Resurgence Structure to All Orders of Multi-bions in Deformed SUSY Quantum Mechanics

Toshiaki Fujimori (Keio University)

based on arXiv:1607.04205, Phys.Rev. D94 (2016)
arXiv:1702.00589, Phys.Rev. D95 (2017)
arXiv:1705.10483, PTEP (2017)

Syo Kamata (Fudan U.), Tatsuhiro Misumi (Akita U.),
Muneto Nitta (Keio U.), Norisuke Sakai (Keio U.),

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Resurgence Structure

perturbation series

g^2 : **coupling constant**

$$Z = a_0 + a_1 g^2 + a_2 g^4 + \cdots$$

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g^2 : **coupling constant**

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divergent

Resurgence Structure

Borel resummation

g^2 : **coupling constant**

$$Z = \int_0^\infty dt\, e^{-\frac{t}{g^2}} B(t)$$

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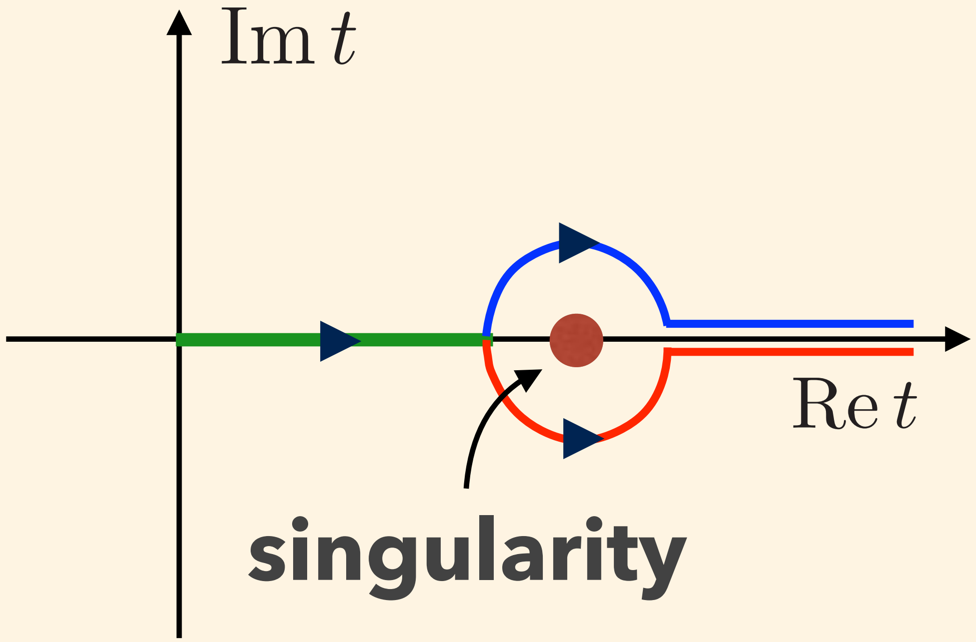
ambiguity



$$Z_+ \neq Z_-$$

Resurgence Structure

resurgent trans-series

g^2 : coupling constant

$$Z = Z_0 + \underline{Z_1 + Z_2 + \cdots}$$

perturbative

non-perturbative with ambiguities

$$Z_p = e^{-\frac{S_p}{g^2}} \left[a_{p,0} + a_{p,2}\, g^2 + a_{p,4}\, g^4 + \cdots \right]$$

cancellation of ambiguities : Z is unambiguous

Explicit Resurgence Structure

- exact results in (localizable) SUSY models

[Russo 2012, Anice-Russo-Schiappa 2014,
Couso-Santamara-Schiappa-Vaz 2015, Honda 2016,
Gukov-Marino-Putrov 2016,…]

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this talk: small SUSY breaking parameter $\delta\epsilon$

[Dunne-Ünsal 2016]

- perturbative expansion w.r.t. $\delta\epsilon$ from the SUSY point
- fully non-perturbative w.r.t. g^2

explicit and non-trivial resurgent structure

CP^{N-1} model

2d CP^{N-1} model : toy model of 4d gauge theory

asymptotic freedom, instanton, large N, etc

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important role in 4d gauge theories [Ünsal 2007 …]

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- small S^1 limit ・・・Quantum mechanics

all-order bion contribution

Plan of Talk

1. CP^1 Quantum Mechanics and ``Exact Results”

2. Bion Solutions : Saddle Points in Complexified CP^1 QM

3. Multi-Bion Contributions and Quasi-Moduli Integral

4. Generalization to CP^{N-1} Quantum Mechanics

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   - dimensional reduction from 2d SUSY CP^1 model
   - ground state energy : trivial resurgence structure
   - SUSY breaking deformation : non-trivial structure
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   - **analytic continuation of action (complexification)**
   - **multi-bion solution ··· not on the original contour**

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    - **kink gas with complexified position and phase**
    - **Lefschetz thimble method : integration contour**

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    - **N-1 types of bions**
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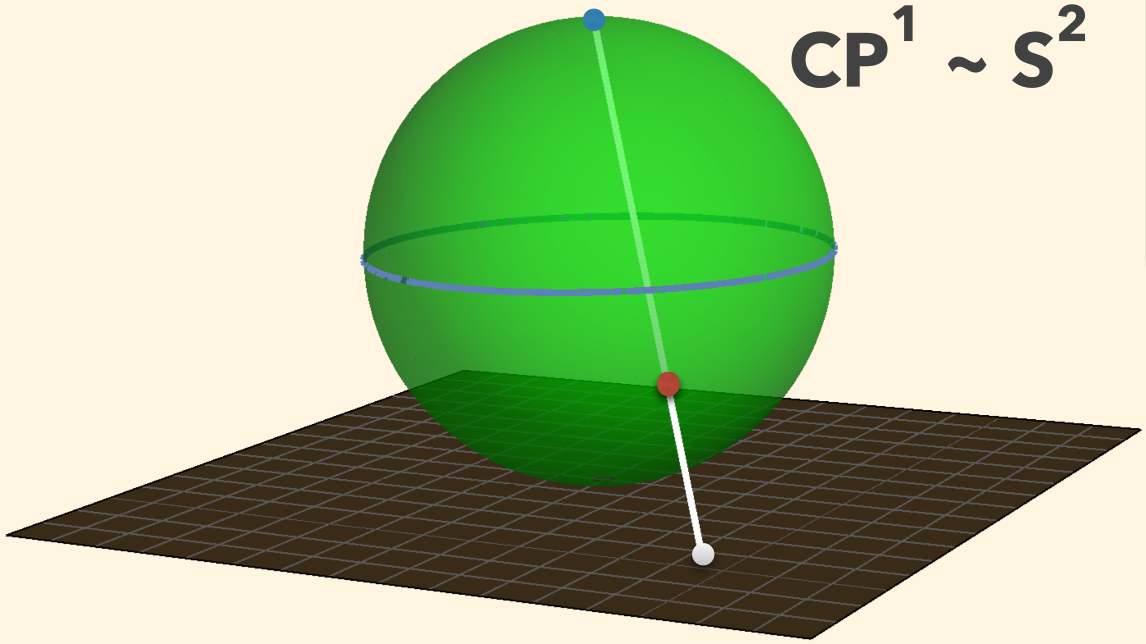
CP¹ model in two dimensions

2d $\mathcal{N} = (2,0)$ supersymmetric CP¹ model

$$\mathcal{L} = \frac{1}{g^2} \frac{1}{(1+|\varphi|^2)^2} \left[|\partial_\mu \varphi|^2 + i\bar{\psi}\mathcal{D}_- \psi \right]$$



complex φ-plane

φ : **inhomogeneous coordinate**

ψ : **fermion**

$$\mathcal{D}_- = \frac{1}{2}(\mathcal{D}_t - \mathcal{D}_x)$$

$$\mathcal{D}_i \psi = \left(\partial_i - \frac{2\bar{\varphi}}{1+|\varphi|^2} \partial_i \varphi \right) \psi$$

CP^1 model in two dimensions

2d $\mathcal{N} = (2,0)$ **supersymmetric CP^1 model**

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$(t, x) \sim (t, x + 2\pi R)$ **: coordinate on cylinder** $\mathbb{R} \times S^1$

twisted boundary condition

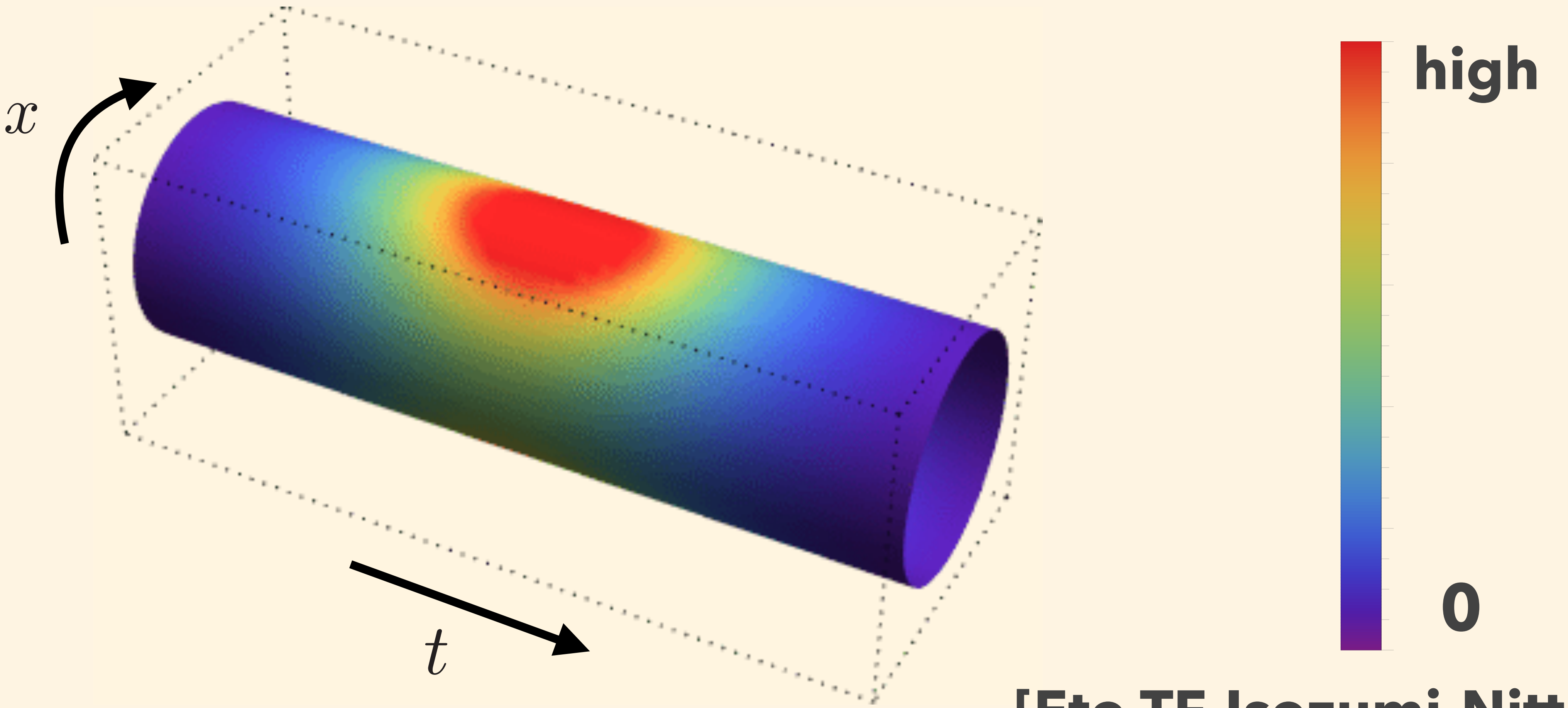
$$\varphi(x + 2\pi R) = e^{imx}\varphi(x)$$

$$\psi(x + 2\pi R) = e^{imx}\psi(x)$$

$m \in \mathbb{R}$

Fractional Instanton

- Euclidean action density of instanton on $\mathbb{R} \times S^1$



[Eto-TF-Isozumi-Nitta-Ohashi-Ohta-Sakai, 2006]

