Resurgent Transseries in String Theory

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RIMS–iTHEMS International Workshop on Resurgence Theory
06–08 September 2017, RIKEN Kobe
Based on work in collaboration with:

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arXiv: Upcoming...
Perturbative Expansions and Borel Resummation

- Perturbative series asymptotic ⇒ Coefficients grow as $F_g \sim g!...$

$$F'(z) \sim \sum_{g=0}^{+\infty} \frac{F_g}{z^{g+1}}.$$ 

- Borel transform “removes” factorial growth

$$B \left[ \frac{1}{z^{\alpha+1}} \right](s) = \frac{s^\alpha}{\Gamma(\alpha + 1)}.$$ 

- Borel resummation given by inverse Borel transform

$$S_\theta F(z) = \int_0^{e^{i\theta}} ds \, B[F](s) \, e^{-zs}.$$ 

Only defined if $B[F](s)$ has no singularities along direction $\theta$.

Perturbation theory non-Borel resummable along any Stokes line.
What class of singularities does one usually find?

**Resurgent Function**: a formal (asymptotic) series whose Borel transform has *endless analytic continuation* (too broad...).

**Simple Resurgent Function**: a resurgent function whose Borel singularities restrict to *simple poles and logarithmic branch-points* (here $\omega$ is a simple singularity, $S_\omega \in \mathbb{C}$, and $\Phi_\omega$ some other sector):

$$\mathcal{B}[F](s) = S_\omega \times \mathcal{B}[\Phi_\omega](s - \omega) \frac{\log(s - \omega)}{2\pi i} + \text{holomorphic}.$$ 

Can be precise about what happens upon crossing a Stokes line!
Discontinuity upon Crossing a Stokes Line

\[ S_{\theta^+} F - S_{\theta^-} F = - \sum_{\{\omega_n\}} S_{\omega_n} e^{-\omega_n z} S_{\theta^-} \Phi_{\omega_n} \equiv -S_{\theta^-} \circ \text{Disc}_{\theta} F \]

\[ \Rightarrow \quad \text{Disc}_{\theta} F = \sum_{\{\omega_n\}} S_{\omega_n} e^{-\omega_n z} \Phi_{\omega_n}. \]

- All sectors \( \Phi_{\omega_n} \) must be included in full solution, as the perturbative series not enough! \( \Rightarrow \) Leads to transseries and to resurgence...
Applications to String Theory and Large N Duality

- Gravity and spacetime out from large $N$ gauge theory. [Maldacena]
- B-model on local Calabi–Yau is large $N$ dual to matrix model [Dijkgraaf-Vafa]. Class of target CY geometries is
  \[ uv = \mathcal{H}(X, Y). \]
- Non-trivial information about this geometry encoded in Riemann surface $\Sigma$ described by $\mathcal{H}(X, Y) = 0$.
- Spectral curve of matrix model is precisely Riemann surface $\Sigma$.
- Special geometry of CY, solving tree-level closed strings on this background, further yields planar solution to hermitian matrix model.
Nonperturbative Strings and Large N Duality?

- Extend to all genera? (Perturbative) large $N$ expansion matches (perturbative) closed string-theory free-energy [Eynard-Mariño-Orantin].

- Can one go beyond the perturbative large $N$ expansion?
  - Establish large $N$ duality at full nonperturbative level?
    - Nonperturbative large $N$ gauge theory?
    - Nonperturbative closed string theory?
  - Better understand the gauge theory $\leftrightarrow$ gravity/spacetime map?
  - Find new phenomena or new phases at large or complex couplings?
  - ......
1. Nonperturbative Holomorphic Anomalies
2. Resurgent Asymptotics in String Theory
3. Asymptotics of Enumerative Invariants
4. The Resurgence of the Large $N$ Expansion
5. Large $N$ Expansion: Stokes versus Anti-Stokes Phases
6. Outlook
Outline

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String Theory on Calabi–Yau Geometries

- Consider topological string B-model on local Calabi–Yau, mirror to some toric threefold. Non-trivial information about CY geometry encoded in Riemann surface $\leftrightarrow$ mirror curve of the geometry.

