Resurgent Transseries in String Theory

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Based on work in collaboration with:

Inês Aniceto, Ricardo Couso-Santamaría, José Edelstein, Marcos Mariño, Sara Pasquetti, Jorge Russo, Ricardo Vaz, Marcel Vonk, Marlene Weiss

`0711.1954, 0809.2619, 0907.4082, 1106.5922, 1302.5138, 1308.1115, 1308.1695, 1407.4821, 1410.5834, 1501.01007, 1605.07473, 1610.06782`

I. Aniceto, S. Codesido, M. Mariño, R. Vaz, M. Vonk,

`arXiv: Upcoming...`

Perturbative Expansions and Borel Resummation

- Perturbative series asymptotic $\Rightarrow$ Coefficients grow as $F_g \sim g!$...

$$F(z) \simeq \sum_{g=0}^{+\infty} \frac{F_g}{z^{g+1}}.$$

- Borel transform "removes" factorial growth

$$\mathcal{B}\left[\frac{1}{z^{\alpha+1}}\right](s) = \frac{s^\alpha}{\Gamma(\alpha+1)}.$$

- Borel resummation given by inverse Borel transform

$$\mathcal{S}_\theta F(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{d}s\, \mathcal{B}[F](s)\, \mathrm{e}^{-zs}.$$

Only defined if $\mathcal{B}[F](s)$ has no singularities along direction θ!...
Perturbation theory non-Borel resummable along *any* Stokes line.

Resurgent Functions versus *Simple* Resurgent Functions

- What class of singularities does one usually find?
- Resurgent Function: a formal (asymptotic) series whose Borel transform has *endless analytic continuation* (too broad...).
- *Simple* Resurgent Function: a resurgent function whose Borel singularities restrict to simple poles and logarithmic branch-points (here ω is a simple singularity, $\mathsf{S}_\omega \in \mathbb{C}$, and Φ_ω some other sector):

$$\mathcal{B}[F](s) = \mathsf{S}_\omega \times \mathcal{B}[\Phi_\omega]\,(s-\omega)\,\frac{\log\left(s-\omega\right)}{2\pi\mathrm{i}} + \text{holomorphic}.$$

  - Can be precise about what happens upon crossing a Stokes line!

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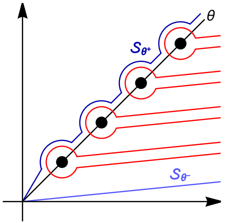
Discontinuity upon Crossing a Stokes Line



$$\mathcal{S}_{\theta^+} F - \mathcal{S}_{\theta^-} F = - \sum_{\{\omega_n\}} \mathsf{S}_{\omega_n} \, \mathrm{e}^{-\omega_n z} \, \mathcal{S}_{\theta^-} \Phi_{\omega_n} \equiv - \mathcal{S}_{\theta^-} \circ \mathrm{Disc}_{\,\theta} F$$

$$\Rightarrow \qquad \mathrm{Disc}_{\,\theta} F = \sum_{\{\omega_n\}} \mathsf{S}_{\omega_n} \, \mathrm{e}^{-\omega_n z} \, \Phi_{\omega_n}.$$

- All sectors Φ_{ω_n} must be included in full solution, as the perturbative series not enough! $\Rightarrow$ Leads to *transseries* and to *resurgence*...

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Applications to String Theory and Large N Duality

- Gravity and spacetime out from large N gauge theory. [Maldacena]
- B-model on local Calabi–Yau is large N dual to matrix model [Dijkgraaf-Vafa]. Class of target CY geometries is

$$uv = \mathcal{H}(X,Y).$$

  - Non-trivial information about this geometry encoded in Riemann surface Σ described by $\mathcal{H}(X,Y) = 0$.
- Spectral curve of matrix model is precisely Riemann surface Σ.
- Special geometry of CY, solving tree-level closed strings on this background, further yields planar solution to hermitian matrix model.

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Nonperturbative Strings and Large N Duality?

- Extend to all genera? (Perturbative) large N expansion matches (perturbative) closed string-theory free-energy [Eynard-Mariño-Orantin].

- Can one go beyond the perturbative large N expansion?
  - Establish large N duality at full nonperturbative level?
    - Nonperturbative large N gauge theory?
    - Nonperturbative closed string theory?
  - Better understand the gauge theory $\Leftrightarrow$ gravity/spacetime map?
  - Find new phenomena or new phases at large or complex couplings?
  - ......

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Outline

1. Nonperturbative Holomorphic Anomalies
2. Resurgent Asymptotics in String Theory
3. Asymptotics of Enumerative Invariants
4. The Resurgence of the Large N Expansion
5. Large N Expansion: Stokes versus Anti-Stokes Phases
6. Outlook

Outline

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String Theory on Calabi–Yau Geometries

- Consider topological string B-model on local Calabi–Yau, mirror to some toric threefold. Non-trivial information about CY geometry encoded in Riemann surface ⇔ mirror curve of the geometry.
- Compute the string free energy using the holomorphic anomaly!

$$F \simeq \sum_{g=0}^{+\infty} F_g(z)\, g_{\mathsf{s}}^{2g-2}.$$

  - Genus-g free energy depends on complex structure moduli z...
  - ...but it is not holomorphic ⇒ $F_g \equiv F_g(z,\bar{z})$! [Bershadsky-Cecotti-Ooguri-Vafa]
  - Moduli space of genus-g surfaces has a boundary:

$$\frac{\partial F_g}{\partial \bar{z}} \sim \int_{\mathfrak{M}_g} \mathrm{d}\mathfrak{m}\, \frac{\partial}{\partial \mathfrak{m}} \left(\cdots\cdots \right) \neq 0.$$

