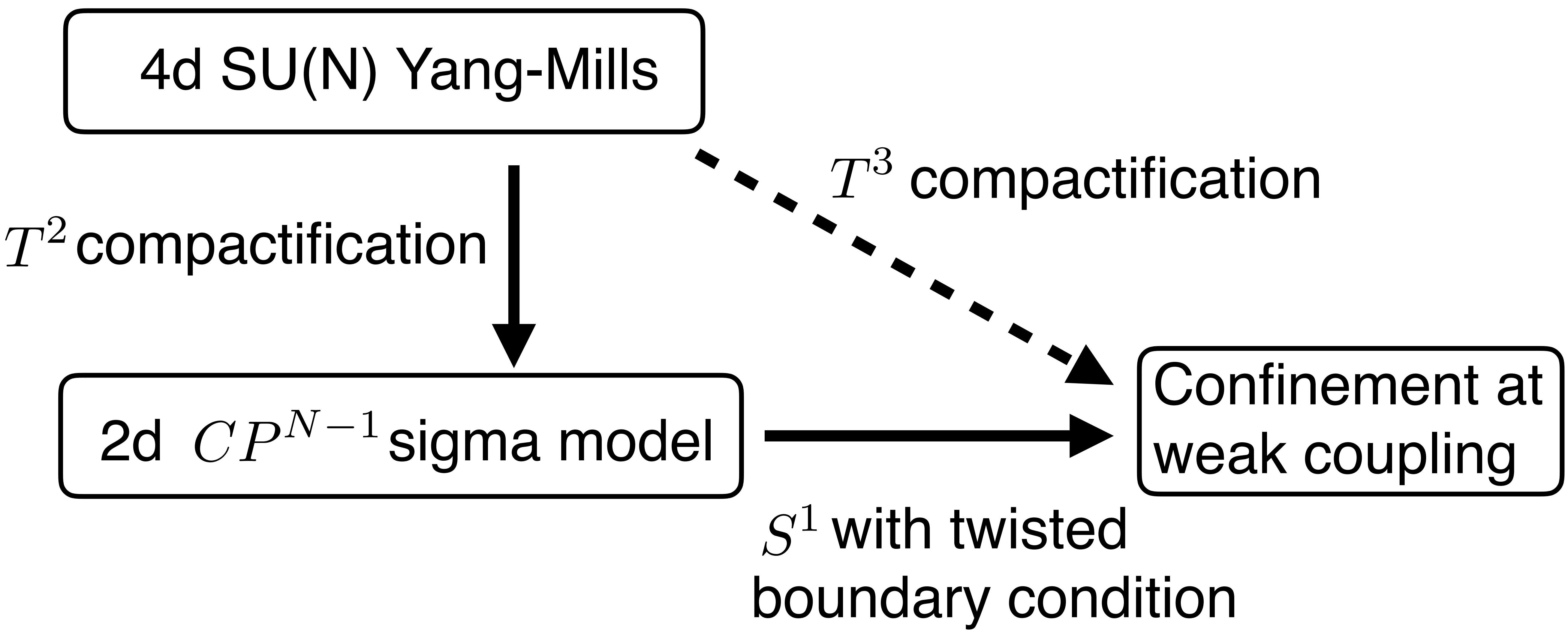
Reminder of Situation



Yang-Mills theory is put on $R_{\text{time}} \times T^3$

Yang-Mills flat connections

What are the states $|P_k\rangle$ in Yang-Mills in the box T^3 ?