- instanton ➡ two fractional instantons (N for CP^{N-1})

kinks in small S^1 limit ••• quantum mechanics

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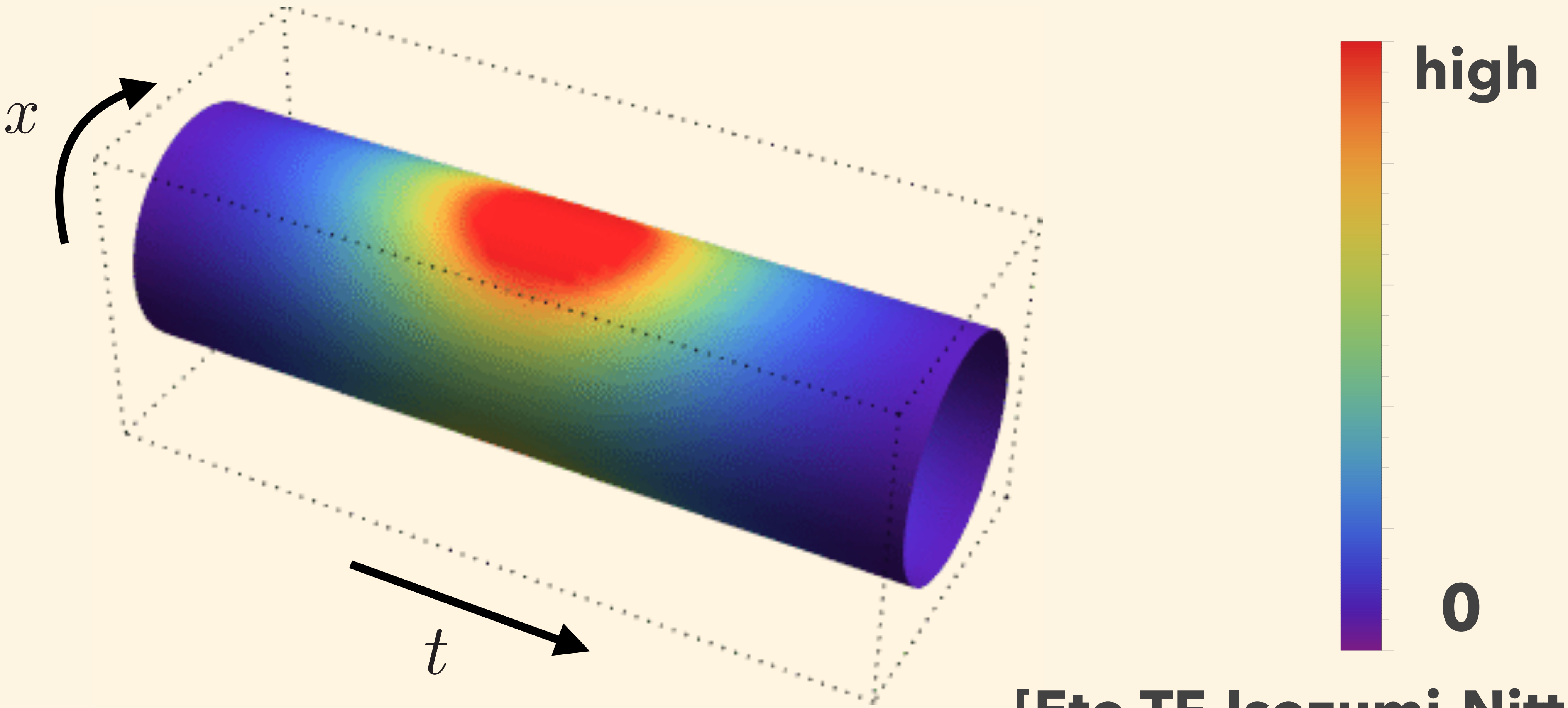

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Dimensional Reduction

2d $\mathcal{N} = (2,0)$ supersymmetric CP^1 model

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twisted boundary condition $m \in \mathbb{R}$

small S^1 limit $\cdots$ dimensional reduction

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↓

twisted boundary condition $m \in \mathbb{R}$

small S¹ limit ・・・ dimensional reduction

supersymmetric CP¹ quantum mechanics

$$L = \frac{1}{g^2}\frac{1}{(1+|\varphi|^2)^2}\left[|\dot{\varphi}|^2 - m^2|\varphi|^2 + i\bar{\psi}\mathcal{D}_t\psi - m\frac{1-|\varphi|^2}{1+|\varphi|^2}\bar{\psi}\psi\right]$$

Fermion Projection

- **Fermionic part of Lagrangian**

$$L = \cdots + i\bar{\psi}\partial_t\psi + A(\varphi, \partial_t\varphi)\bar{\psi}\psi$$

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$$H = H_{f=0} \oplus H_{f=1} \quad \Rightarrow \quad Z = Z_{f=0} + Z_{f=1}$$

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- **partition function of f=0 sector**

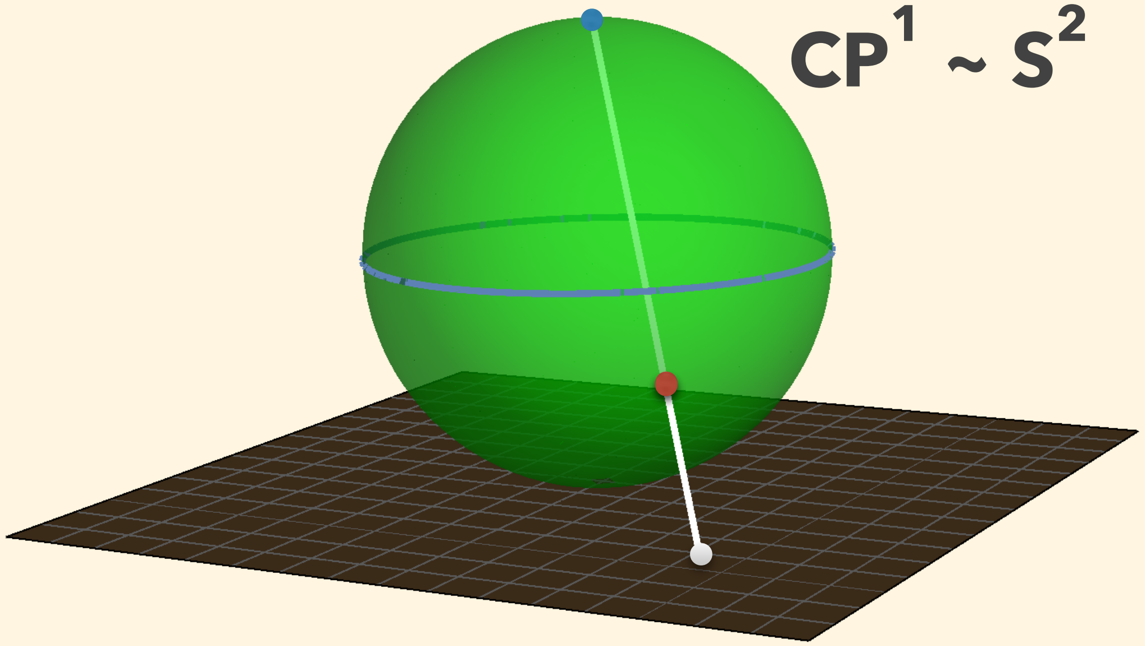
$$Z_{f=0} = \int \mathcal{D}\varphi \, \exp\left[- \int d\tau (L + V_f) \right]$$

induced potential

CP^1 Quantum Mechanics

Quantum mechanics of particle on sphere

$$S = \frac{1}{g^2}\int dt\,\left[\frac{|\dot{\varphi}|^2}{(1+|\varphi|^2)^2} - V(|\varphi|)\right]$$



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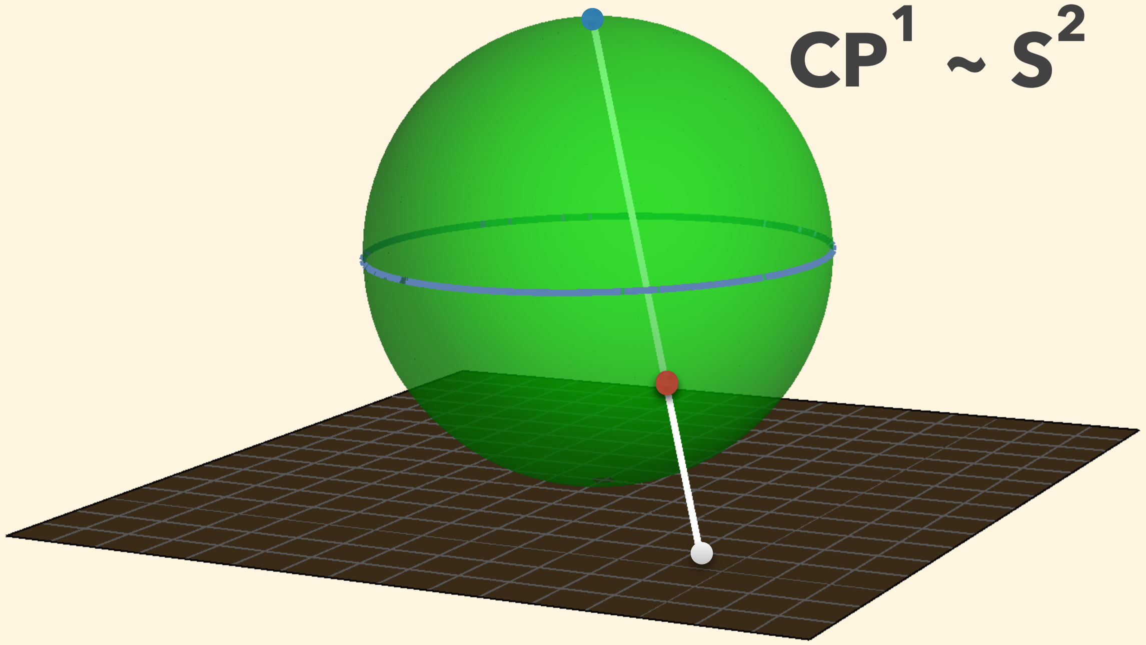