- Compute the string free energy using the holomorphic anomaly!

$$ F \sim \sum_{g=0}^{+\infty} F_g(z) g_s^{2g-2}. $$

- Genus-$g$ free energy depends on complex structure moduli $z$...
- ...but it is not holomorphic $\Rightarrow F_g \equiv F_g(z, \bar{z})!$ [Bershadsky-Cecotti-Ooguri-Vafa]

- Moduli space of genus-$g$ surfaces has a boundary:

$$ \frac{\partial F_g}{\partial \bar{z}} \sim \int_{\mathcal{M}_g} \mathcal{D}m \frac{\partial}{\partial m} \left( \cdots \right) \neq 0. $$
Holomorphic Anomaly Equations: Idea

- **The boundary** of genus-$g$ surface moduli space corresponds to those moduli which make the genus-$g$ surface **degenerate**:

- This leads to the **holomorphic anomaly equations**: [BCOV]

\[
\begin{align*}
\frac{\partial F_g}{\partial \bar{z}} \sim D^2 F_{g-1} + \sum_{h=1}^{g-1} \frac{\partial F_{g-h}}{\partial F_h}.
\end{align*}
\]
Will consider the example of B-model on mirror of $\mathbb{KP}^2 = \mathcal{O}(-3) \rightarrow \mathbb{P}^2$ ⇒ Fully solved perturbatively [Haghighat-Klemm-Rauch] via holomorphic anomaly equations [Bershadsky-Cecotti-Ooguri-Vafa]

$$\frac{\partial F_{g}^{(0)}}{\partial S_{ij}} = \frac{1}{2} \left( D_{i} D_{j} F_{g-1}^{(0)} + \sum_{h=1}^{g-1} \partial_{i} F_{g-h}^{(0)} \partial_{j} F_{h}^{(0)} \right), \quad g \geq 2.$$ 

Here $D_{i}$ covariant derivative in complex structure moduli space (holomorphic dependence); $S_{ij}$ propagators or “potentials” for Yukawa couplings (also anti-holomorphic dependence).
Nonperturbative Holomorphic Anomaly Equations?

- $F^{(0)}_g(z_i, \bar{z}_i)$ depends on holomorphic and anti-holomorphic complex structure moduli ⇒ What is large-order behavior?

\[
F^{(0)}_g(z_i, \bar{z}_i) \quad \text{versus} \quad F^{(0)}_g(z_i)
\]

- Rewrite holomorphic anomaly equations for partition function $Z$ ⇒ Naturally solved with transseries ansatz [Cousso-Edelstein-RS-Vonk]

\[
Z = \exp \left( \sum_n \sigma^n F^{(n)} \right).
\]
Complex structure moduli space of dimension one (single $z$ and $S$).

Master equation for standard holomorphic anomaly:

$$\left( \frac{\partial}{\partial S} - \frac{1}{2} g_s^2 D_z^2 \right) Z + U D_z Z = \left( \frac{1}{g_s^2} W + V \right) Z,$$

with “initial data”

$$U \equiv D_z F_0^{(0)},$$

$$W \equiv \frac{\partial F_0^{(0)}}{\partial S} + \frac{1}{2} \left( D_z F_0^{(0)} \right)^2,$$

$$V \equiv \frac{\partial F_1^{(0)}}{\partial S} - \frac{1}{2} \left( D_z^2 F_0^{(0)} \right).$$

Inserting $Z = \exp F^{(0)}$ yields back holomorphic anomaly equations.
Nonperturbative Holomorphic Anomaly Equations

- Inserting $Z = \exp \left( \sum_{n} \sigma^n e^{-n \frac{A}{g_s}} F^{(n)}(g_s) \right)$ leads to recursion for nonperturbative transseries components!
- Instanton action is holomorphic: $\partial S A = 0$. [Couso-Edelstein-RS-Vonk]
- Instanton action holomorphic $\Rightarrow$ Can still compute $A$ as appropriate combinations of periods in the geometry. [Drukker-Mariño-Putrov]
- Nonperturbative holomorphc anomaly equations ($A^{(n)} \equiv nA$):

$$
\left( \partial_S - \frac{1}{2} \left( \partial_z A^{(n)} \right)^2 \right) F_g^{(n)} = - \sum_{h=1}^{g} D_h^{(n)} F_{g-h}^{(n)} + \\
+ \frac{1}{2} \sum_{m=1}^{n-1} \sum_{h=0}^{g-1} \left( \partial_z F_{h-1}^{(m)} - \partial_z A^{(m)} F_{h}^{(m)} \right) \left( \partial_z F_{g-2-h}^{(n-m)} - \partial_z A^{(n-m)} F_{g-1-h}^{(n-m)} \right).
$$
Interlude: *Refined* Holomorphic Anomaly