Lowest energy states

$$\downarrow$$

$$F_{\mu\nu} = 0$$

Flat connections on $T^3 = S^1_A \times S^1_B \times S^1_C$

Characterized by $N \times N$ matrices

(U_A, U_B, U_C) :Wilson lines

$$U_{A,B,C} = \exp i \int_{S_{A,B,C}} A_\mu dx^\mu$$

1-form center symmetry

Z_N 1-form center symmetry

- 1-form symmetry acts on line operators like Wilson lines (ordinary “0-form” symmetry acts on local operators)

- center symmetry is, roughly, the symmetry

$$U \to e^{2\pi i/N} U$$

U : Wilson line

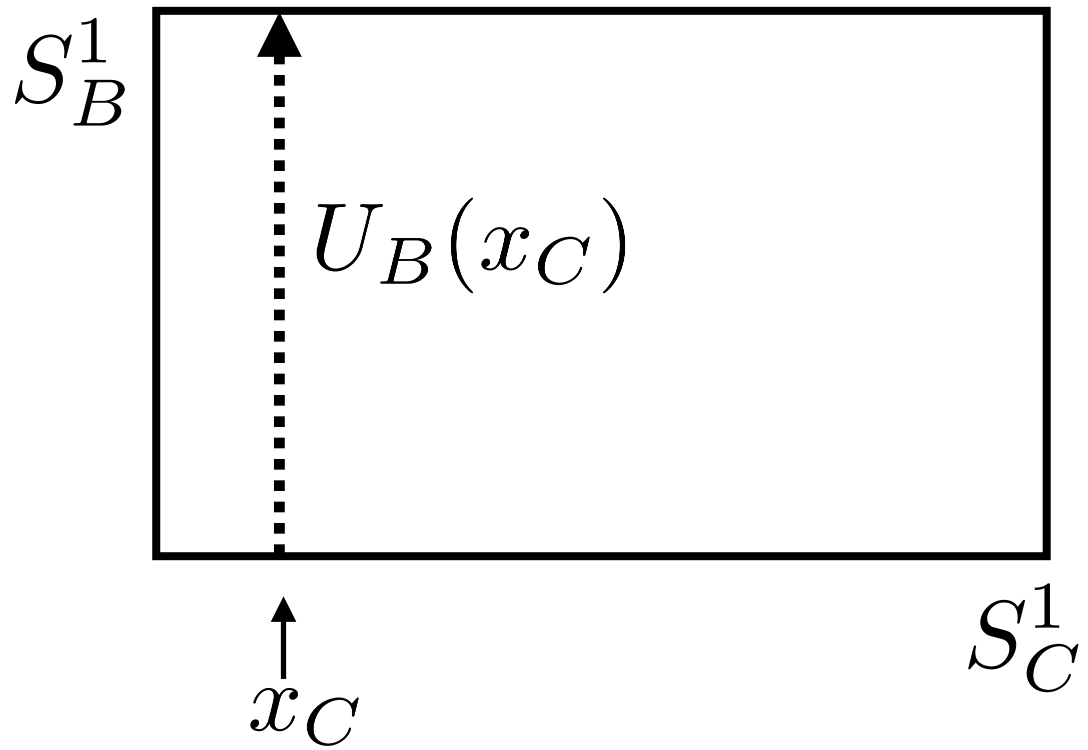
$e^{2\pi i/N}$: in the center Z_N of SU(N)

Twist by 1-form symmetry

How can we implement twisted compactification by the 1-form center symmetry? ['t Hooft,1979]

I'm going to give a rough explanation.
Please don't care subtle details (because I don't remember).