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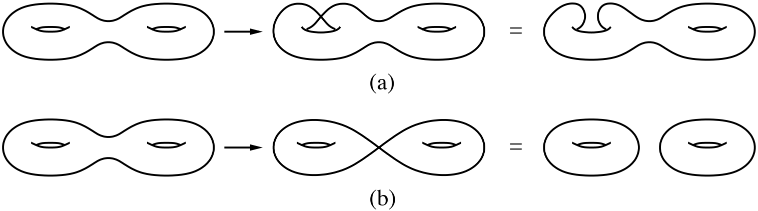
Holomorphic Anomaly Equations: Idea

- The boundary of genus-g surface moduli space corresponds to those moduli which make the genus-g surface degenerate:



- This leads to the holomorphic anomaly equations: [BCOV]

$$\frac{\partial F_g}{\partial \bar{z}} \sim \underbrace{D^2 F_{g-1}}_{\text{(a)}} + \sum_{h=1}^{g-1} \underbrace{\partial F_{g-h}\, \partial F_h}_{\text{(b)}} .$$

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Holomorphic Anomaly Equations: Formalism

- Will consider the example of B-model on mirror of $\mathbb{K}_{\mathbb{P}^2} = \mathcal{O}(-3) \to \mathbb{P}^2$
  $\Rightarrow$ Fully solved perturbatively [Haghighat-Klemm-Rauch] via holomorphic anomaly equations [Bershadsky-Cecotti-Ooguri-Vafa]

$$\frac{\partial F_g^{(0)}}{\partial S^{ij}} = \frac{1}{2} \left(D_i D_j F_{g-1}^{(0)} + \sum_{h=1}^{g-1} \partial_i F_{g-h}^{(0)} \, \partial_j F_h^{(0)} \right), \qquad g \geq 2.$$

Here D_i covariant derivative in complex structure moduli space (holomorphic dependence); S^{ij} propagators or "potentials" for Yukawa couplings (also anti-holomorphic dependence).

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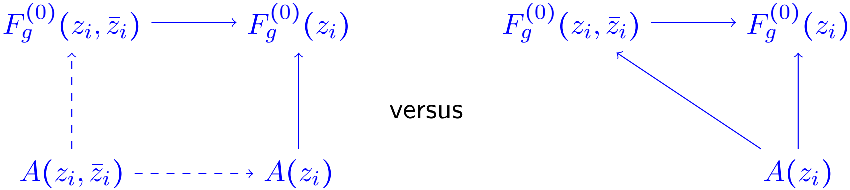
Nonperturbative Holomorphic Anomaly Equations?

- $F_g^{(0)}(z_i, \bar{z}_i)$ depends on holomorphic and anti-holomorphic complex structure moduli ⇒ What is large-order behavior?



- Rewrite holomorphic anomaly equations for partition function Z ⇒ Naturally solved with transseries *ansatz* [Couso-Edelstein-RS-Vonk]

$$Z = \exp\left(\sum_{\boldsymbol{n}} \boldsymbol{\sigma}^{\boldsymbol{n}} F^{(\boldsymbol{n})}\right).$$

Holomorphic Anomaly: Master Equation

- Complex structure moduli space of dimension one (single z and S).
- Master equation for standard holomorphic anomaly:

$$\left(\frac{\partial}{\partial S} - \frac{1}{2} g_s^2 D_z^2\right) Z + U\, D_z Z = \left(\frac{1}{g_s^2} W + V\right) Z,$$

with "initial data"

$$\begin{aligned} U &\equiv D_z F_0^{(0)}, \\ W &\equiv \frac{\partial F_0^{(0)}}{\partial S} + \frac{1}{2}\left(D_z F_0^{(0)}\right)^2, \\ V &\equiv \frac{\partial F_1^{(0)}}{\partial S} - \frac{1}{2}\left(D_z^2 F_0^{(0)}\right). \end{aligned}$$

- Inserting $Z = \exp F^{(0)}$ yields back holomorphic anomaly equations.

Nonperturbative Holomorphic Anomaly Equations

- Inserting $Z = \exp\left(\sum_n \sigma^n \, \mathrm{e}^{-n\frac{A}{g_s}} \, F^{(n)}(g_s)\right)$ leads to recursion for nonperturbative transseries components!
- Instanton action is holomorphic: $\partial_{S} A = 0$. [Couso-Edelstein-RS-Vonk]
- Instanton action holomorphic $\Rightarrow$ Can *still* compute A as appropriate combinations of periods in the geometry. [Drukker-Mariño-Putrov]
- Nonperturbative holomorphic anomaly equations ($A^{(n)} \equiv nA$):

$$\left(\partial_S - \frac{1}{2}\left(\partial_z A^{(n)}\right)^2\right) F_g^{(n)} = -\sum_{h=1}^{g} \mathcal{D}_h^{(n)} F_{g-h}^{(n)} +$$

$$+\frac{1}{2} \sum_{m=1}^{n-1} \sum_{h=0}^{g-1} \left(\partial_z F_{h-1}^{(m)} - \partial_z A^{(m)} F_h^{(m)}\right) \left(\partial_z F_{g-2-h}^{(n-m)} - \partial_z A^{(n-m)} F_{g-1-h}^{(n-m)}\right).$$