2d $\mathcal{N}=(2,0)$ SUSY CP¹ model

dim. reduction w/ twisted bc

projection to fixed fermion number

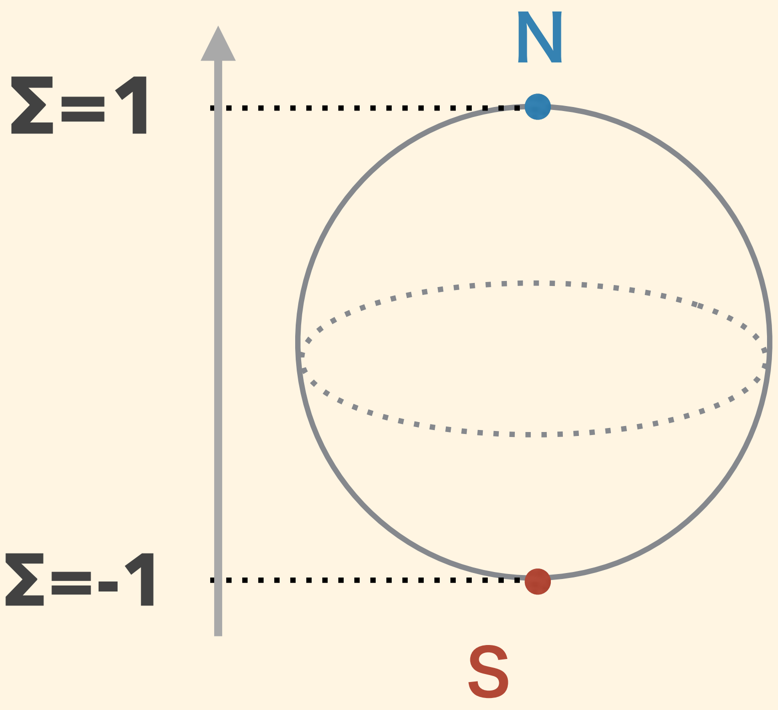
$$H = \boxed{H_{f=0}} \oplus H_{f=1}$$

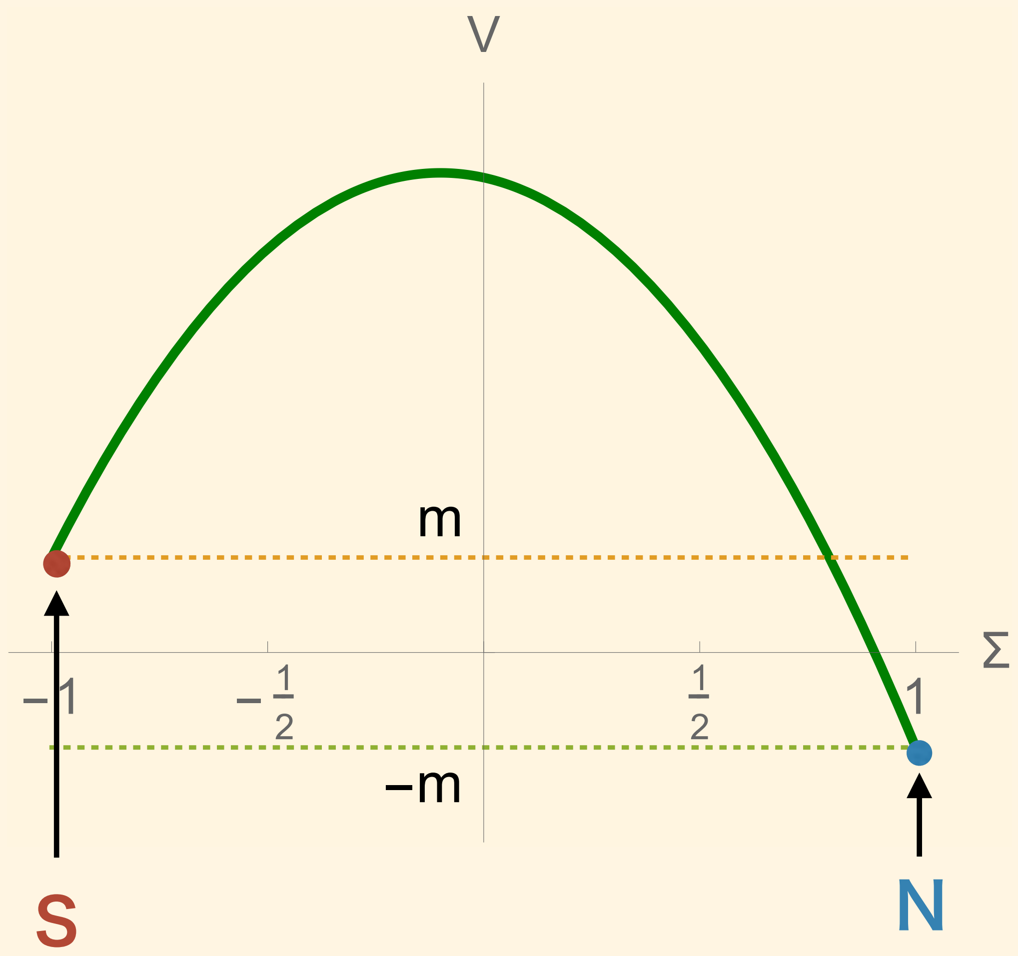
Potential on Sphere

- potential ••• twisted b.c. + fermion projection

$$V = -m\left[\frac{m}{4}\Sigma^2 + g^2\Sigma + const.\right]$$

$$\Sigma = \frac{1-|\varphi|^2}{1+|\varphi|^2}$$: height of S²





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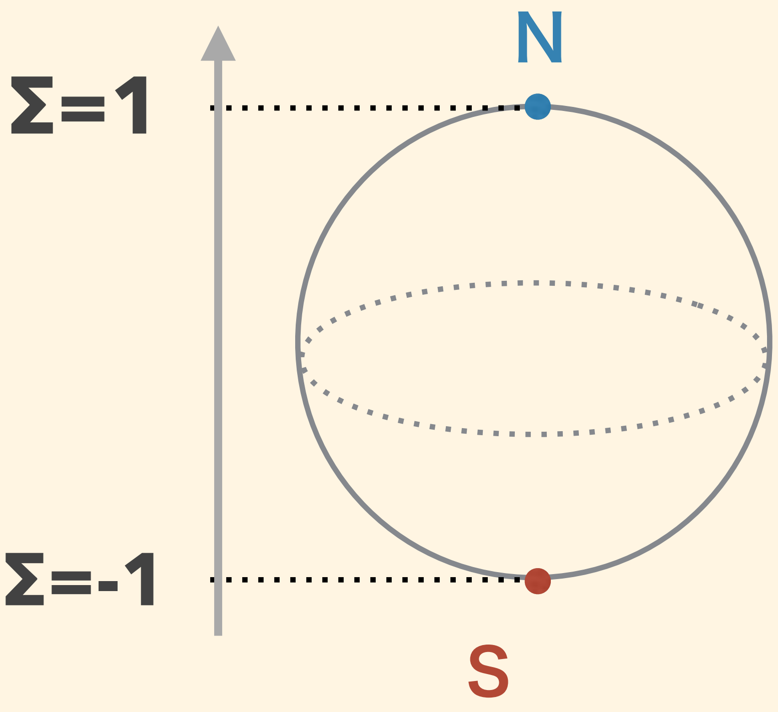

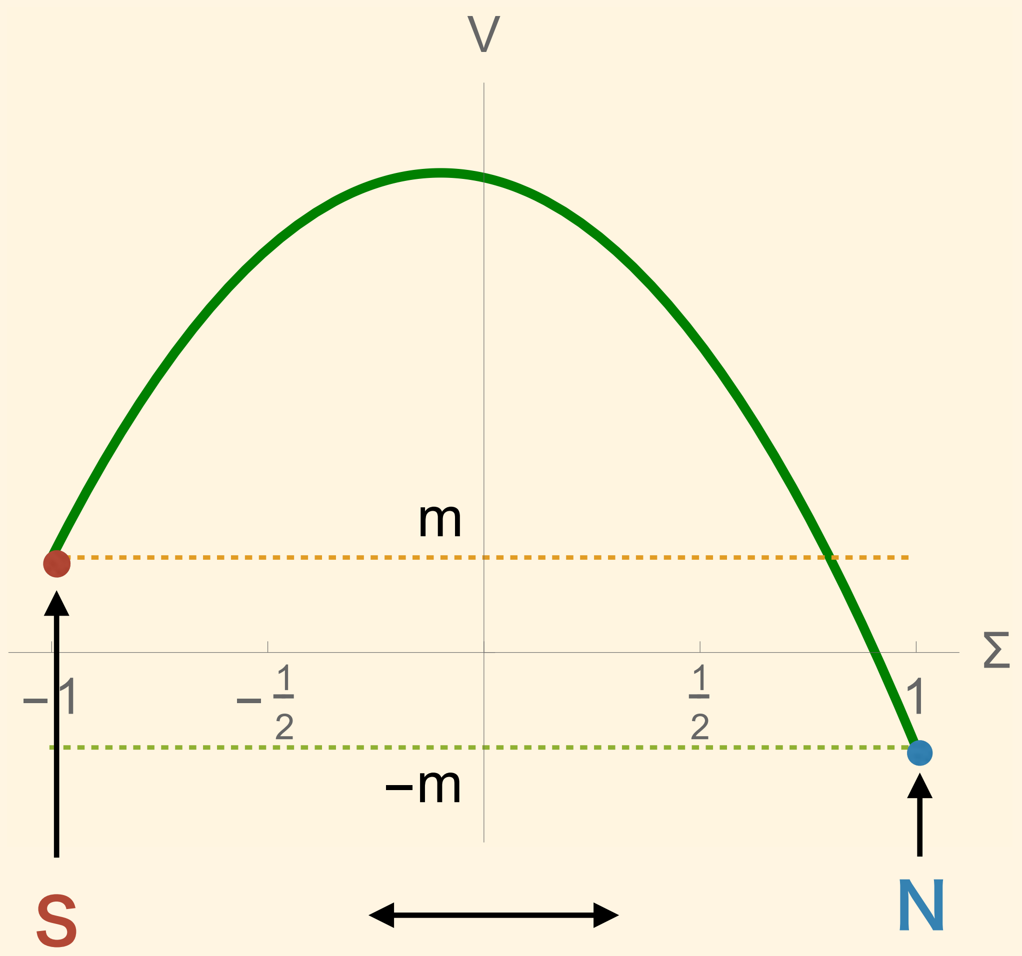

tunneling

(kink) $S_{\text{kink}} = \frac{m}{g^2}$

Ground State Energy

Schrödinger equation

$$H\Psi = \left[-g^2 \Delta + V \right] \Psi$$

Laplacian on S^2

$$\Psi = \exp\left(\frac{m}{2g^2} \Sigma \right) \qquad E = 0$$

ground state wave function

ground state energy is exactly zero due to SUSY

SUSY Breaking Deformation

- **deformation of potential**

$$V = -\frac{m^2}{4g^2}\Sigma^2 - m\Sigma$$

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- **response of E under deformation**

δε=ε-1 expansion

$$E^{(n)} = \frac{1}{n!}\frac{\partial^n}{\partial\epsilon^n}E\Big|_{\epsilon=1}$$

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- **Rayleigh-Schrödinger perturbation theory**

$$E^{(1)} = \frac{\langle 0|H^{(1)}|0\rangle}{\langle 0|0\rangle}, \qquad E^{(2)} = -\frac{\langle\Psi^{(1)}|H^{(0)}|\Psi^{(1)}\rangle}{\langle 0|0\rangle}, \qquad \cdots$$

First Order Coefficient

- **First order perturbation with respect to** $\delta\epsilon = \epsilon - 1$

$$E^{(1)} = \frac{\langle 0|H^{(1)}|0\rangle}{\langle 0|0\rangle} \quad \textbf{with} \quad H^{(1)} = -m\Sigma = -m\frac{1-|\varphi|^2}{1+|\varphi|^2}$$

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- **using explicit form of ground state wave function**

$$\langle 0|0\rangle = \int \frac{d^2\varphi}{(1+|\varphi|^2)^2} \exp\left(\frac{m}{g^2}\Sigma\right)$$

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- **relation between** $\langle 0|0\rangle$ **and** $E^{(1)}$

$$E^{(1)} = -g^2 m \frac{\partial}{\partial m} \log\langle 0|0\rangle$$

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$$E^{(1)} = -g^2 m \frac{\partial}{\partial m} \log\langle 0|0\rangle \quad \Rightarrow \quad \boxed{E^{(1)} = g^2 - m\coth\frac{m}{g^2}}$$