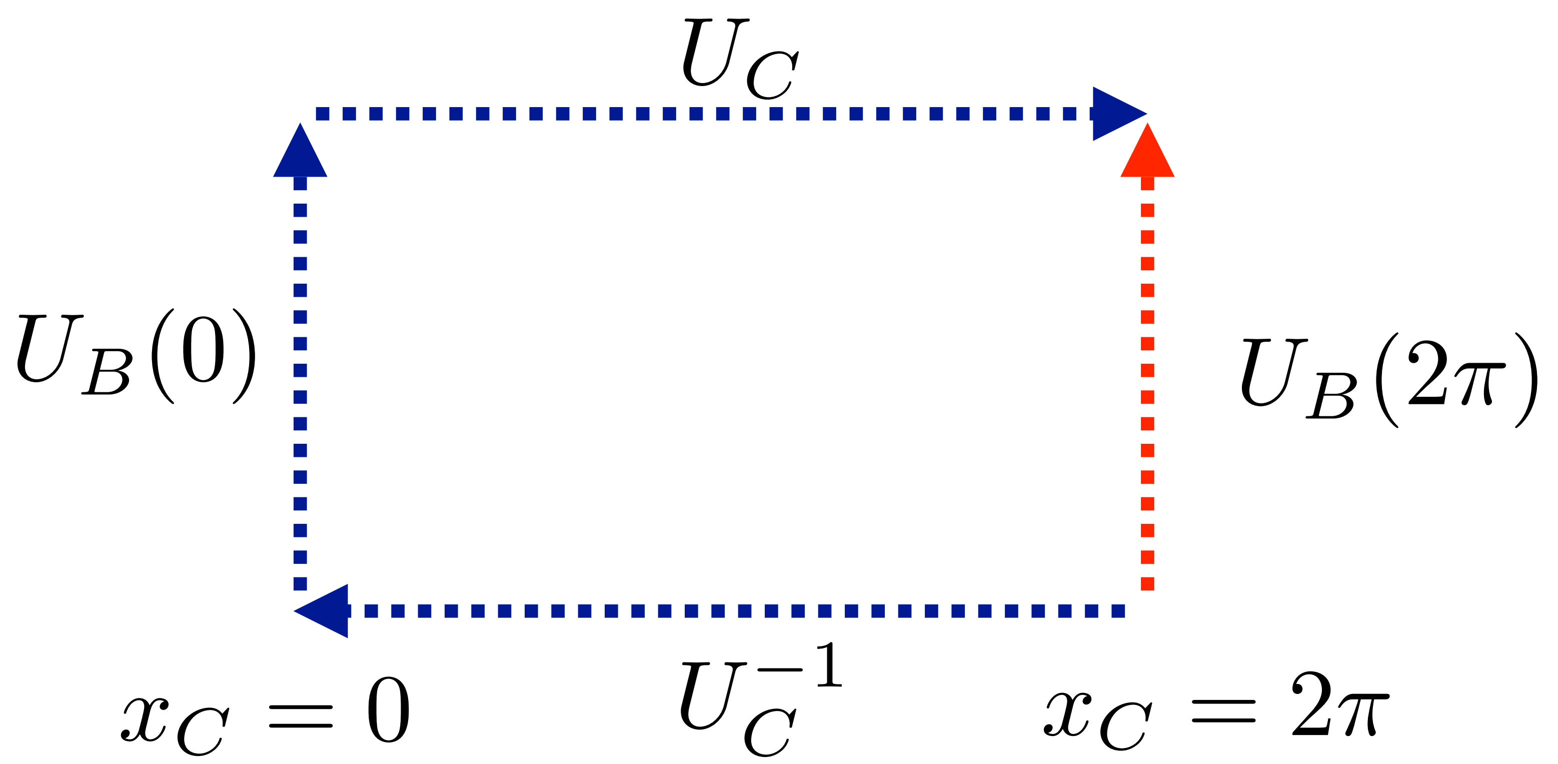
Twist



Twisted boundary condition by 1-form symmetry

$$U_B(x_C + 2\pi) = e^{2\pi i/N} U_B(x_C)$$

Twist



$$U_B(2\pi) = U_C U_B(0) U_C^{-1} \qquad \text{(if } F_{\mu\nu} = 0\text{)}$$

Twist

- $U_B(2\pi) = e^{2\pi i/N} U_B(0)$
- $U_B(2\pi) = U_C U_B(0) U_C^{-1}$

Combining them and defining $U_B := U_B(0)$, we get

$$\boxed{\begin{aligned} U_C U_B &= e^{2\pi i/N} U_B U_C \\ U_A U_B &= U_B U_A \\ U_A U_C &= U_C U_A \end{aligned}}$$

(The U_A have trivial commutation relations in our twist.)