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Interlude: *Refined* Holomorphic Anomaly

- Refined holomorphic anomaly equations compute Nekrasov partition function in Ω-background

$$F(z;\epsilon_1,\epsilon_2) \simeq \sum_{n,g\geq 0} (\epsilon_1+\epsilon_2)^{2n} (\epsilon_1\epsilon_2)^{g-1} F_{(n,g)}(z),$$

via [Krefl-Walcher, Huang-Klemm]

$$\frac{\partial F_{(n,g)}}{\partial S} = \frac{1}{2}\left(D_z^2 F_{(n,g-1)} + {\sum_{m,h}}' D_z F_{(m,h)}\, D_z F_{(n-m,g-h)} \right).$$

- Standard topological string limit:

$$F_{\text{Top}} := F(z;\epsilon_1,\epsilon_2)\Big|_{\epsilon_1+\epsilon_2=0,\, \epsilon_1\epsilon_2=g_s^2} \simeq \sum_{g\geq 0} g_s^{2g-2} F_{(0,g)}(z)$$

- Refined NS-limit: [Nekrasov-Shatashvili]

$$F_{\text{NS}} := \lim_{\epsilon_1\to 0} \epsilon_1\, F(z;\epsilon_1,\epsilon_2=\hbar) \simeq \sum_{n\geq 0} \hbar^{2n-1} F_{(n,0)}(z)$$

Interlude: *Refined* Master Equation

- Master equation for refined holomorphic anomaly: [Codesido-Mariño-RS]

$$\left(\frac{\partial}{\partial S} - \frac{1}{2}\epsilon_1\epsilon_2 D_z^2\right) Z + U\, D_z Z = \left(\frac{1}{\epsilon_1\epsilon_2} W + V\right) Z,$$

  with "initial data"

$$U \equiv D_z F^{(0)}_{(0,0)}, \quad W \equiv \sum_{n=0}^{+\infty} \left(\epsilon_1+\epsilon_2\right)^{2n} W_n, \quad V \equiv \sum_{n=0}^{+\infty} \left(\epsilon_1+\epsilon_2\right)^{2n} V_n.$$

- Standard topological string limit $\epsilon_1+\epsilon_2=0$, $\epsilon_1\epsilon_2=g_s^2$ recovers standard master equation.
- Refined NS-limit yields the master equation:

$$\frac{\partial F}{\partial S} - \frac{1}{2}\hbar \left(D_z F\right)^2 + U\, D_z F = \frac{1}{\hbar} W,$$

  reproducing the (refined) NS-limit holomorphic anomaly equations,

$$\frac{\partial F_{(n,0)}}{\partial S} = \frac{1}{2} \sum_{m=1}^{n-1} D_z F_{(m,0)}\, D_z F_{(n-m,0)}.$$

Interlude: *Refined* Nonperturbative Holomorphic Anomaly

- Inserting $Z = \exp\left(\sum_n \sigma^n \, \mathrm{e}^{-n\frac{A}{\hbar}} \, F^{(n)}(\hbar)\right)$ leads to recursion for nonperturbative transseries components!
- Instanton action is holomorphic: $\partial_S A = 0$. [Codesido-Mariño-RS]
- Refined nonperturbative holomorphic anomaly equations ($A^{(n)} \equiv nA$):

$$\frac{\partial F_g^{(n)}}{\partial S} = -\sum_{h=1}^{g} \mathfrak{D}_h^{(n)} F_{g-h}^{(n)} +$$

$$+\frac{1}{2} \sum_{m=1}^{n-1} \sum_{h=-1}^{g-1} \left(\partial_z F_h^{(m)} - \partial_z A^{(m)} F_{h+1}^{(m)}\right) \left(\partial_z F_{g-2-h}^{(n-m)} - \partial_z A^{(n-m)} F_{g-1-h}^{(n-m)}\right).$$

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On the Structure of Nonperturbative Solutions

- Perturbative topological string: [Yamaguchi-Yau]

$$F_g^{(0)} = \mathsf{Pol}\left(S; 3g-3\right).$$

- Nonperturbative topological string:

$$\begin{aligned} F_g^{(1)} &= \mathrm{e}^{\frac{1}{2}(\partial_z A)^2 S}\, \mathsf{Pol}\left(S; 3g\right), \\ F_g^{(2)} &= \mathrm{e}^{(\partial_z A)^2 S}\, \mathsf{Pol}\left(S; 3g-3\right) + \mathrm{e}^{2(\partial_z A)^2 S}\, \mathsf{Pol}\left(S; 3g\right), \quad \cdots \end{aligned}$$

- Perturbative refined NS-limit:

$$F_n^{(0)} = \mathsf{Pol}\left(S; 2n-3\right).$$

- Nonperturbative refined NS-limit:

$$\begin{aligned} F_n^{(1)} &= \mathsf{Pol}\left(S; n\right), \\ F_n^{(2)} &= \mathsf{Pol}\left(S; n+1\right), \quad \cdots \end{aligned}$$

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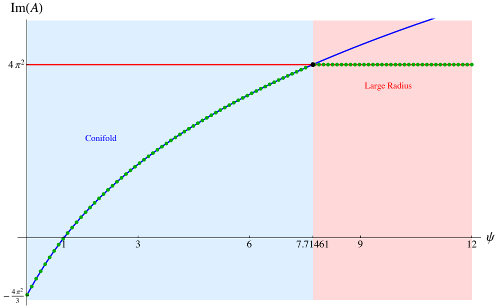
Outline