Bion Expansion of E$^{(1)}$

- **First order δε expansion coefficient**

$$\boxed{E^{(1)} = g^2 - m \coth \frac{m}{g^2}}$$

$\Big\downarrow$ $e^{-S_{\text{bion}}} = e^{-\frac{2m}{g^2}}$ **expansion**

$$\boxed{E^{(1)} = E_0^{(1)} + E_1^{(1)} + E_2^{(1)} + \cdots}$$

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expansion

$$\boxed{E^{(1)} = E_0^{(1)} + E_1^{(1)} + E_2^{(1)} + \cdots}$$

$$E_0^{(1)} = g^2 - m$$

: perturbative part

$$E_p^{(1)} = -2m\, e^{-\frac{2pm}{g^2}}$$

: p-bion ••• semi-classical only

Second Order Coefficient

$|\Psi\rangle = |0\rangle + \delta\epsilon|\Psi^{(1)}\rangle + \mathcal{O}(\delta\epsilon^2)$ **: δε expansion of ground state**

$$H^{(0)}|\Psi^{(1)}\rangle = (E^{(1)} - H^{(1)})|0\rangle$$

↓ **perturbed Schrödinger eq**

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perturbed Schrödinger eq

$$\boxed{\Psi^{(1)} = -\Psi^{(0)}\,\frac{f\left(\Sigma+1\right) - e^{-\frac{2m}{g^2}} f\left(\Sigma-1\right)}{1 - e^{-\frac{2m}{g^2}}}}$$

$$f(-z) = \int_0^z dt\,\frac{e^{-\frac{m}{g^2}t} - 1}{t} = \mathrm{Ei}\left(\frac{mz}{g^2}\right) - \log\left(\frac{mz}{g^2}\right) - \gamma$$

Second Order Coefficient

- **First order ground state wave funciton**

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$$E^{(2)} = -\frac{\langle \Psi^{(1)} | H^{(0)} | \Psi^{(1)} \rangle}{\langle 0|0\rangle}$$

standard perturbation theory

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standard perturbation theory

2nd order coefficient

$$E^{(2)} = g^2 - 2m \coth \frac{m}{g^2} \int_0^m \frac{d\mu}{\mu} \frac{\sinh^2 \frac{\mu}{g^2}}{\sinh^2 \frac{m}{g^2}}$$

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standard perturbation theory

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$$E^{(2)} = g^2 - m\frac{\coth\frac{m}{g^2}}{\sinh^2\frac{m}{g^2}}\left[\mathrm{Chi}\left(\frac{2m}{g^2}\right) - \log\frac{2m}{g^2} - \gamma\right]$$

Bion Expansion of E$^{(2)}$

2nd order coefficient

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$$E^{(2)} = E_0^{(2)} + E_1^{(2)} + E_2^{(2)} + \cdots$$

$$E_0^{(2)} = -g^2\sum_{n=1}^{\infty} n!\left(\frac{g^2}{2m}\right)$$

perturbative part

Bion Expansion of $E^{(2)}$

2nd order coefficient

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$$E^{(2)} = E_0^{(2)} + E_1^{(2)} + E_2^{(2)} + \cdots$$

$$E_0^{(2)} = g^2 - 2m\int_0^\infty dt\,\frac{e^{-t}}{t - \frac{2m}{g^2} \pm i0}$$

Borel resummation

Bion Expansion of E$^{(2)}$

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$$\Downarrow \quad E^{(2)} = E_0^{(2)} + E_1^{(2)} + E_2^{(2)} + \cdots$$

$$E_0^{(2)} = g^2 + 2m\, e^{\frac{2m}{g^2}}\,\mathrm{Ei}\left(\frac{2m}{g^2}\right) \pm 2\pi i m e^{-\frac{2m}{g^2}}$$

ambiguity

Bion Expansion of E$^{(2)}$

2nd order coefficient

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$$E^{(2)} = E_0^{(2)} + E_1^{(2)} + E_2^{(2)} + \cdots$$

$$E_0^{(2)} = g^2 + 2m\,e^{\frac{2m}{g^2}}\,\mathrm{Ei}\left(\frac{2m}{g^2}\right) \boxed{\pm 2\pi i m e^{-\frac{2m}{g^2}}}$$

ambiguity

Bion Expansion of E$^{(2)}$

2nd order coefficient

$$E^{(2)} = g^2 - 2m \coth \frac{m}{g^2} \int_0^m \frac{d\mu}{\mu} \frac{\sinh^2 \frac{\mu}{g^2}}{\sinh^2 \frac{m}{g^2}}$$

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ambiguity

$$E_1^{(2)} = -4m \left(\gamma + \log \frac{2m}{g^2} \pm \frac{\pi i}{2}\right) e^{-\frac{2m}{g^2}} + \mathcal{O}(g^2)$$

single bion

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ambiguity

cancellation

$$E_1^{(2)} = -4m\left(\gamma + \log\frac{2m}{g^2}\boxed{\pm\frac{\pi i}{2}}\right)e^{-\frac{2m}{g^2}} + \mathcal{O}(g^2)$$

single bion

Bion Expansion of E$^{(2)}$

multi bion contribution

$$E_p^{(2)} = -4mp^2 \left(\gamma + \log \frac{2m}{g^2} \pm \frac{\pi i}{2} \right) e^{-\frac{2pm}{g^2}}$$

: semi-classical

$$+ \mathcal{O}_p(g^2)\, e^{-\frac{2pm}{g^2}}$$

: fluctuation around p-bion config

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$$\mathcal{O}_p(g^2) = 2m\int_0^\infty dt\, e^{-t}\left\{\frac{(p+1)^2}{t - \frac{2m}{g^2} \pm i0} + \frac{(p-1)^2}{t + \frac{2m}{g^2}}\right\}$$

Bion Expansion of E$^{(2)}$

multi bion contribution

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resurgence : cancellation with semi-classical (p+1)-bion part

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Bion Expansion of $E^{(2)}$

multi bion contribution

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$$\mathcal{O}_p(g^2) = 2m\int_0^\infty dt\, e^{-t}\left\{\frac{(p+1)^2}{t-\frac{2m}{g^2}\pm i0} + \frac{(p-1)^2}{t+\frac{2m}{g^2}}\right\}$$

m ↔ -m symmetry

Bion Expansion of $E^{(2)}$

multi bion contribution

$$E^{(2)}_p = -4mp^2 \left(\gamma + \log \frac{2m}{g^2} \pm \frac{\pi i}{2} \right) e^{-\frac{2pm}{g^2}}$$

: semi-classical

$$+\, \mathcal{O}_p(g^2)\, e^{-\frac{2pm}{g^2}}$$

: fluctuation around p-bion config

resurgence : cancellation with semi-classical (p+1)-bion part

$$\mathcal{O}_p(g^2) = 2m \int_0^\infty dt\, e^{-t} \left\{ \frac{(p+1)^2}{t - \frac{2m}{g^2} \pm i0} + \frac{(p-1)^2}{t + \frac{2m}{g^2}} \right\}$$

determined by semi-classical part

m ↔ -m symmetry

Plan of Talk

1. CP^1 Quantum Mechanics and ``Exact Results”
2. Bion Solutions : Saddle Points in Complexified CP^1 QM
3. Multi-Bion Contributions and Quasi-Moduli Integral
4. Generalization to $CP^{N\text{-}1}$ Quantum Mechanics

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Saddle Point Method

- **partition function in path integral formalism**

$$Z = \int \mathcal{D}\varphi \, e^{-S_E[\varphi,\bar{\varphi}]}$$

➡

ground state energy

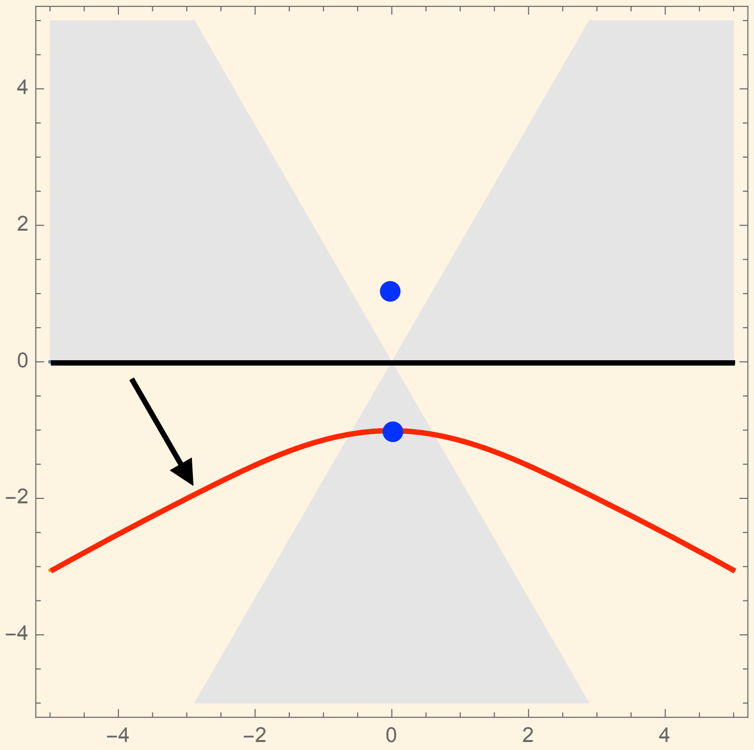
$$E = -\lim_{\beta\to\infty}\frac{1}{\beta}\log Z$$

saddle points: not only on original contour

- **e.g. Airy function**

$$\mathrm{Ai}(g^{-2}) = \int_{-\infty}^{\infty} d\varphi \, \exp\left[-i\left(\frac{\varphi^3}{3}+\frac{\varphi}{g^2}\right)\right]$$

$$\approx \sqrt{\frac{g}{4\pi}}\exp\left(-\frac{2}{3g^3}\right)$$



Complexification

- **real and imaginary parts of** φ ➡ **complex**

$$\varphi = \varphi_R + i\varphi_I$$

$$\bar{\varphi} = \varphi_R - i\varphi_I \quad \to \quad \tilde{\varphi}$$

$$\left(\frac{SU(2)}{U(1)} \to \frac{SU(2)^{\mathbb{C}}}{U(1)^{\mathbb{C}}} \right)$$

$$\mathbb{C}P^1 \qquad T^*\mathbb{C}P^1$$

- **analytic continuation of Euclidean action**

$$S_E[\varphi, \bar{\varphi}] \quad ➡ \quad S_E[\varphi, \tilde{\varphi}]$$

holomorphic

- **saddle point eq. = Euclidean e.o.m**

$$\boxed{\frac{\delta S_E}{\delta \varphi} = \frac{\delta S_E}{\delta \tilde{\varphi}} = 0}$$

Symmetry of Euclidean Action

time shift

$$\tau \to \tau + a$$

➡

"energy" conservation

$$\frac{|\dot{\varphi}|^2}{(1+|\varphi|^2)^2} - V = \textbf{const.}$$

phase rotation

$$\varphi \to e^{i\alpha}\varphi$$

azimuthal angle

➡

angular momentum

$$\frac{i(\bar{\varphi}\dot{\varphi} - \dot{\bar{\varphi}}\varphi)}{(1+|\varphi|^2)^2} = \textbf{const.}$$

Saddle Point Solutions

solution (up to sym.)

$$\varphi = \tilde{\varphi} = A\,\mathrm{cs}(\Omega\tau, k)$$ **(Jacobi elliptic function)**