Classical vacua

Classical vacua are solutions of the commutation relations up to gauge transformations:

$$\boxed{\begin{gathered} U_C U_B = e^{2\pi i/N} U_B U_C \\ U_A U_B = U_B U_A \\ U_A U_C = U_C U_A \end{gathered}}$$

Algebra: U_C may be regarded as "lowering operator"

If $U_B \vec{v} = e^{i\alpha} \vec{v}$ then $U_B (U_C \vec{v}) = e^{i\alpha - 2\pi i/N} (U_C \vec{v})$

Classical vacua

Solutions up to gauge transformation [Witten,1982]

$$U_B = \mathrm{diag}(1, e^{2\pi i/N}, e^{4\pi i/N} ...)$$

$$U_C = (\delta_{i+1,j})$$

$$U_A = e^{2\pi i k/N}, \quad k = 1, 2, \cdots, N$$

It turns out that N classical vacua are

$$\boxed{U_A = e^{2\pi i k/N} \quad \longleftrightarrow \quad |P_k\rangle}$$

Confinement

Definition:

confinement $\longleftrightarrow$ center symmetry unbroken

How about in our case?

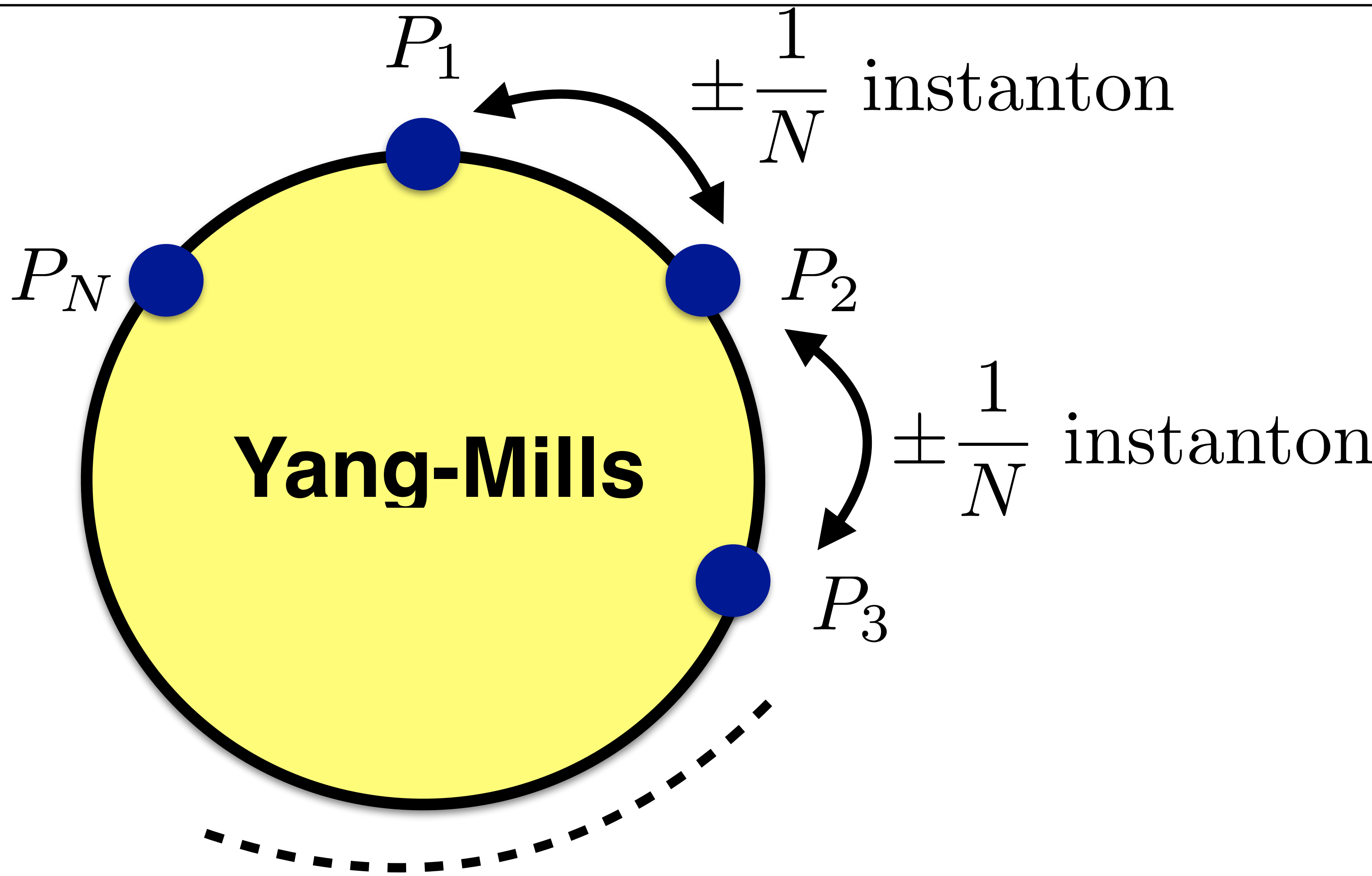
(Part of) relevant gauge invariant operators:

$$\mathrm{tr}(U_B) = \mathrm{tr}(U_C) = 0$$

$$\mathrm{tr}(U_A) = N e^{2\pi i k/N} \neq 0$$

The center symmetry is broken by the nonzero vacuum expectation value of $\mathrm{tr}(U_A)$ at each classical vacua $|P_k\rangle$

Confinement



Quantum vacua: $|\ell\rangle = \sum_{k=1}^{N} e^{2\pi i k\ell/N}|P_k\rangle \qquad \ell \in \mathbb{Z}_N$

$\langle \ell|\mathrm{tr}(U_A)|\ell\rangle = 0$ **:confinement**

Remarks

The center symmetry restoration itself is not surprising because all the spatial directions are compactified.

The points are:

- We realized it in completely weakly coupled regime
- Our results are expected to be continued to large volume (assuming resurgence and mass gap)
- I will later discuss an example of center symmetry breaking in the presence of fermions.

Contents

Adjoint fermions

Let's include fermions λ in the adjoint representation.

Just for simplicity, in this talk I discuss the case of a single adjoint fermion

$\longrightarrow$ N=1 Super-Yang-Mills

Axial symmetry

$$Z_{2N}: \ \lambda \to e^{2\pi i/2N}\lambda$$

Axial current

Naive axial current $J^\mu \sim \bar{\lambda}\gamma_5\gamma^\mu\lambda$

$$\partial_\mu J^\mu = N \cdot \frac{1}{16\pi^2}\mathrm{tr}F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}$$