Local $\mathbb{P}^2$: Checks of Conifold Instanton Actions

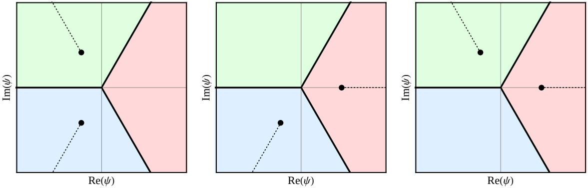


- Near conifold $z = -\frac{1}{27}$ use coordinate $\psi^{-3} = -27z \Rightarrow 3$ conifold points at cubic roots of unity $\Rightarrow$ Instanton actions $A_i(\psi) = \frac{2\pi \mathrm{i}}{\sqrt{3}}\, t_{\mathsf{c},i}(\psi)$, with $t_{\mathsf{c},1}(\psi) = t_{\mathsf{c}}(\psi)$, $t_{\mathsf{c},2}(\psi) = t_{\mathsf{c}}(\mathrm{e}^{-2\pi\mathrm{i}/3}\,\psi)$, $t_{\mathsf{c},3}(\psi) = t_{\mathsf{c}}(\mathrm{e}^{+2\pi\mathrm{i}/3}\,\psi)$.

$$\left(t_{\mathsf{c}} = \frac{2\pi}{\sqrt{3}} \left(\frac{3\psi}{\Gamma\left(\frac{2}{3}\right)^3}\; {}_3F_2\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{2}{3}, \frac{4}{3} \,\middle|\, \psi^3 \right) - \frac{\frac{9}{2}\psi^2}{\Gamma\left(\frac{1}{3}\right)^3}\; {}_3F_2\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}; \frac{4}{3}, \frac{5}{3} \,\middle|\, \psi^3 \right) - 1 \right) \right)$$

Local $\mathbb{P}^2$: Structure of Instanton Actions

- Branch points and cuts of actions A_1, A_2, A_3, in complex ψ plane. Each wedge $2\pi/3$ in correspondence with one full complex z plane.



- Large-radius point $z=0$, $\psi^{-1}=0$ use mirror map to write Kähler parameter as

$$T(\psi) = -\frac{1}{2\pi \mathrm{i}} \frac{\sqrt{3}}{2\pi} G^{22}_{33} \left(\begin{array}{ccc|c} \frac{1}{3} & \frac{2}{3} & 1 & \multirow{2}{*}{$-\frac{1}{\psi^3}$} \\ 0 & 0 & 0 & \end{array} \right).$$

$\Rightarrow$ In region of moduli space associated to large-radius point, dominant instanton action is $A_{\mathsf{K}}(\psi) = 4\pi^2 \mathrm{i}\, T(\psi)$.

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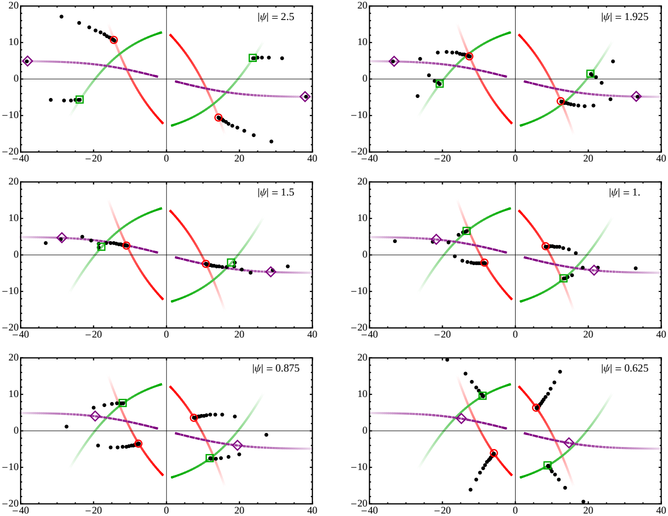
Local $\mathbb{P}^2$: Conifold versus Kähler Actions on Borel Plane

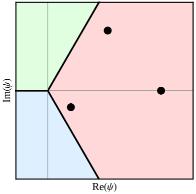


Local $\mathbb{P}^2$: Checks of Conifold One-Instanton Sector I

- Conifold one-instanton: $F_g^{(\mathbf{e}_1)} = \frac{\mathrm{i}\pi}{S_{1,1}}\, \mathrm{e}^{\frac{1}{2}(\partial_z A_1)^2 (S - S_{1,\mathrm{hol}})}\, \mathrm{Pol}\,(S; 3g)$.
- Testable at large-order using the sequence:

$$\frac{S_{1,1}}{\mathrm{i}\pi} F_h^{(\mathbf{e}_1)} = \lim_{g\to\infty} \frac{A_1^{2g-1-h}}{\Gamma(2g-1-h)} \left(F_g^{(\mathbf{0})} - \sum_{h'=0}^{h-1} \frac{\Gamma\left(2g-1-h'\right)}{A_1^{2g-1-h'}} \frac{S_{1,1}}{\mathrm{i}\pi} F_{h'}^{(\mathbf{e}_1)} \right).$$

- Fig: for $h = 0, 1, 2, 3$, at three different points in ψ-moduli space:



- Fig: x changes value of propagator around its holomorphic value.
- Fig: numerical blue, green dots correspond to real, imaginary.