(A, Ω, k) **: complex constants depending on**

p, q **: integers labeling solutions**

$$p \geq 0 \qquad 0 \leq q \leq p-1$$

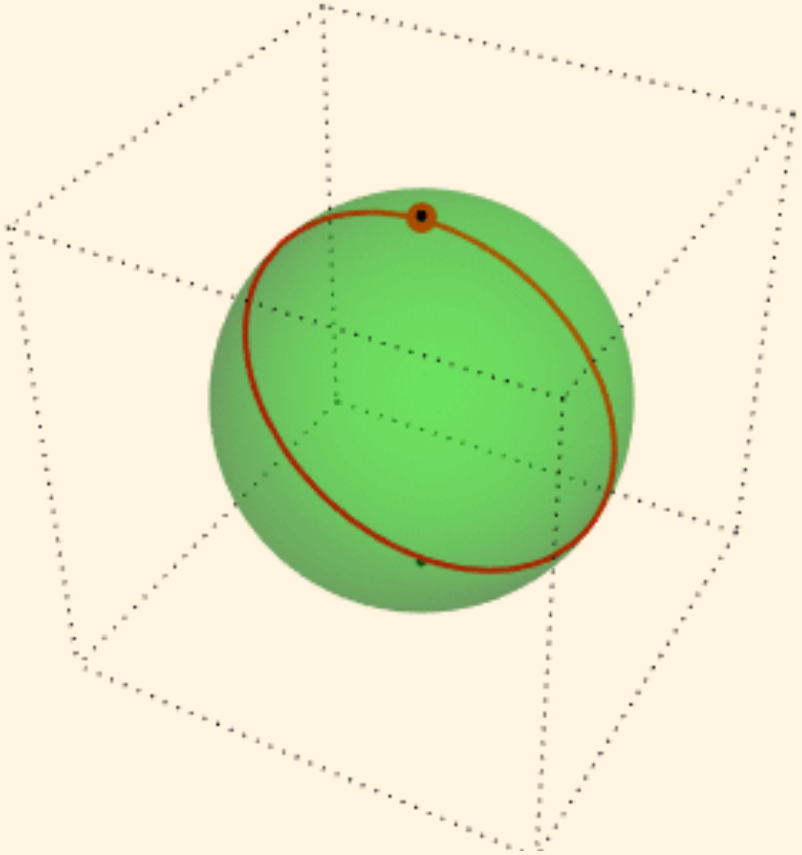
- **complex saddle points** $\tilde{\varphi} \neq \bar{\varphi}$ **for generic** p, q

Real Bion Solution (p=1)

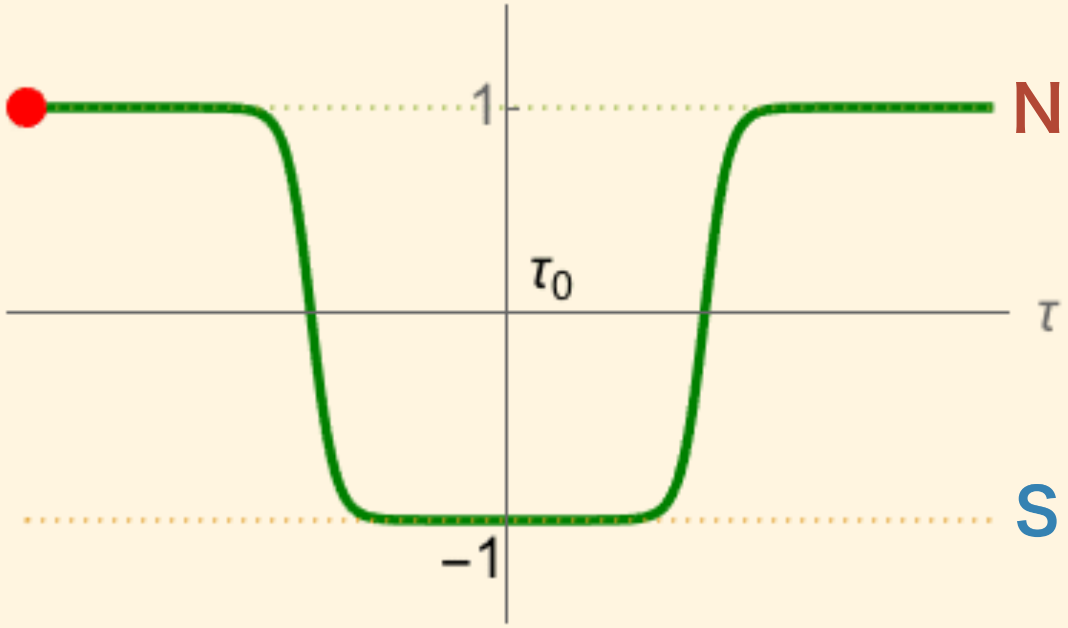
- **single bion solution in $\beta \to \infty$ limit**

$$\varphi = \tilde{\varphi} = \sqrt{\frac{\omega^2}{\omega^2 - m^2}} \frac{1}{\sinh \omega\tau} \quad \textbf{with} \quad \omega = m\sqrt{1 + \frac{2\epsilon g^2}{m}}$$

real bion : saddle point on original integration contour



$$\Sigma = \frac{1 - |\varphi|^2}{1 + |\varphi|^2}$$ **: height**



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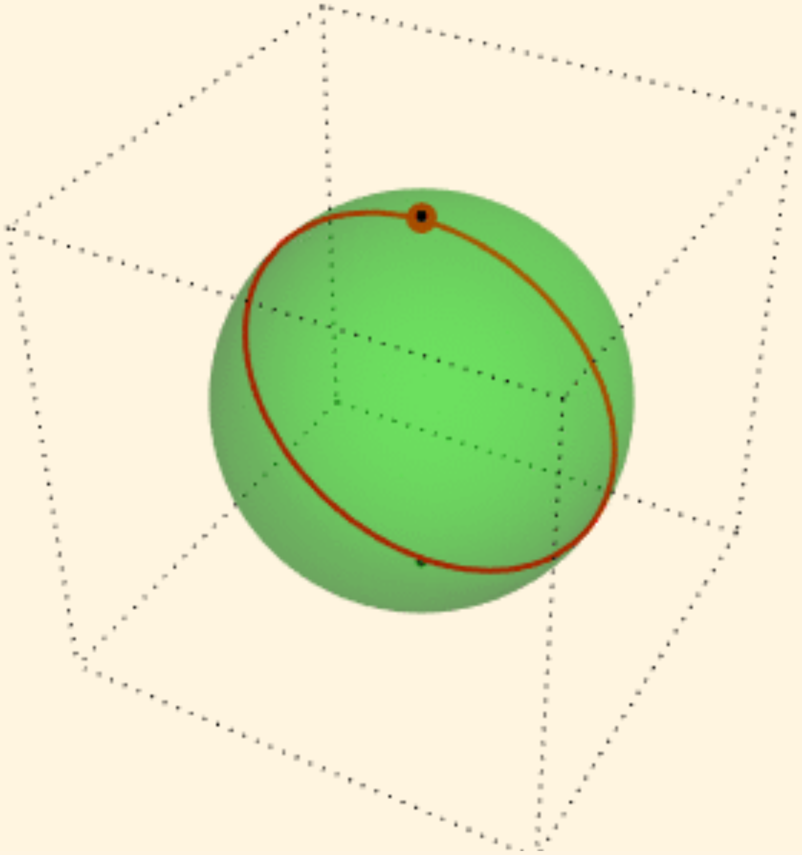
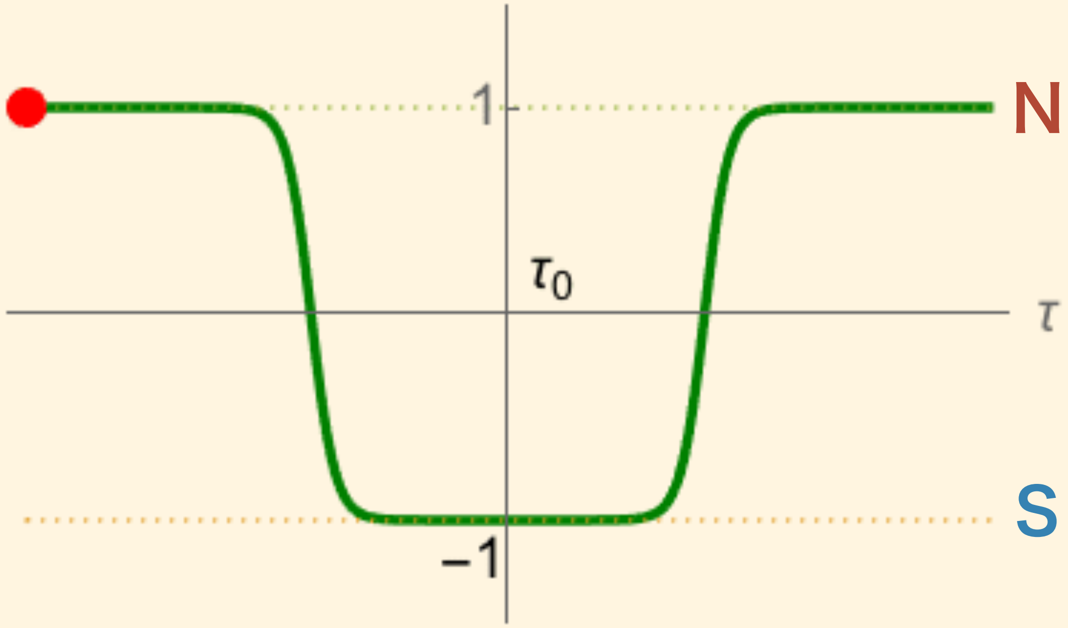


$$\Sigma = \frac{1 - |\varphi|^2}{1 + |\varphi|^2}$$ **: height**



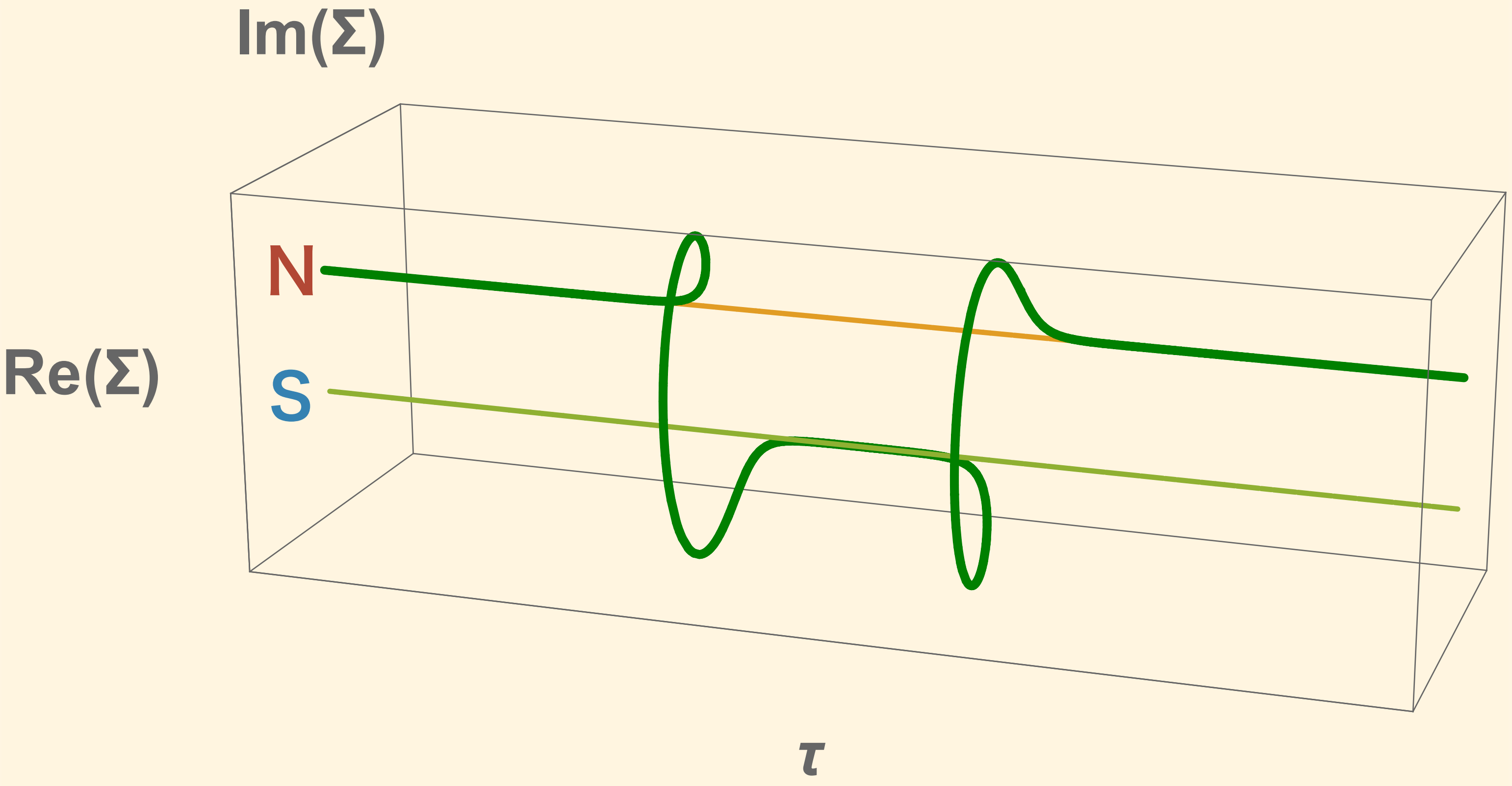
Complex Bion Solution (p=1)