$$\Downarrow$$

Conserved axial current

$$\partial_\mu \tilde{J}^\mu = 0$$

$$\tilde{J}^\mu = J^\mu - N \cdot \mathrm{CS}^\mu$$

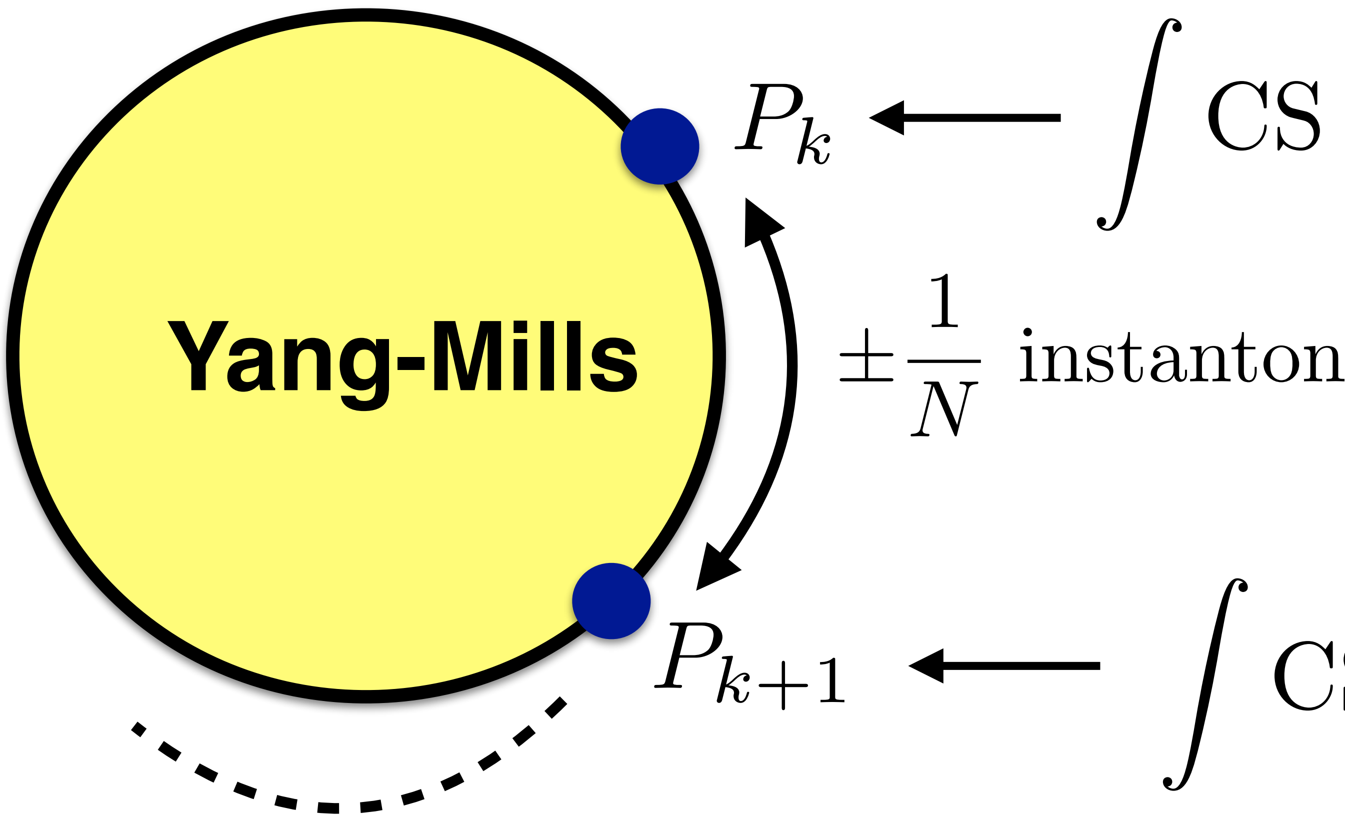
CS^μ : Chern-Simons

$$\partial_\mu(\mathrm{CS}^\mu) = \frac{1}{16\pi^2}\mathrm{tr}F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}$$

Axial charge

Axial charge: $Q = \int d^3x \tilde{J}^0$

$$= \int d^3x J^0 - N \int \mathrm{CS}$$



$$P_k \longleftarrow \int \mathrm{CS} = \frac{k}{N} \mod 1$$

$$P_{k+1} \longleftarrow \int \mathrm{CS} = \frac{k+1}{N} \mod 1$$

Axial charge

Q : defined only modulo N (due to Chern-Simons)