- **Refined** holomorphic anomaly equations compute Nekrasov partition function in $\Omega$-background

\[
F(z; \epsilon_1, \epsilon_2) \simeq \sum_{n,g \geq 0} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} F(n,g)(z),
\]

via [Krefl-Walcher, Huang-Klemm]

\[
\frac{\partial F(n,g)}{\partial S} = \frac{1}{2} \left( D_z^2 F(n,g-1) + \sum_{m,h} D_z F(m,h) D_z F(n-m,g-h) \right).
\]

- **Standard** topological string limit:

\[
F_{\text{Top}} := F(z; \epsilon_1, \epsilon_2) \bigg|_{\epsilon_1+\epsilon_2=0, \epsilon_1 \epsilon_2 = g_s^2} \simeq \sum_{g \geq 0} g_s^{2g-2} F(0,g)(z)
\]

- **Refined NS-limit**: [Nekrasov-Shatashvili]

\[
F_{\text{NS}} := \lim_{\epsilon_1 \to 0} \epsilon_1 F(z; \epsilon_1, \epsilon_2 = \hbar) \simeq \sum_{n \geq 0} \hbar^{2n-1} F(n,0)(z)
\]
Interlude: *Refined* Master Equation

- Master equation for *refined* holomorphic anomaly: \([\text{Codesido-Mariño-RS}]\)
  \[
  \left( \frac{\partial}{\partial S} - \frac{1}{2} \epsilon_1 \epsilon_2 D^2_z \right) Z + U D_z Z = \left( \frac{1}{\epsilon_1 \epsilon_2} W + V \right) Z, 
  \]
  with “initial data”
  \[
  U \equiv D_z F^{(0)}_{(0,0)}, \quad W \equiv \sum_{n=0}^{+\infty} (\epsilon_1 + \epsilon_2)^{2n} W_n, \quad V \equiv \sum_{n=0}^{+\infty} (\epsilon_1 + \epsilon_2)^{2n} V_n. 
  \]

- Standard topological string limit \(\epsilon_1 + \epsilon_2 = 0, \epsilon_1 \epsilon_2 = g_s^2\) recovers standard master equation.
- Refined *NS-limit* yields the master equation:
  \[
  \frac{\partial F}{\partial S} - \frac{1}{2} \hbar (D_z F)^2 + U D_z F = \frac{1}{\hbar} W, 
  \]
  reproducing the (refined) *NS-limit* holomorphic anomaly equations,
  \[
  \frac{\partial F^{(n,0)}}{\partial S} = \frac{1}{2} \sum_{m=1}^{n-1} D_z F^{(m,0)} D_z F^{(n-m,0)}. 
  \]
Interlude: *Refined* Nonperturbative Holomorphic Anomaly

- Inserting \( Z = \exp \left( \sum_n \sigma^n e^{-n \frac{A}{\hbar}} F^{(n)}(\hbar) \right) \) leads to recursion for nonperturbative transseries components!

- Instanton action is holomorphic: \( \partial_S A = 0 \). [Codesido-Mariño-RS]

- Refined nonperturbative holomorphic anomaly equations (\( A^{(n)} \equiv nA \)):

\[
\frac{\partial F^{(n)}_g}{\partial S} = - \sum_{h=1}^{g} \mathcal{D}^{(n)}_h F^{(n)}_{g-h} + \\
+ \frac{1}{2} \sum_{m=1}^{n-1} \sum_{h=-1}^{g-1} \left( \partial_z F^{(m)}_h - \partial_z A^{(m)} F^{(m)}_{h+1} \right) \left( \partial_z F^{(n-m)}_{g-2-h} - \partial_z A^{(n-m)} F^{(n-m)}_{g-1-h} \right).
\]
On the Structure of Nonperturbative Solutions

- **Perturbative topological string:** \([\text{Yamaguchi-Yau}]\)

\[ F_g^{(0)} = \text{Pol} (S; 3g - 3). \]

- **Nonperturbative topological string:**

\[ F_g^{(1)} = e^{\frac{1}{2} \left( \partial_z A \right)^2 S} \text{Pol} (S; 3g), \]
\[ F_g^{(2)} = e^{\left( \partial_z A \right)^2 S} \text{Pol} (S; 3g - 3) + e^{2 \left( \partial_z A \right)^2 S} \text{Pol} (S; 3g), \quad \ldots \]