- **complex bion : not on the original contour** $\tilde{\varphi} = \bar{\varphi}$

$$\boxed{\varphi = -\tilde{\varphi} = \sqrt{\frac{\omega^2}{\omega^2 - m^2}\frac{1}{\cosh \omega\tau}}}$$

$$\Sigma = \frac{1 - \varphi\tilde{\varphi}}{1 + \varphi\tilde{\varphi}}$$

complexified height



Real and Complex Bion

- **kink-antikink ansatz**

$$\varphi = \Big[\underbrace{\exp\left(-\omega\tau - z\right)}_{\text{kink}} + \underbrace{\exp\left(\omega\tau + z\right)}_{\text{antikink}} \Big]^{-1}$$

$z = \omega\Delta\tau + i\Delta\phi$ **: relative kink position and phase**

real bion

$$\Delta\tau = \frac{1}{2\omega} \log \frac{\omega^2}{\omega^2 - m^2}$$

$$\Delta\phi = \pi$$

complex bion

$$\Delta\tau = \frac{1}{2\omega} \log \left(- \frac{\omega^2}{\omega^2 - m^2} \right)$$

$$\Delta\phi = 0$$

Real and Complex Bion

- **kink-antikink ansatz**

$$\varphi = \Big[\underbrace{\exp\left(-\omega\tau - z\right)}_{\textbf{kink}} + \underbrace{\exp\left(\omega\tau + z\right)}_{\textbf{antikink}} \Big]^{-1}$$

$z = \omega\Delta\tau + i\Delta\phi$ **: relative kink position and phase**

- **in the weak coupling limit** $g \to 0$

$$\Delta\tau \sim \frac{1}{2m} \log \frac{m}{2\epsilon g} \quad \to \infty$$

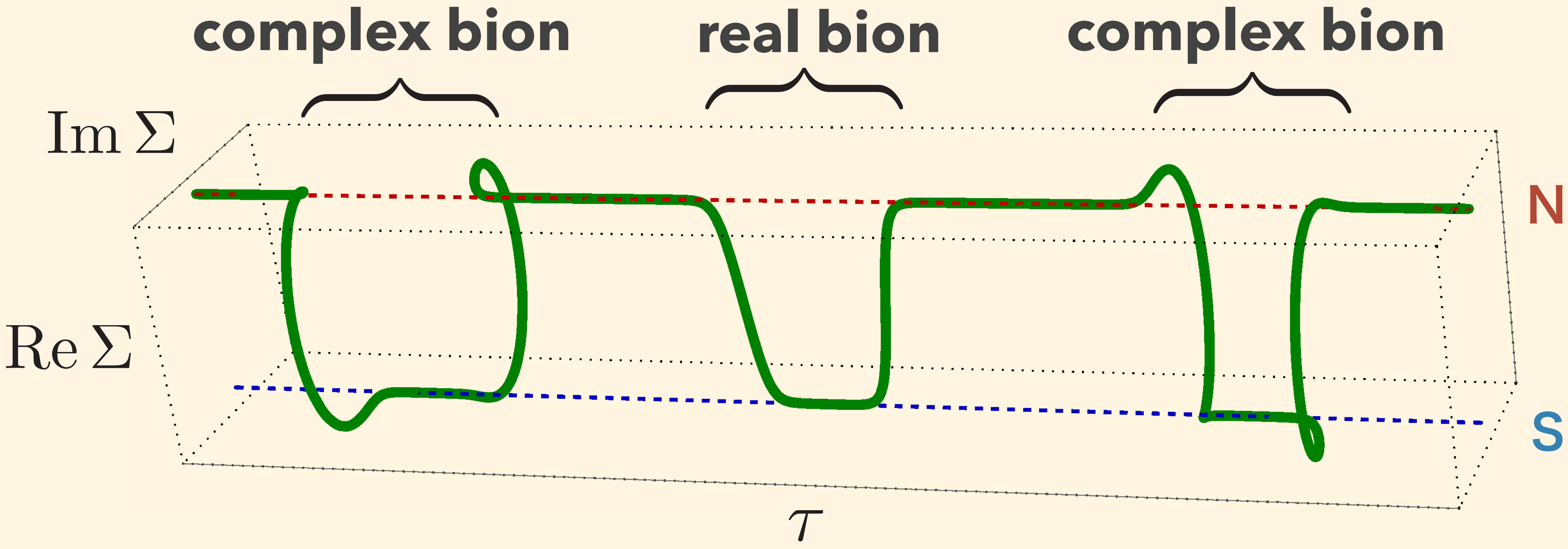
- **loosely bounded pair of kink-antikink**

$(\Delta\tau, \Delta\phi)$ **: quasi-moduli**

Kink profile of multi-bion

- **p : number of bions**

- **q : label saddle points in p-bion sector**



$(p, q) = (3, 1)$ **: multi bion solution**

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Lefschetz thimble method

- **decomposition of integration contour**

$$\mathcal{C} = \sum_{\sigma \in \mathfrak{S}} n_\sigma \mathcal{J}_\sigma$$

$\mathfrak{S}$ **: set of saddle points**

flow equation

$$\frac{d\varphi}{dt} = -\overline{\frac{\delta S_E}{\delta \varphi}}$$

thimble $\mathcal{J}_\sigma$ **: flow to σ**

dual thimble $\mathcal{K}_\sigma$ **: flow from σ**

intersection # $n_\sigma = \langle \mathcal{K}_\sigma , \mathcal{C} \rangle$

Reduction to Quasi Moduli Space

- **semi-classical limit** $g \to 0$
- **zero temperature** $\beta \to \infty$

$$\varphi_{k\bar{k}} = \Big[\underbrace{\exp(-\omega\tau - z)}_{\textbf{kink}} + \underbrace{\exp(\omega\tau + z)}_{\textbf{antikink}} \Big]^{-1}$$

$z = \omega\Delta\tau + i\Delta\phi$ **: relative kink position and phase**

$$\varphi = \varphi_{k\bar{k}} + g^2 \delta\varphi$$

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quasi moduli parameters

``nearly flat directions"

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quasi moduli parameters

``nearly flat directions"

$$\varphi = \varphi_{k\bar{k}} + g^2\delta\varphi$$

``massive modes"

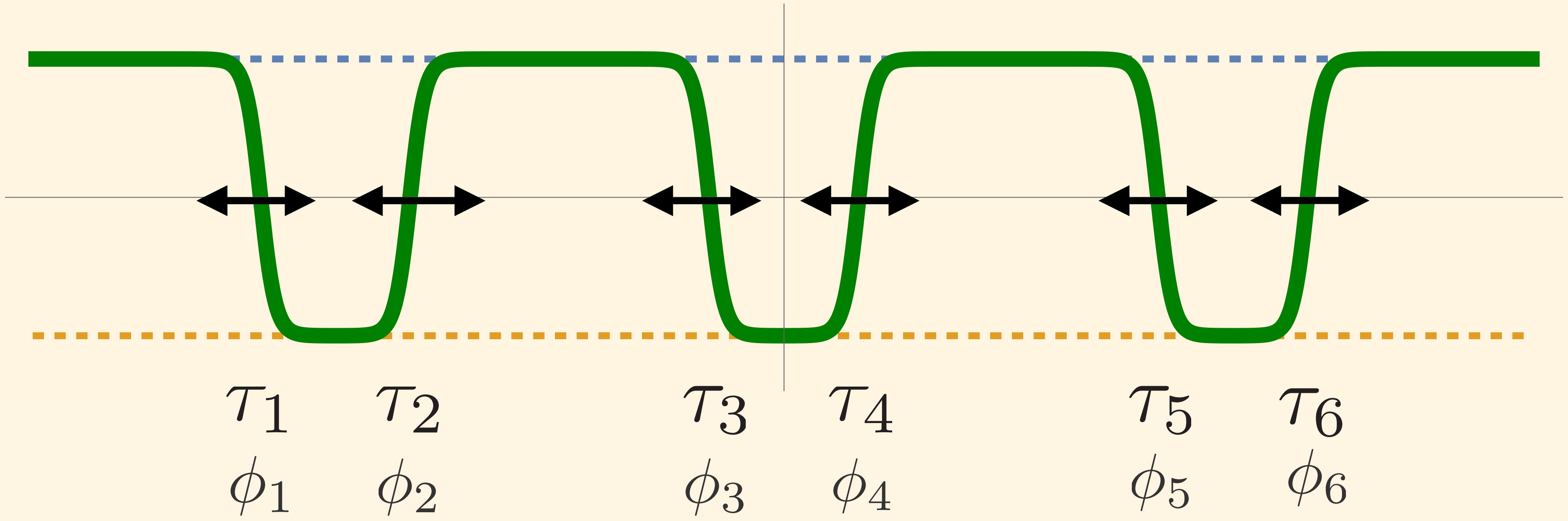
➡ $S[\varphi] = S_{\text{eff}}(z) + \delta\tilde{\varphi}\,\Delta\,\delta\varphi + \mathcal{O}(g^2)$