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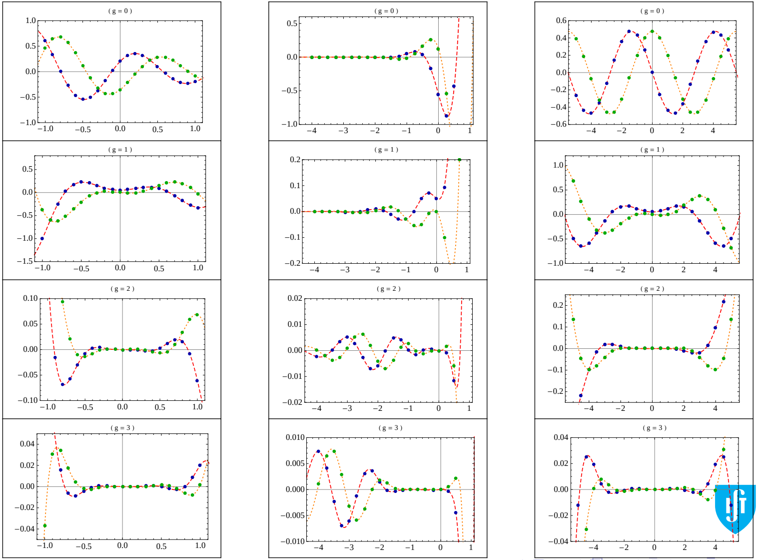
Local $\mathbb{P}^2$: Checks of Conifold One-Instanton Sector II



Local $\mathbb{P}^2$: Checks of Conifold Two-Instanton Sectors I

- At two-instanton level find a "pure" contribution

$$\widehat{F}_h^{(2\mathbf{e}_1)} = \mathrm{e}^{(\partial_z A_1)^2 (S - S_{1,\mathsf{hol}})}\, \mathsf{Pol}\,(S; 3h) + \mathrm{e}^{2(\partial_z A_1)^2 (S - S_{1,\mathsf{hol}})}\, \mathsf{Pol}\,(S; 3h),$$

alongside a "mixed" contribution

$$\widehat{F}_h^{(\mathbf{e}_{1,1})} = \mathrm{e}^{(\partial_z A_1)^2 (S - S_{1,\mathsf{hol}})}\, \mathsf{Pol}\left(S; \frac{5h}{2}\right) + \mathsf{Pol}\left(S; \frac{3h}{2} - 1\right).$$

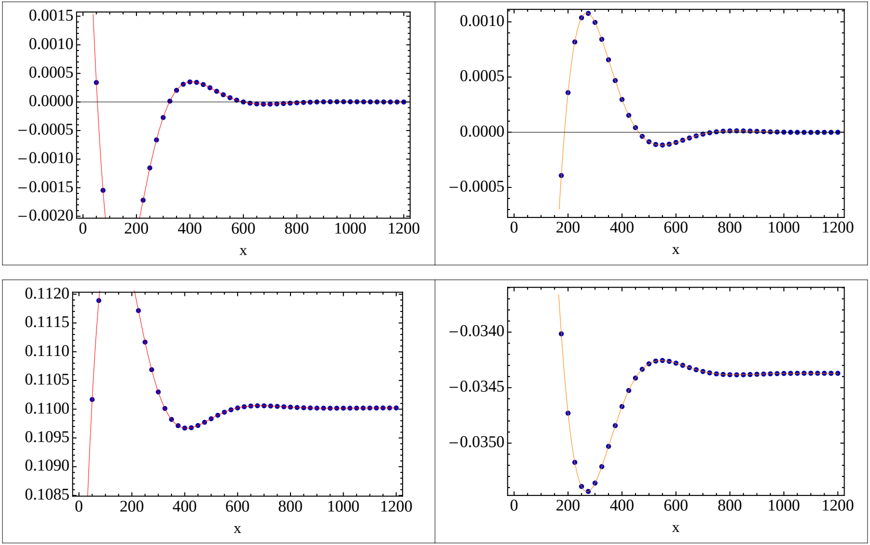
- These are testable at large-order using the sequence:

$$F_g^{(\mathbf{e}_1)} \simeq \sum_{h=0}^{+\infty} \left\{ \frac{\Gamma(g+1-h)}{(+A_1)^{g+1-h}}\, \frac{S_{1,1}}{\mathrm{i}\pi}\, \widehat{F}_h^{(2\mathbf{e}_1)} + \frac{\Gamma(g+1-h)}{(-A_1)^{g+1-h}}\, \frac{\widetilde{S}_{-1,1}}{2\pi\mathrm{i}}\, \widehat{F}_h^{(\mathbf{e}_{1,1})} \right\} + \cdots.$$

- Top Fig: real and imaginary parts of $\frac{S_{1,1}^2}{(\mathrm{i}\pi)^2}\, \widehat{F}_0^{(2\mathbf{e}_1)}$.
- Bottom Fig: real and imaginary parts of $\frac{S_{1,1}}{\mathrm{i}\pi}\, \frac{\widetilde{S}_{-1,1}}{2\pi\mathrm{i}}\, \widehat{F}_0^{(\mathbf{e}_{1,1})}$.