- **Perturbative refined NS-limit:**

\[ F_n^{(0)} = \text{Pol} (S; 2n - 3). \]

- **Nonperturbative refined NS-limit:**

\[ F_n^{(1)} = \text{Pol} (S; n), \]
\[ F_n^{(2)} = \text{Pol} (S; n + 1), \quad \ldots \]
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Near conifold $z = -\frac{1}{27}$ use coordinate $\psi^{-3} = -27z \Rightarrow 3$ conifold points at cubic roots of unity $\Rightarrow$ Instanton actions $A_i(\psi) = \frac{2\pi i}{\sqrt{3}} t_{c,i}(\psi)$, with $t_{c,1}(\psi) = t_c(\psi)$, $t_{c,2}(\psi) = t_c(e^{-2\pi i/3} \psi)$, $t_{c,3}(\psi) = t_c(e^{+2\pi i/3} \psi)$.

$$ t_c = \frac{2\pi}{\sqrt{3}} \left( \frac{3\psi}{\Gamma\left(\frac{2}{3}\right)^3} \right) \, _3F_2 \left( \left[ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{2}{3}, \frac{4}{3} \right] \left| \psi^3 \right) - \frac{9}{2} \psi^2 \right) \, _3F_2 \left( \left[ \frac{2}{3}, \frac{2}{3}, \frac{2}{3}; \frac{4}{3}, \frac{5}{3} \right] \left| \psi^3 \right) - 1 \right) $$
**Local \( \mathbb{P}^2 \): Structure of Instanton Actions**

- **Branch points and cuts** of actions \( A_1, A_2, A_3 \), in complex \( \psi \) plane. Each wedge \( 2\pi/3 \) in correspondence with one full complex \( z \) plane.

![Graphs showing branch points and cuts of actions](image)

- **Large-radius point** \( z = 0, \psi^{-1} = 0 \) use mirror map to write Kähler parameter as

\[
T(\psi) = -\frac{1}{2\pi i} \frac{\sqrt{3}}{2\pi} G_{33}^{22} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{\psi^3}.
\]

⇒ In region of moduli space associated to large-radius point, dominant instanton action is \( A_K(\psi) = 4\pi^2 i T(\psi) \).
Local $\mathbb{P}^2$: Conifold versus Kähler Actions on Borel Plane
Local \( \mathbb{P}^2 \): Checks of Conifold One-Instanton Sector

- **Conifold one-instanton:**
  \[ F_g^{(e_1)} = \frac{i\pi}{S_{1,1}} e^{\frac{1}{2}(\partial_z A_1)^2 (S-S_{1,\text{hol}})} \text{Pol}(S;3g). \]

- **Testable at large-order** using the sequence:

\[
\frac{S_{1,1}}{i\pi} F_h^{(e_1)} = \lim_{g \to \infty} \frac{A_1^{2g-1-h}}{\Gamma(2g-1-h)} \left( F_g^{(0)} - \sum_{h'=0}^{h-1} \frac{\Gamma(2g-1-h')}{A_1^{2g-1-h'}} \frac{S_{1,1}}{i\pi} F_{h'}^{(e_1)} \right). 
\]

- **Fig:** for \( h = 0, 1, 2, 3 \), at three different points in \( \psi \)-moduli space:

Fig: \( x \) changes value of propagator around its holomorphic value.

Fig: numerical blue, green dots correspond to real, imaginary.
Local $\mathbb{P}^2$: Checks of Conifold One-Instanton Sector II
At two-instanton level find a “pure” contribution

\[ \hat{F}_{h}^{(2e_1)} = e(\partial_z A_1)^2 (S-S_{1,\text{hol}}) \text{Pol}(S; 3h) + e^2(\partial_z A_1)^2 (S-S_{1,\text{hol}}) \text{Pol}(S; 3h), \]

alongside a “mixed” contribution

\[ \hat{F}_{h}^{(e_1,1)} = e(\partial_z A_1)^2 (S-S_{1,\text{hol}}) \text{Pol}\left(S; \frac{5h}{2}\right) + \text{Pol}\left(S; \frac{3h}{2} - 1\right). \]