$e^{2\pi i Q/N}$: well-defined charge for discrete Z_{2N}

$|P_k\rangle$ are eigenstates of the axial charge $e^{2\pi i Q/N}$

$$e^{2\pi i Q/N}|P_k\rangle = e^{2\pi i k/N}|P_k\rangle$$

Axial v.s. center symmetry

$|P_k\rangle$: eigenstates of axial charge

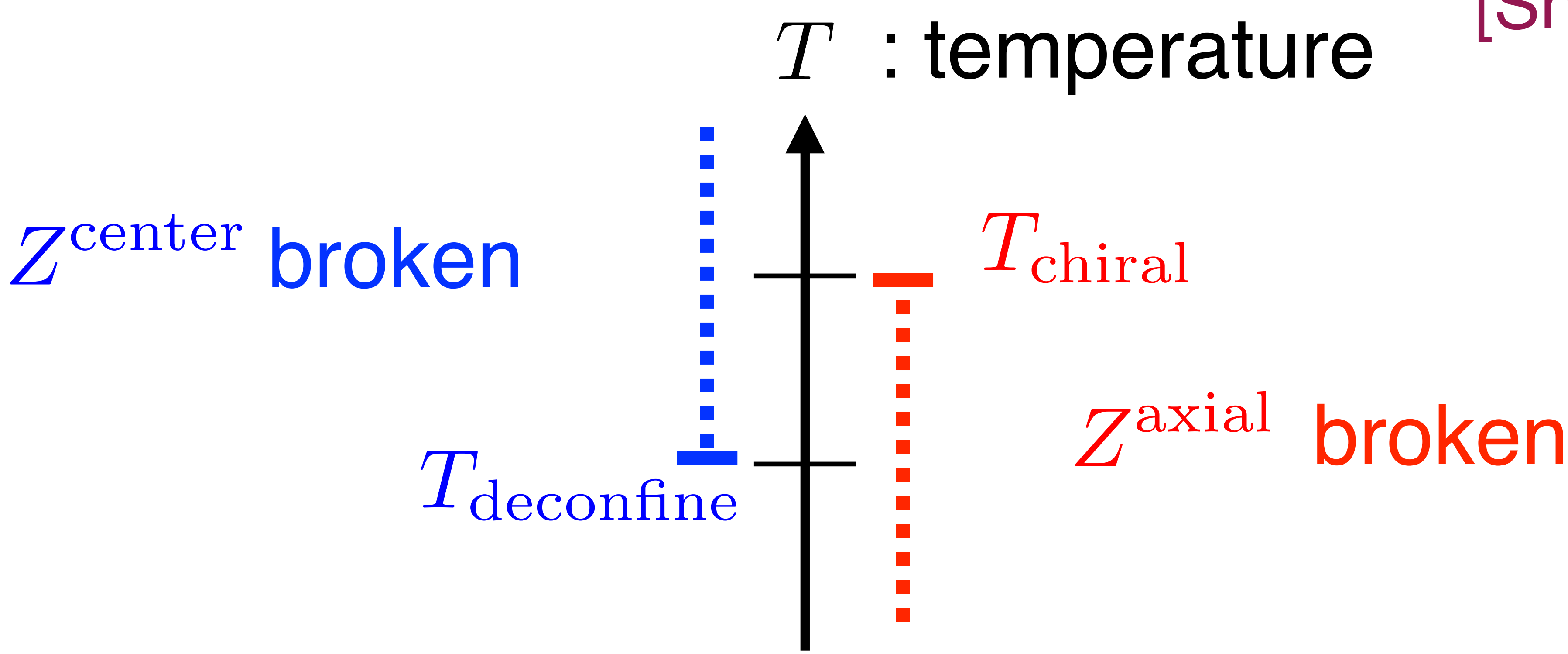
$$|\ell\rangle = \sum_{k=1}^{N} e^{2\pi i k \ell / N} |P_k\rangle$$: eigenstates of center symmetry (confinement)

In Hilbert space, there is no simultaneous eigenstate of the axial symmetry and the center symmetry!
One of them (or both of them) is always spontaneously broken.

Thermal phase transition

Finite temperature scenario

[Komargodski-Sulejmanpasic-Unsal]
[Shimizu-Yonekura]



$$T_{\text{deconfine}} \leq T_{\text{chiral}}$$

The equality is possible only for 1st order transition

Remarks

- More deep reason behind it is a mixed anomaly between axial and center symmetry

[Gaiotto-Kapustin-Seiberg-Willett]

- It is also possible to constrain phase transitions with fermions in the fundamental representation (massless QCD!). [Shimizu-Yonekura]

- For details, please see our paper

Contents

Summary

- A setup which is free from IR divergences reproduce qualitative features of confinement in Yang-Mills even at weak coupling regime.

- Vacuum structure has rich phenomena such as
  - Nontrivial θ angle dependence of vacuum energy
  - Relation between confinement and axial symmetry