(numerical tests at fixed $\psi = 2\,\mathrm{e}^{-\mathrm{i}\pi/36}$ and varying $S = 10^{-8}\,(1 + \mathrm{i}x)$)

Local $\mathbb{P}^2$: Checks of Conifold Two-Instanton Sectors II



Local $\mathbb{P}^2$: Nonperturbative Completions

- For the case of strings on local $\mathbb{P}^2$ there is a nonperturbative definition in the literature! [Grassi-Hatsuda-Mariño, Kashaev-Mariño, Mariño-Zakany]
  - Actually extends for toric Calabi–Yau geometries...
- This is based upon the quantization of mirror curve:

$$\hat{\mathcal{O}}_{\mathbb{P}^2}(\hat{x},\hat{p}) = \mathrm{e}^{\hat{p}} + \mathrm{e}^{\hat{x}} + \mathrm{e}^{-\hat{p}-\hat{x}},$$

  with commutation relations $[\hat{x},\hat{p}] = \mathrm{i}\hbar$.
- Inverse operator $\rho = \hat{\mathcal{O}}^{-1}$ acting on $L^2(\mathbb{R})$ is trace class and positive definite $\Rightarrow$ Partition function (in conifold frame) follows from spectral-trace expansion of Fredholm determinant:

$$\det\left(1+\kappa\,\rho\right) = 1 + \sum_{N=1}^{+\infty} Z(N,\hbar)\,\kappa^N.$$

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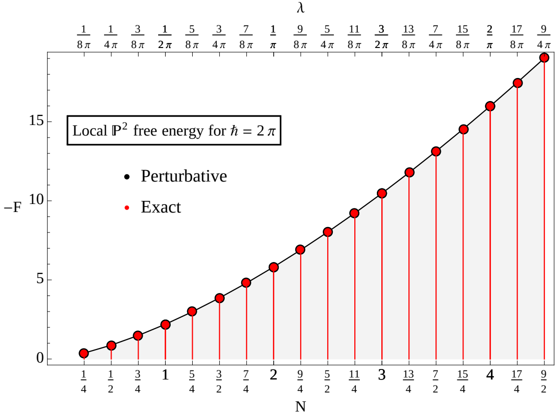
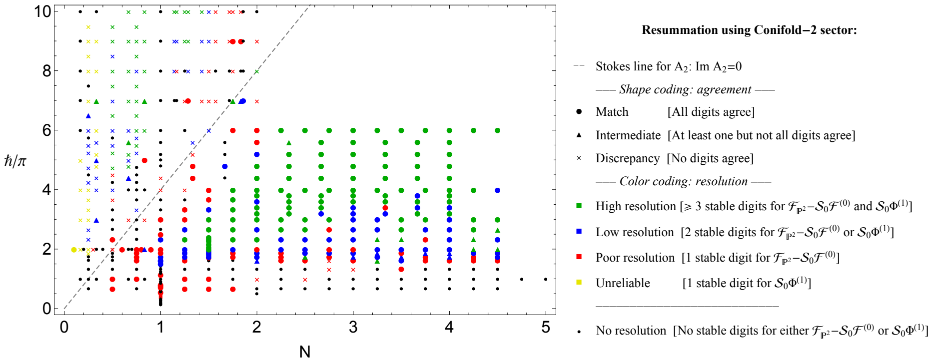
Local $\mathbb{P}^2$: Resummation of the String Free Energy



- Matching perturbative results... [Couso-Mariño-RS]

Local $\mathbb{P}^2$: Resummations and Stokes Phenomena I



- Matching nonperturbative one-instanton results before Stokes line...
- Notation is $\hbar = \dfrac{4\pi^2}{g_s}$ and $\lambda = \dfrac{N}{\hbar}$ (with $\lambda = \dfrac{\sqrt{3}}{12\pi^2}\, t_c$).

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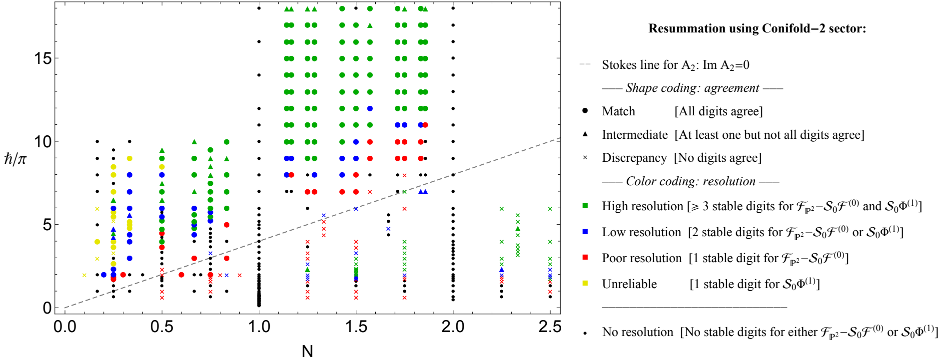
Local $\mathbb{P}^2$: Resummations and Stokes Phenomena II



- Crossing Stokes line and still find matching results! [Couso-Mariño-RS]
- Stokes jump is: $2\pi\mathrm{i}\,\mathrm{e}^{2\pi\mathrm{i}N} \to 2\pi\mathrm{i}\,\mathrm{e}^{2\pi\mathrm{i}N} - 2\pi\mathrm{i}$.

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Local $\mathbb{P}^1 \times \mathbb{P}^1$: Different Nonperturbative Completions?