These are testable at large-order using the sequence:

\[ F_g^{(e_1)} = \sum_{h=0}^{+\infty} \left\{ \frac{\Gamma(g+1-h)}{(A_1)^{g+1-h}} \frac{S_{1,1}}{i\pi} \hat{F}_{h}^{(2e_1)} + \frac{\Gamma(g+1-h)}{(-A_1)^{g+1-h}} \frac{\tilde{S}_{-1,1}}{2\pi i} \hat{F}_{h}^{(e_1,1)} \right\} + \ldots. \]

Top Fig: real and imaginary parts of \( \frac{S_{1,1}^2}{(i\pi)^2} \hat{F}_0^{(2e_1)}. \)

Bottom Fig: real and imaginary parts of \( \frac{S_{1,1}}{i\pi} \frac{\tilde{S}_{-1,1}}{2\pi i} \hat{F}_0^{(e_1,1)}. \)

(numerical tests at fixed \( \psi = 2 e^{-i\pi/36} \) and varying \( S = 10^{-8} (1 + i\bar{x}) \))
Local $\mathbb{P}^2$: Checks of Conifold Two-Instanton Sectors II
Local $\mathbb{P}^2$: Nonperturbative Completions

- For the case of strings on local $\mathbb{P}^2$ there is a nonperturbative definition in the literature! [Grassi-Hatsuda-Mariño, Kashaev-Mariño, Mariño-Zakany]
  - Actually extends for toric Calabi–Yau geometries...
- This is based upon the quantization of mirror curve:
  \[
  \hat{O}_{\mathbb{P}^2}(\hat{x}, \hat{p}) = e^{\hat{p}} + e^{\hat{x}} + e^{-\hat{p} - \hat{x}},
  \]
  with commutation relations $[\hat{x}, \hat{p}] = i\hbar$.
- Inverse operator $\rho = \hat{O}^{-1}$ acting on $L^2(\mathbb{R})$ is trace class and positive definite $\Rightarrow$ Partition function (in conifold frame) follows from spectral-trace expansion of Fredholm determinant:
  \[
  \det (1 + \kappa \rho) = 1 + \sum_{N=1}^{+\infty} Z(N, \hbar) \kappa^N.
  \]
Local $\mathbb{P}^2$: Resummation of the String Free Energy

- Perturbative
- Exact

Matching perturbative results... [Couso-Mariño-RS]
Local $\mathbb{P}^2$: Resummations and Stokes Phenomena I

- **Matching nonperturbative one-instanton results before Stokes line.**
- **Notation is** $\hbar = \frac{4\pi^2}{g_s}$ and $\lambda = \frac{N}{\hbar}$ (with $\lambda = \frac{\sqrt{3}}{12\pi^2} t_c$).
Local $\mathbb{P}^2$: Resummations and Stokes Phenomena II

- Crossing Stokes line and still find matching results! [Couso-Mariño-RS]

- Stokes jump is: $2\pi i e^{2\pi i N} \rightarrow 2\pi i e^{2\pi i N} - 2\pi i.$

Resummation using Conifold–2 sector:

- Stokes line for $A_2$: $\text{Im } A_2=0$
  
  --- Shape coding: agreement ---

- Match [All digits agree]

- Intermediate [At least one but not all digits agree]

- Discrepancy [No digits agree]
  
  --- Color coding: resolution ---

- High resolution [$\geq 3$ stable digits for $F_{p^2}-S_0 F^{(0)}$ and $S_0 \Phi^{(1)}$]

- Low resolution [2 stable digits for $F_{p^2}-S_0 F^{(0)}$ or $S_0 \Phi^{(1)}$]

- Poor resolution [1 stable digit for $F_{p^2}-S_0 F^{(0)}$]

- Unreliable [1 stable digit for $S_0 \Phi^{(1)}$]

- No resolution [No stable digits for either $F_{p^2}-S_0 F^{(0)}$ or $S_0 \Phi^{(1)}$]
Local $\mathbb{P}^1 \times \mathbb{P}^1$: Different Nonperturbative Completions?

- For the case of strings on local $\mathbb{P}^1 \times \mathbb{P}^1$ there are actually two distinct nonperturbative definitions in the literature!

- Quantization of mirror curve: $\hat{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\hat{x}, \hat{p}) = e^{\hat{p}} + e^{-\hat{p}} + e^{\hat{x}} + m e^{-\hat{x}}$, partition function out of spectral trace expansion of Fredholm determinant $\det(1 + \kappa \rho) = 1 + \sum_{N=1}^{+\infty} Z(N, \hbar) \kappa^N$ (with $\rho = \hat{O}^{-1}$).