Contribution from Saddles

- semi-classical limit $g \to 0$
- zero temperature $\beta \to \infty$

➡ massive modes : one-loop determinant $\det \Delta$

- integral along nearly flat directions : quasi-moduli



kink gas with complexified quasi-moduli

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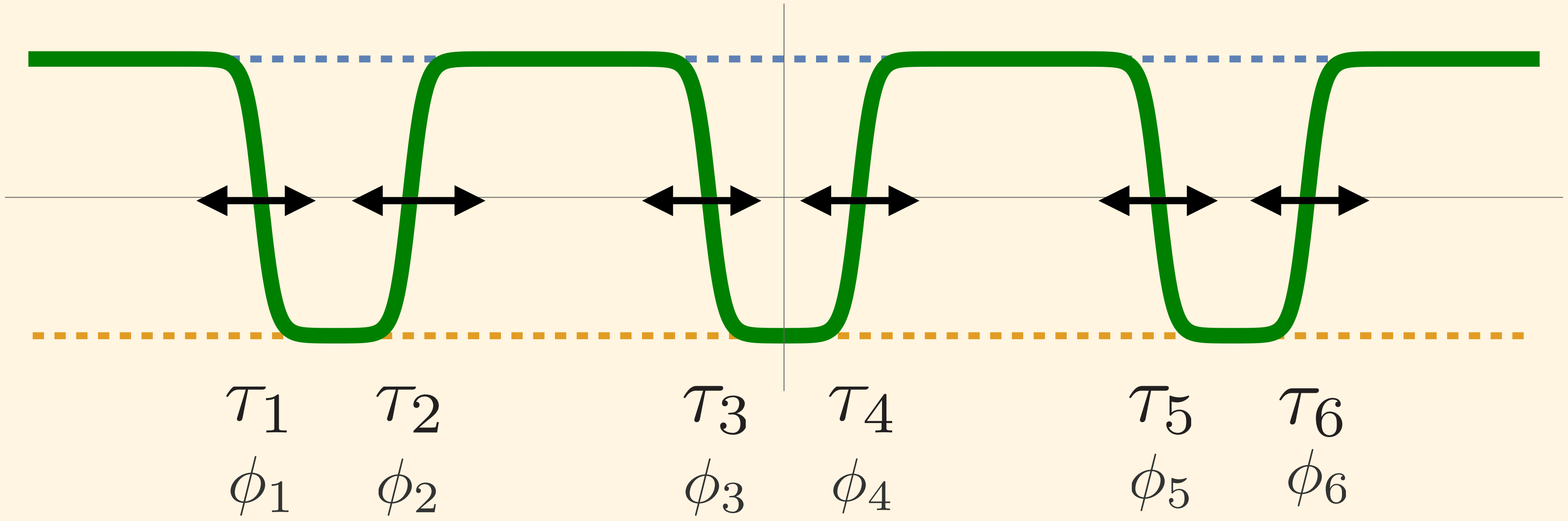


kink gas with complexified quasi-moduli

Quasi-Moduli Integral

- **semi-classical p-bion contribution to the partition function**

integral over nearly flat directions

$$Z_p = \int \prod_{i=1}^{2p} d\tau_i d\phi_i \ \det \Delta^{-1} \ \exp\left(-S_{\text{eff}}\right)$$

effective action $z_i = m(\tau_i - \tau_{i-1}) + i(\phi_i - \phi_{i-1})$

$$S_{\text{eff}} = \sum_{i=1}^{2p} \left(-\frac{2m}{g^2} e^{-z_i} + \epsilon_i z_i \right) + c.c.$$

- **interaction between nearest neighbor pair of kinks**

Thimble $\mathcal{J}_\sigma$ and Dual Thimble $\mathcal{K}_\sigma$

- **single bion case (a pair of kink-antikink)**

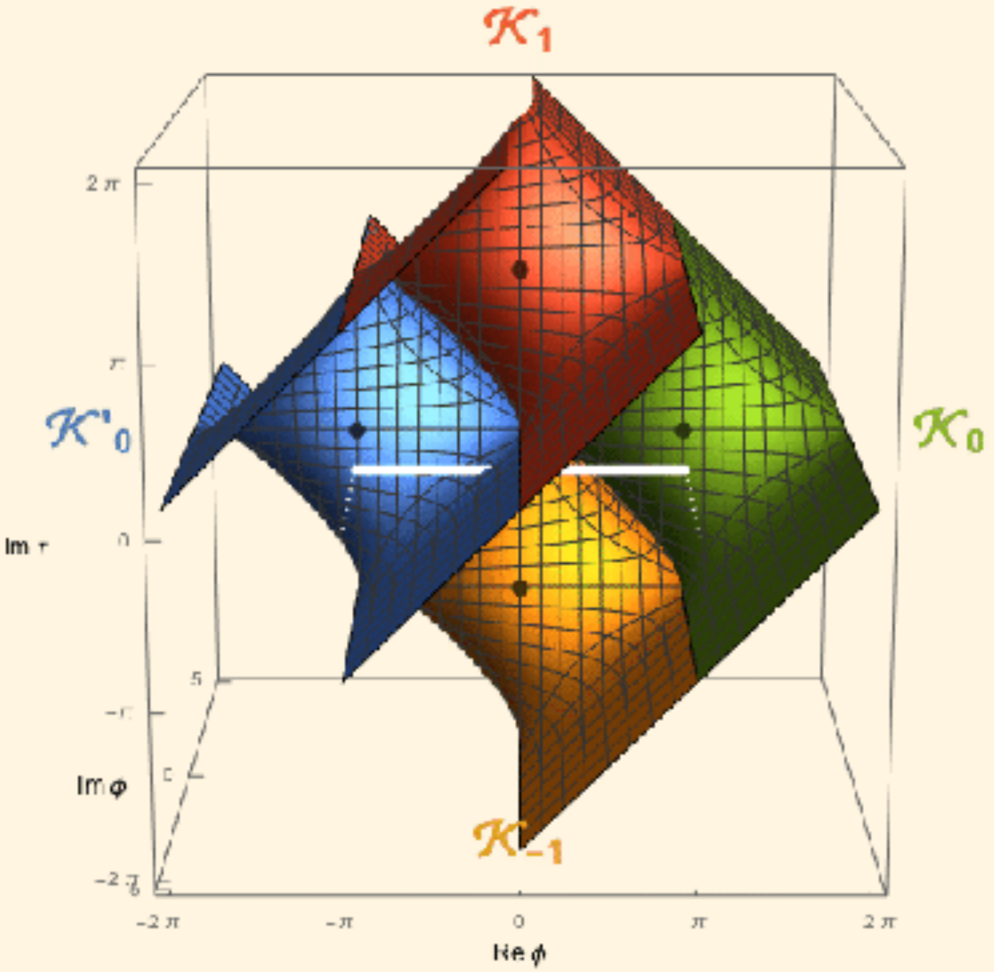
$$S_{\text{eff}} = -\frac{4m}{g^2} e^{-m\tau} \cos\phi + 2\epsilon m \tau$$ ➡ **saddles = bions**

thimble J_σ

$$\tau_I = \frac{1}{m}(\sigma\pi - \arg g) \quad \phi_R = -(\sigma - 1)\pi$$

dual thimble K_σ

$$m\tau_R \pm \phi_I = \log\left[\frac{2m}{\epsilon g^2}\frac{\sin(m\tau_I \pm \phi_R + a_{\pm\sigma})}{m\tau_I \pm \phi_R + a_{\pm\sigma}}\right]$$



3d projection from $(\tau, \phi) \in \mathbb{C}^2$

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$$S_{\text{eff}} = -\frac{4m}{g^2} e^{-m\tau} \cos\phi + 2\epsilon m\tau$$ → **saddles = bions**

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$$\tau_I = \frac{1}{m}(\sigma\pi - \arg g) \quad \phi_R = -(\sigma - 1)\pi$$

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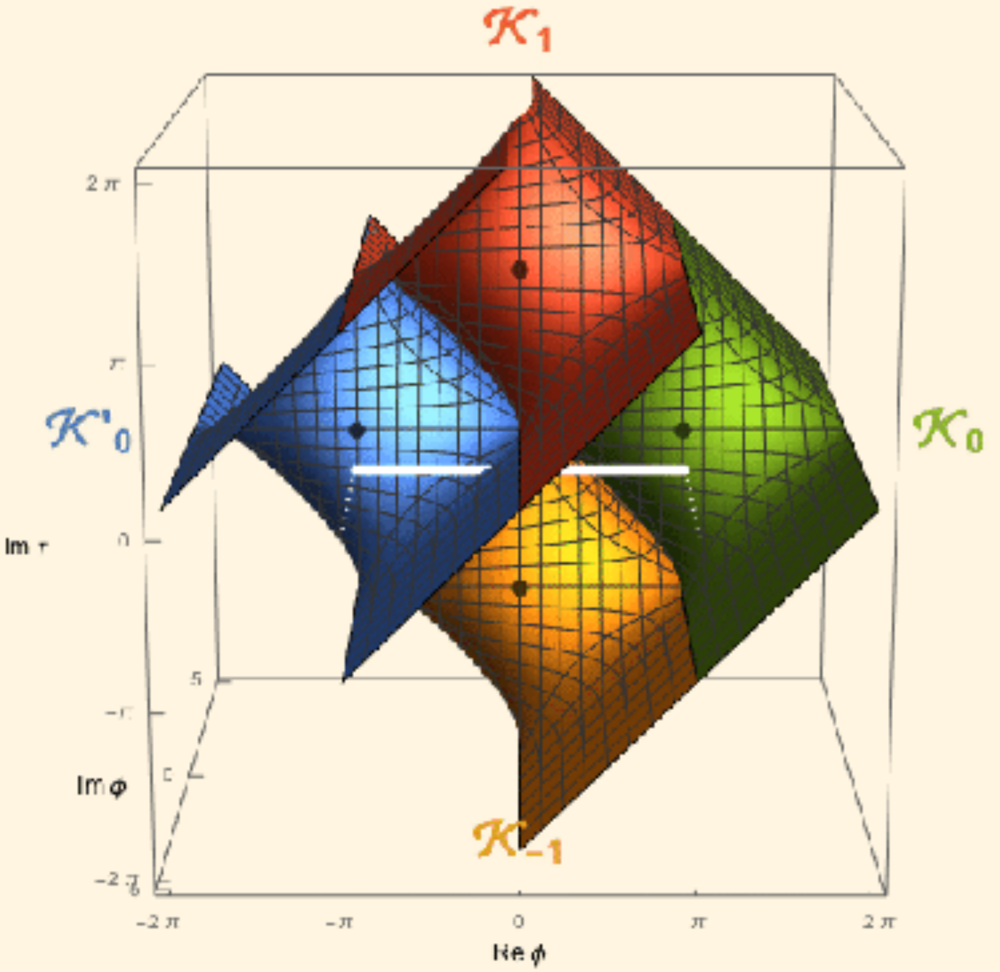

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- **single bion case (a pair of kink-antikink)**

$$S_{\text{eff}} = -\frac{4m}{g^2} e^{-m\tau} \cos\phi + 2\epsilon m\tau$$

- **discontinuity of intersection numbers at** $\operatorname{Im} g = 0$

$$\mathcal{C} = \begin{cases} \mathcal{J}_0 - \mathcal{J}_{-1} & \text{for } \operatorname{Im} g < 0 \\ \mathcal{J}_1 - \mathcal{J}_0 & \text{for } \operatorname{Im} g > 0 \end{cases}$$

↓ **Stokes phenomenon**

$$\boxed{Z_1 = \frac{\pi}{m} \left(\frac{g^2}{2m}\right)^{2\epsilon} \frac{\Gamma(\epsilon)}{\Gamma(1-\epsilon)} \begin{cases} e^{\pi i \epsilon} & \text{for } \operatorname{Im} g < 0 \\ e^{-\pi i \epsilon} & \text{for } \operatorname{Im} g > 0 \end{cases}}$$

Multi-Bion case

- **complexified relative quasi-moduli parameters**

$$z_i = m(\tau_i - \tau_{i-1}) + i(\phi_i - \phi_{i-1})$$

$$\tilde{z}_i = m(\tau_i - \tau_{i-1}) - i(\phi_i - \phi_{i-1})$$

- **constraint (sum of relative kink positions = period of S¹)**

$$\delta\left(\sum_{i=1}^{2p} \frac{z_i + \tilde{z}_i}{2} - \beta\right) = \int \frac{d\sigma}{2\pi} \exp\left[i\sigma\left(\sum_{i=1}^{2p} \frac{z_i + \tilde{z}_i}{2} - \beta\right)\right]$$