- For the case of strings on local $\mathbb{P}^1 \times \mathbb{P}^1$ there are actually two distinct nonperturbative definitions in the literature!
  - Quantization of mirror curve: $\hat{\mathcal{O}}_{\mathbb{P}^1 \times \mathbb{P}^1}(\hat{x}, \hat{p}) = \mathrm{e}^{\hat{p}} + \mathrm{e}^{-\hat{p}} + \mathrm{e}^{\hat{x}} + m\, \mathrm{e}^{-\hat{x}}$, partition function out of spectral trace expansion of Fredholm determinant $\det(1 + \kappa\, \rho) = 1 + \sum_{N=1}^{+\infty} Z(N, \hbar)\, \kappa^N$ (with $\rho = \hat{\mathcal{O}}^{-1}$).
  - Chern–Simons partition function on lens space $L(2,1) \simeq \mathbb{S}^2/\mathbb{Z}_2$ (localized to 2-cuts matrix model). [Gopakumar-Vafa, Aganagic-Klemm-Mariño-Vafa]

- In what / how much do these definitions differ?
- Can the same transseries match both results?
- How would the corresponding semi-classical decodings differ?

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Outline

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Mirror Map to A-Model and Gromov–Witten Invariants

- Mirror map from B-model (with complex structure moduli z) to the A-model mirror geometry (with Kähler moduli t) is

$$-t = \log z - 6z + 45z^2 - 560z^2 + \cdots.$$

- In the A-model, the genus-g free energies are

$$F_g(t) = \sum_{d=1}^{+\infty} N_{g,d}\, \mathrm{e}^{-d\,t},$$

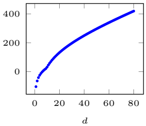
where the $N_{g,d}$ are the enumerative Gromov–Witten invariants.

- If free energies are asymptotic, how does this translate to the growth of enumerative GW invariants in genus and in degree?

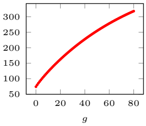
Growth of GW Invariants: Local $\mathbb{P}^2$ Geometry

$\log\lvert N_{g,d}\rvert$	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
20	111.4	117.1	122.6	128.0	133.2	138.3	143.4	148.3	153.2	157.9	162.6	167.3	171.8	176.3	180.8	185.2	189.5	193.9	198.1	202.3	206.5
21	113.7	119.6	125.3	130.8	136.2	141.4	146.6	151.6	156.6	161.5	166.3	171.0	175.7	180.3	184.8	189.3	193.7	198.1	202.5	206.8	211.1
22	116.0	122.0	127.9	133.5	139.1	144.5	149.7	154.9	160.0	165.0	169.9	174.7	179.5	184.2	188.8	193.4	197.9	202.4	206.8	211.2	215.5
23	118.2	124.4	130.4	136.2	141.9	147.4	152.8	158.1	163.3	168.4	173.4	178.4	183.2	188.0	192.8	197.4	202.0	206.6	211.1	215.6	220.0
24	120.4	126.7	132.9	138.9	144.7	150.3	155.9	161.3	166.6	171.8	177.0	182.0	187.0	191.8	196.7	201.4	206.1	210.7	215.3	219.9	224.4
25	122.5	129.0	135.3	141.5	147.4	153.2	158.9	164.4	169.9	175.2	180.4	185.6	190.6	195.6	200.5	205.4	210.1	214.9	219.5	224.2	228.7
26	124.5	131.2	137.7	144.0	150.1	156.0	161.8	167.5	173.1	178.5	183.9	189.1	194.3	199.3	204.3	209.3	214.1	219.0	223.7	228.4	233.0
27	126.5	133.4	140.0	146.5	152.7	158.8	164.8	170.6	176.2	181.8	187.2	192.6	197.8	203.0	208.1	213.1	218.1	223.0	227.8	232.6	237.3
28	128.5	135.5	142.3	148.9	155.3	161.5	167.6	173.5	179.3	185.0	190.6	196.0	201.4	206.7	211.9	217.0	222.0	227.0	231.9	236.8	241.6
29	130.4	137.6	144.6	151.3	157.9	164.2	170.4	176.5	182.4	188.2	193.9	199.4	204.9	210.3	215.6	220.8	225.9	231.0	236.0	240.9	245.8
30	132.2	139.6	146.7	153.7	160.4	166.9	173.2	179.4	185.4	191.3	197.1	202.8	208.4	213.8	219.2	224.5	229.7	234.9	240.0	245.0	249.9
31	134.0	141.6	148.9	155.9	162.8	169.5	175.9	182.2	188.4	194.4	200.3	206.1	211.8	217.4	222.9	228.2	233.6	238.8	244.0	249.1	254.1
32	135.7	143.5	151.0	158.2	165.2	172.0	178.6	185.1	191.4	197.5	203.5	209.4	215.2	220.9	226.4	231.9	237.3	242.7	247.9	253.1	258.2
33	137.4	145.4	153.0	160.4	167.6	174.5	181.3	187.8	194.3	200.5	206.6	212.7	218.5	224.3	230.0	235.6	241.1	246.5	251.8	257.1	262.3
34	139.0	147.2	155.0	162.6	169.9	177.0	183.9	190.6	197.1	203.5	209.7	215.9	221.9	227.7	233.5	239.2	244.8	250.3	255.7	261.0	266.3
35	140.6	149.0	157.0	164.7	172.2	179.4	186.4	193.3	199.9	206.4	212.8	219.0	225.1	231.1	237.0	242.8	248.4	254.0	259.5	265.0	270.3
36	142.2	150.7	158.9	166.8	174.4	181.8	188.9	195.9	202.7	209.3	215.8	222.2	228.4	234.5	240.4	246.3	252.1	257.7	263.3	268.8	274.3
37	143.7	152.4	160.7	168.8	176.6	184.1	191.4	198.5	205.5	212.2	218.8	225.3	231.6	237.8	243.8	249.8	255.7	261.4	267.1	272.7	278.2
38	145.1	154.0	162.6	170.8	178.7	186.4	193.9	201.1	208.2	215.0	221.8	228.3	234.7	241.0	247.2	253.3	259.2	265.1	270.9	276.5	282.1
39	146.6	155.6	164.3	172.7	180.8	188.7	196.3	203.7	210.8	217.8	224.7	231.4	237.9	244.3	250.6	256.7	262.8	268.7	274.6	280.3	286.0
40	147.9	157.2	166.1	174.6	182.9	190.9	198.6	206.2	213.5	220.6	227.6	234.3	241.0	247.5	253.9	260.1	266.3	272.3	278.3	284.1	289.9