- Chern–Simons partition function on lens space $L(2, 1) \simeq S^2/\mathbb{Z}_2$ (localized to 2-cuts matrix model). [Gopakumar-Vafa, Aganagic-Klemm-Mariño-Vafa]

- In what / how much do these definitions differ?
- Can the same transseries match both results?
- How would the corresponding semi-classical decodings differ?
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Mirror map from B-model (with complex structure moduli $z$) to the A-model mirror geometry (with Kähler moduli $t$) is

$$-t = \log z - 6z + 45z^2 - 560z^2 + \ldots.$$  

In the A-model, the genus-$g$ free energies are

$$F_g(t) = \sum_{d=1}^{+\infty} N_{g,d} e^{-dt},$$

where the $N_{g,d}$ are the enumerative Gromov–Witten invariants.

If free energies are asymptotic, how does this translate to the growth of enumerative GW invariants in genus and in degree?
For fixed genus $g$ or degree $d$ there is no factorial growth...

Only when both genus and degree grow,

$$d = a_0(t) + a_1(t) g.$$
Universal growth of GW invariants with $d = \frac{2g-3}{t}$, controlled by the Kähler instanton action $A_K = 2\pi t$, [Couso-RS-Vaz]

\[ N_{g,d}^{\mathbb{P}^2} Q^d \bigg|_{g=\frac{t}{2}d+q} \sim \sum_{n=1}^{+\infty} \sum_{h=0}^{+\infty} \frac{\Gamma\left(2g - \frac{3}{2} - h\right)}{(nA_K)^{2g - \frac{3}{2} - h}} \frac{n^{(1)}_0 t^{\frac{3}{2} - h}}{2^{2h+1} \pi^{h+2} n^{\frac{3}{2} + h}} \mathcal{P}_h(q), \]

where

\[ \mathcal{P}_0(q) = 1, \]

\[ \mathcal{P}_1(q) = -\frac{71}{12} + 12q - 4q^2, \]

\[ \mathcal{P}_2(q) = \frac{11545}{288} - 131q + \frac{419q^2}{3} - \frac{176q^3}{3} + 8q^4, \]

... 

All geometries tested: resolved conifold, local $\mathbb{P}^2$, local $\mathbb{P}^1 \times \mathbb{P}^1$, local curve $X_p = \mathcal{O}(p-2) \oplus \mathcal{O}(-p) \to \mathbb{P}^1$, Hurwitz theory, mirror quintic.
But Contributions from Other Instanton Actions...

- There are further contributions arising from other instanton actions!

\[ |N_{g,d} Q^d / F_{g}^{(0)}| \]

Local \(\mathbb{P}^2\) for \(g=60, t=11.625\)

\[ \frac{2g-3}{t} \leftrightarrow \text{Kähler action} \]

\[ a_0(Q) + a_1(Q)g \leftrightarrow \text{conifold action} \]

- Construct \(a_0(t), a_1(t)\) numerically but (still) cannot in closed form....
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Random Matrices and 't Hooft Large $N$ Limit

- What is the resurgent, nonperturbative nature of the large $N$ limit?
- Hermitian one-matrix model with polynomial potential $V(z)$,

$$Z = \frac{1}{\text{vol}(U(N))} \int dM \exp\left(-\frac{1}{g_s} \text{Tr} V(M)\right).$$

- Consider limit $N \to +\infty$ while $t = g_s N$ fixed [‘t Hooft]. In this case free energy $F = \log Z$ has perturbative genus expansion,

$$F \simeq \sum_{g=0}^{+\infty} F_g(t) g_s^{2g-2}.$$

- Large-order behavior $F_g \sim (2g)!$ renders topological genus expansion as asymptotic approximation [Shenker].
Interacting Theory: Quartic Matrix Model

- Potential $V(z) = \frac{1}{2} z^2 - \frac{\lambda}{24} z^4$ generically three-cut solution.
- One-cut $y^2 = \left(1 - \frac{\lambda}{6} (z^2 + 2\alpha^2)\right)^2 (z^2 - 4\alpha^2)$. [Mariño, Aniceto-RS-Vonk]
- Two-cut $\mathbb{Z}_2$-symmetric $y^2 = \frac{1}{36} \lambda^2 z^2 (z^2 - a^2) (z^2 - b^2)$. [RS-Vaz]
Resurgent Solution around One-Cut Background