- **factorization into single bion contributions**

$$Z_p \propto \int \frac{d\sigma}{2\pi} e^{-i\sigma\beta} \prod_{i=1}^{2p} I_i \quad \textbf{with} \quad I_i = \int dz_i d\tilde{z}_i \, e^{-\mathcal{V}(z_i) - \tilde{\mathcal{V}}(\tilde{z}_i)}$$

Multi-Bion case

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$$z_i = m(\tau_i - \tau_{i-1}) + i(\phi_i - \phi_{i-1})$$

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single bion result

Non-Perturbative Contributions

$$\frac{Z_p}{Z_0} = -\frac{2im\beta}{p} \int \frac{d\sigma}{2\pi i} e^{-im\beta\sigma} \prod_{i=1}^{2p} I_i(\sigma)$$

$$I_i(\sigma) = \left(\frac{2m}{g^2} e^{\pm \frac{\pi i}{2}} \right)^{i\sigma - \epsilon_i} \frac{\Gamma\big((\epsilon_i - i\sigma)/2\big)}{\Gamma\big(1 - (\epsilon_i - i\sigma)/2\big)}$$

bion expansion of partition function $e^{-S_{\text{bion}}} = e^{-\frac{2m}{g^2}}$

$$Z = Z_0 + Z_1 e^{-\frac{2m}{g^2}} + Z_2 e^{-\frac{4m}{g^2}} + \cdots$$

Non-Perturbative Contributions

$$\frac{Z_p}{Z_0} = -\frac{2im\beta}{p}\int \frac{d\sigma}{2\pi i}\, e^{-im\beta\sigma}\prod_{i=1}^{2p} I_i(\sigma)$$

$$I_i(\sigma) = \left(\frac{2m}{g^2}\boxed{e^{\pm\frac{\pi i}{2}}}\right)^{i\sigma-\epsilon_i}\frac{\Gamma\big((\epsilon_i - i\sigma)/2\big)}{\Gamma\big(1-(\epsilon_i - i\sigma)/2\big)}$$

imaginary ambiguity

bion expansion of partition function $e^{-S_{\rm bion}} = e^{-\frac{2m}{g^2}}$

$$Z = Z_0 + Z_1 e^{-\frac{2m}{g^2}} + Z_2 e^{-\frac{4m}{g^2}} + \cdots$$

SUSY and Near SUSY Cases

SUSY case

$$E_p^{(0)} \sim \frac{1}{\Gamma(1-\epsilon)^p} \to 0$$

consistent with the exact result (SUSY : E=0)

……

SUSY and Near SUSY Cases

SUSY case

$$E_p^{(0)} \sim \frac{1}{\Gamma(1-\epsilon)^p} \to 0$$

consistent with the exact result (SUSY : E=0)

……

1st order

$$E^{(1)}_{\text{pert}} = -m + g^2, \quad E_p^{(1)} = -2m$$

➡ $$E^{(1)} = g^2 - m \coth \frac{m}{g^2}$$

exact result

Bion Expansion of E$^{(2)}$

multi bion semi-classical contribution

$$E_p^{(2)} = -4mp^2 \left(\gamma + \log \frac{2m}{g^2} \pm \frac{\pi i}{2} \right) e^{-\frac{2pm}{g^2}} + \mathcal{O}_p(g^2)\, e^{-\frac{2pm}{g^2}}$$

resurgence + m ↔ -m symmetry gives

$$\mathcal{O}_p(g^2) = 2m \int_0^\infty dt\, e^{-t} \left\{ \frac{(p+1)^2}{t - \frac{2m}{g^2} \pm i0} + \frac{(p-1)^2}{t + \frac{2m}{g^2}} \right\}$$

consistent with the exact results

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CP^{N-1} Quantum Mechanics

Lagrangian

$$L = \frac{1}{g^2} G_{i\bar{j}}\, \dot{\varphi}^i \dot{\bar{\varphi}}^j - V$$

with $V = G^{i\bar{j}} \left(\frac{1}{g^2} \partial_i \mu\, \partial_{\bar{j}} \mu - \epsilon\, \partial_i \partial_{\bar{j}} \mu \right)$

$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log \left(1 + \sum_i |\varphi|^2 \right)$ **: Fubini-Study metric (i=1,$\cdots$,N-1)**

$\mu = \frac{\sum_i m_i |\varphi^i|^2}{1 + \sum_i |\varphi^i|^2}$ **: moment map of** $U(1) \subset SU(N)$

$\mathrm{CP}^{N\text{-}1}$ Quantum Mechanics

$$\Psi^{(0)} = e^{-\frac{\mu}{g^2}} \qquad \Psi^{(1)} = -\frac{N}{2}\Psi^{(0)} \log\left(1 + \sum_i |\varphi|^2\right) + \cdots$$

need to solve PDE

$$\boxed{E^{(0)} = 0 \qquad \textbf{(supersymmetry)}}$$

$$\boxed{E^{(1)} = \frac{N(N-1)}{2} g^2 - \sum_i m_i \left(1 + \frac{N A_i e^{-\frac{2m_i}{g^2}}}{1 - \sum_j A_j e^{-\frac{2m_i}{g^2}}}\right)} \quad A_i = \prod_{j \neq i} \frac{m_j}{m_j - m_i}$$

$$\boxed{E^{(2)} = \frac{N^2}{4}\left[g^2 + \sum_{i=1}^{N-1} 2m_i A_i \int_0^\infty dt\, \frac{e^{-t}}{t - \frac{2m_i}{g^2} \pm i0}\right] + O(e^{-\frac{2m_i}{g^2}})}$$

Bions CP^{N-1} Model

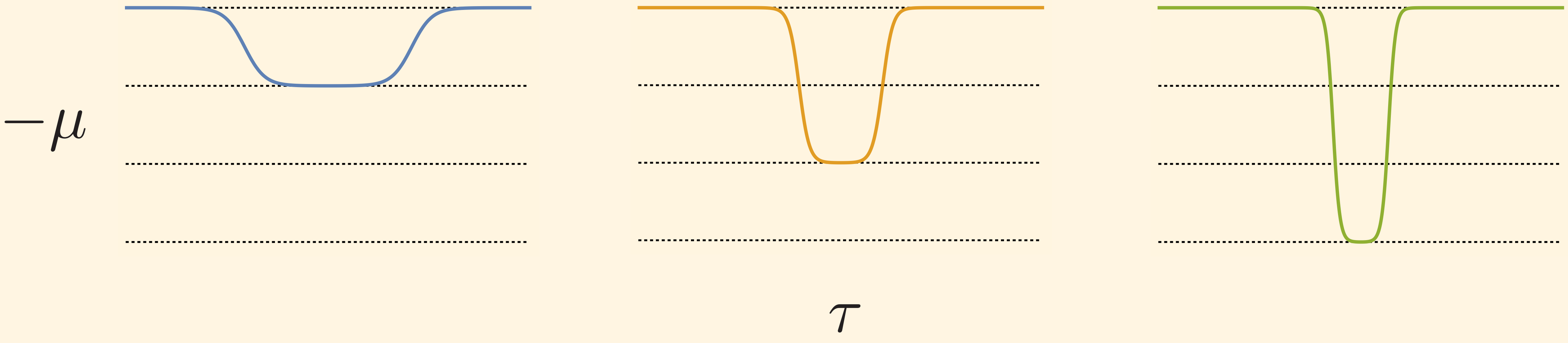
- **Embedding CP^1 bion**

N-1 types of bion

$\varphi^i =$ **CP^1 bion solution with m = m_i**

$$\varphi^j = 0 \quad (j \neq i)$$

- **e.g. $\mathbb{C}P^3$ case (N=4) $\cdots$ three bions**



Quasi-Moduli Integral

- **quasi-moduli integral for single i-th bion**

$$Z_i = \int dV \ \det \Delta^{-1} \exp\left[\frac{2m_i}{g^2} e^{-z} - \frac{N}{2}\epsilon z + (c.c.)\right]$$

- **quasi-moduli integral for single i-th bion**

$$\det \Delta = A_i \, e^{(N-2)\,\mathrm{Re}\, z}$$ **× CP¹ determinant**

quantum correction to ε

$$\epsilon' = \epsilon + \frac{1}{2}(N-2)(\epsilon - 1)$$

Single Bion Contributions

- **(N-1) types of bion** → **consistent results**

$$E_1^{(0)} = 0$$ ($\forall\, \epsilon' \in \mathbb{Z}^+$, **quasi-exactly solvable)**

$$E_1^{(1)} = -N \sum_{i=1}^{N-1} m_i A_i e^{-\frac{2m_i}{g^2}}$$

$$E_1^{(2)} = N^2 \sum_{i=1}^{N-1} m_i A_i \left[\gamma + \log \frac{2m_i}{g^2} \pm \frac{\pi i}{2} + \mathcal{O}(g^2) \right] e^{-\frac{2m_i}{g^2}}$$

cancellation with $\mathrm{Im}\, E_0^{(2)} = \mp \frac{\pi i}{2} N^2 \sum_{i=1}^{N-1} m_i A_i e^{-\frac{2m_i}{g^2}}$

Multi Bion Contributions

assumption for multi-bion configurations

well-separated kink-antikink pairs
each of which is either one of the N – 1 types of bions

All order multi-bion contributions

$$E^{(0)} = 0$$

(consistent with supersymmetry)

$$E^{(1)} = \frac{N(N-1)}{2} g^2 - \sum_i m_i \left(1 + \frac{N A_i e^{-\frac{2m_i}{g^2}}}{1 - \sum_j A_j e^{-\frac{2m_i}{g^2}}} \right)$$

exact results are reproduced

Multi Bion Contributions

- **second order coefficient of ground state energy**

$$E^{(2)} = -N^2 \sum_{i=1}^{N-1} m_i A_i e^{-s_i} \left[f(s_i) Y_{ii} + \sum_{j=1}^{N-1} m_j A_j Y_{ij} X_{ij} e^{-s_i} \right]$$

$$s_i = \frac{2m_i}{g^2} \qquad Y_{ij} = \frac{R_i R_j}{1 - \sum_k A_k e^{-s_k}} \quad R_i = \frac{1 - \sum_k \frac{m_i - m_k}{m_i} A_k e^{-s_i}}{1 - \sum_k A_k e^{-s_k}}$$

resurgence + m ↔ -m symmetry

$$f(s) = \log \frac{2m}{g^2} + \gamma \pm \frac{\pi i}{2} + \mathcal{O}(g^2) \xrightarrow[\text{CP}^1 \text{ case}]{} \log \frac{2m}{g^2} + \gamma - \mathrm{Chi}\left(\frac{2m}{g^2}\right)$$

Multi Bion Contributions

- **second order coefficient of ground state energy**

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$$s_i = \frac{2m_i}{g^2} \qquad Y_{ij} = \frac{R_i R_j}{1 - \sum_k A_k e^{-s_k}} \qquad R_i = \frac{1 - \sum_k \frac{m_i - m_k}{m_i} A_k e^{-s_i}}{1 - \sum_k A_k e^{-s_k}}$$

resurgence + m ↔ -m symmetry

$$f(s) = \log \frac{2m}{g^2} + \gamma \pm \frac{\pi i}{2} + \mathcal{O}(g^2)$$

→ **prediction of resurgence**

CP^{N-1} case

Summary

- Explicit resurgence structure in CP^N quantum mechanics
- Small SUSY breaking deformation
- Complex saddle points : multi bion solution
- All order multi bion contributions
- Consistent with exact results and resurgence structure

future work

- generalization to field theory : 2d NLσM, 4d gauge theory