- Transseries solution to (quartic) string equation:

\[
\mathcal{R}(x) \left\{ 1 - \frac{\lambda}{6} \left( \mathcal{R}(x - g_s) + \mathcal{R}(x) + \mathcal{R}(x + g_s) \right) \right\} = x.
\]

- Requires both “instanton” actions \(+A\) and \(-A\), leading to transseries:

\[
\mathcal{R}(x, \sigma_1, \sigma_2) = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sigma_1^n \sigma_2^m e^{-(n-m)A(x)/g_s} \sum_{g=\beta_{nm}}^{+\infty} g^g R_g^{(n|m)}(x).
\]

- Fully nonperturbative solution ⇒ Via Stokes transitions can move anywhere in (multi-cut) phase diagram.

- Extensive resurgent checks of large-order asymptotics on both perturbative and multi-instantonic sectors! [Aniceto-RS-Vonk]
Double-Scaling Limit and the Painlevé I Equation

- DSL yields Painlevé I equation for \( u(z) = -F_{ds}^{\prime\prime}(z) \)

\[
    u^2(z) - \frac{1}{6} u''(z) = z.
\]

- Perturbative solution

\[
    u(z) \approx \sqrt{z} \sum_{g=0}^{+\infty} \frac{u_g}{z^{g/2}},
\]

yields recursion equation; obtain asymptotic expansion

\[
    u(z) \approx \sqrt{z} \left( 1 - \frac{1}{48} z^{-5/2} - \frac{49}{4608} z^{-5} + \ldots \right).
\]

- A second order differential equation \( \Rightarrow \) Yields two instanton actions

\[
    A = \pm \frac{8\sqrt{3}}{5}.
\]
Two-Parameter Transseries Solution

- General two-parameter transseries solution is \( (g_s = z^{-5/4}) \):

\[
\begin{align*}
    u(g_s, \sigma_1, \sigma_2) &= \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sigma_1^n \sigma_2^m e^{-(n-m)} \frac{A}{g_s} \left( \sum_{k=0}^{\min(n,m)} \log^k(g_s) \cdot \Phi_{(n|m)}^k(g_s) \right),
\end{align*}
\]

- Checked nonperturbative sectors via resurgent large-order analysis.
- Resurgence allows extremely accurate tests: at genus \( g = 30 \), including six instantons corrections, results correct up to 60 decimal places!
Outline

1. Nonperturbative Holomorphic Anomalies
2. Resurgent Asymptotics in String Theory
3. Asymptotics of Enumerative Invariants
4. The Resurgence of the Large $N$ Expansion
5. Large $N$ Expansion: Stokes versus Anti-Stokes Phases
6. Outlook
Holomorphic Effective Potential

- What exactly controls saddle-points/asymptotics of matrix integral?
- In diagonal gauge, $M = \text{diag} (\lambda_1, \ldots, \lambda_N)$, holomorphic effective potential

$$V_{h;\text{eff}}(\lambda) = \int_a^\lambda dz \, y(z)$$

appears at leading order in large $N$ expansion of the matrix integral

$$Z \sim \int \prod_{i=1}^N d\lambda_i \exp \left( -\frac{1}{g_s} \sum_{i=1}^N V_{h;\text{eff}}(\lambda_i) + \ldots \right).$$

- Zero-locus $\mathcal{V}_0 = \{ z \in \mathbb{C} \mid \Re V_{h;\text{eff}}(z) = 0 \}$ constructs “spectral network” suitable for (non-linear) steepest-descent analysis.
Multi-Instanton Sectors from B-Cycles

- Instantons from B-cycles \([\text{David, Seiberg-Shih, Mariño-RS-Weiss, RS-Vaz}].\)
- Stokes lines (“jumps” in Borel plane): \(\text{Im} \left( \frac{A(t)}{g_s} \right) = 0.\)
- Anti-Stokes lines (phase boundaries): \(\text{Re} \left( \frac{A(t)}{g_s} \right) = 0.\)
Quartic Phase Diagram at Complex 't Hooft Coupling

- **Anti-Stokes phase** [Bonnet-David-Eynard, Mariño-Pasquetti-Putrov, Aniceto-RS-Vonk].
- **Trivalent-tree phase** [David, Bertola, Couso-RS-Vaz].
Phases and their Transseries Asymptotics

- **Stokes** phases (either 1 or $\mathbb{Z}_2$-symmetric 2 cuts): usual $\frac{1}{N}$ perturbative expansion, alongside $\exp(-N)$ instanton corrections...

- **Anti-Stokes** phase (with 3 cuts): unusual oscillatory asymptotics in $N$, given by theta functions $\Rightarrow$ No clear string dual!

- **Trivalent** phase (here numerical solution $N = 75$, $t = 5$):

  - More intricate theta function asymptotics...
  - Accesses strong ’t Hooft coupling region...
The Modularity of Resurgent Transseries

- Resummation methods help describing different phases of our systems.
- Can reorganize the transseries double-sum

\[ u(z, \sigma) = \sum_{n=0}^{+\infty} \sigma^n e^{-nAz} z^{-n\beta} \sum_{g=1}^{+\infty} \frac{u(g)}{z^g}, \]

summing first over all instanton numbers (transasymptotics [Costin]).

- Introducing variable \( \xi = \sigma z^{-\beta} e^{-Az} \) this yields

\[ u(z, \xi) = \sum_{g=0}^{+\infty} \frac{U_g(\xi)}{z^g} \approx U_0(\xi) + \ldots, \quad U_g(\xi) = \sum_{n=0}^{+\infty} \xi^n u_g(n). \]

Surprisingly, now each \( U_g(\xi) \) results in a convergent series!

- For Painlevé I, find Weierstrass elliptic function in variable \( s = \log \xi \),

\[ (U'_0(s))^2 = 4U^3_0(s) - g_2 U_0(s) - g_3. \]
Finding Poles of Painlevé Solutions

- Painlevé solutions generically have double poles,

\[ u(z) \bigg|_{z=z_0} \approx \frac{1}{(z - z_0)^2} + \ldots, \]

translating to simple zeroes of partition function \( Z(z) \approx (z - z_0) + \ldots. \)

- One-parameter transseries, \( u_0^{(n)} = \frac{n}{12^{n-1}}, \) implying \( \tau = \frac{1}{12} \sigma z^{-\beta} e^{-Az} \)

\[ \mathbb{U}_0(\tau) = \frac{1 + 10\tau + \tau^2}{(1 - \tau)^2}. \]

- Two-parameter transseries harder, now find [RS-Vonk]

\[ \mathbb{U}_0(T) = \frac{1 + 10T + T^2}{(1 - T)^2}, \]

with

\[ T = \frac{\sigma_1}{12\sqrt{z}} \exp \left( -\frac{47\sqrt{3}}{72 z} \sigma_1^2 \sigma_2^2 + \frac{2\sqrt{3}}{3} \sigma_1 \sigma_2 \log z - Az \right). \]
All Painlevé Solutions from Transseries

- Tritronquée solution: perturbative series alone.
- Tronquée solution: one-parameter transseries.
- General solution: full two-parameter transseries [RS-Vonk].
Finding Zeroes of Matrix-Model Partition Function

- Solutions to string equation also have double poles,

\[ R(t) \approx \frac{g_s^2}{(t - t_0)^2} + \cdots, \]

leading to simple zeroes (Lee–Yang zeroes) of the partition function

\[ Z(t) \approx (t - t_0) + \cdots. \]

- For one-parameter transseries, find [RS-Vaz]

\[ R_0(\zeta) = \frac{\zeta}{(1 - \zeta)(1 - X^2\zeta)} + \cdots, \]

where we have defined

\[ \zeta = \frac{1}{18} \sigma^2 \lambda^2 \frac{X(1 + 4X + X^2)^2}{(1 - X)^5 (1 + X)^3} \quad \text{and} \quad X = e^{A'}. \]

- Harder than previous Lambert \( W \)-function but still doable...
Matrix Model Zeroes of the Partition Function
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Wrap-up:

- **Resurgence** and **transseries** definitely powerful tools in string theory! ⇒ Play a **critical** role in nonperturbative constructions.
- **Observables** described by resurgent functions/transseries ⇒ **Define** observables **nonperturbatively** starting out from perturbation theory!
- Nonperturbative structure **fully** encoded in Borel **surface**.
- Constructed and rigorously tested **resurgent** solutions in many models...

Upcoming:

- Deal with **trivalent-tree** phase: resurgent transseries & asymptotics?
- Uncover new phases of **spacetime** via large $N$ duality?
- **Extend** to other toric or non-toric geometries and multi-matrix models... Need fully **general classification** of resurgent asymptotics?
- Systematics as a tool to access **finite** and **complex** coupling!