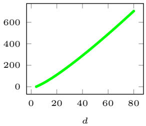
$\log\lvert N_{30,d}\rvert$



$\log\lvert N_{g,30}\rvert$



$\log\lvert N_{d,d}\rvert$



- For fixed genus g or degree d there is no factorial growth...
- Only when both genus and degree grow,

$$d = a_0(t) + a_1(t)\, g.$$

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Universal Growth from Kähler Instanton Action

- Universal growth of GW invariants with $d = \frac{2g-3}{t}$, controlled by the Kähler instanton action $A_{\mathsf{K}} = 2\pi t$, [Couso-RS-Vaz]

$$N_{g,d}^{\mathbb{P}^2} Q^d \Big|_{g=\frac{t}{2}d+q} \sim \sum_{n=1}^{+\infty} \sum_{h=0}^{+\infty} \frac{\Gamma\left(2g-\frac{3}{2}-h\right)}{(nA_{\mathsf{K}})^{2g-\frac{3}{2}-h}} \frac{n_0^{(1)}\, t^{\frac{3}{2}-h}}{2^{2h+1}\, \pi^{h+2}\, n^{\frac{3}{2}+h}} \mathcal{P}_h(q),$$

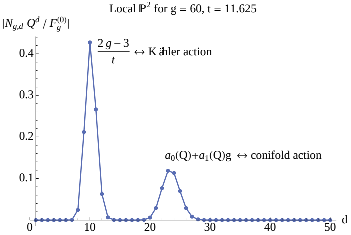
  where

$$\begin{aligned}
\mathcal{P}_0(q) &= 1, \\
\mathcal{P}_1(q) &= -\frac{71}{12} + 12q - 4q^2, \\
\mathcal{P}_2(q) &= \frac{11545}{288} - 131q + \frac{419q^2}{3} - \frac{176q^3}{3} + 8q^4, \\
&\cdots
\end{aligned}$$

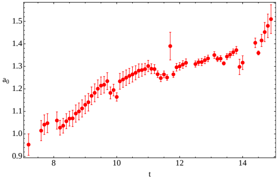
- All geometries tested: resolved conifold, local $\mathbb{P}^2$, local $\mathbb{P}^1 \times \mathbb{P}^1$, local curve $X_p = \mathcal{O}(p-2) \oplus \mathcal{O}(-p) \to \mathbb{P}^1$, Hurwitz theory, mirror quintic.

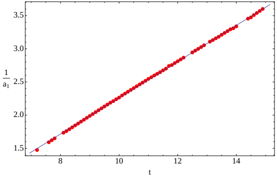
But Contributions from Other Instanton Actions...

- There are further contributions arising from other instanton actions!



- Construct $a_0(t)$, $a_1(t)$ numerically but (still) cannot in closed form....





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Random Matrices and 't Hooft Large N Limit

- What is the resurgent, nonperturbative nature of the large N limit?...
- Hermitian one-matrix model with polynomial potential $V(z)$,

$$Z = \frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int \mathrm{d}M \, \exp\left(-\frac{1}{g_s} \mathrm{Tr} V(M)\right).$$

- Consider limit $N \to +\infty$ while $t = g_s N$ fixed ['t Hooft]. In this case free energy $F = \log Z$ has perturbative genus expansion,

$$F \simeq \sum_{g=0}^{+\infty} F_g(t)\, g_s^{2g-2}.$$

- Large-order behavior $F_g \sim (2g)!$ renders topological genus expansion as asymptotic approximation [Shenker].

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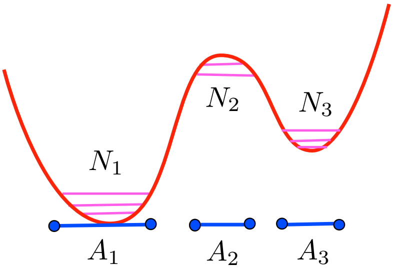
Interacting Theory: Quartic Matrix Model



- Potential $V(z) = \frac{1}{2}z^2 - \frac{\lambda}{24}z^4$ generically three-cut solution.
- One-cut $y^2 = \left(1 - \frac{\lambda}{6}\left(z^2 + 2\alpha^2\right)\right)^2 \left(z^2 - 4\alpha^2\right)$. [Mariño, Aniceto-RS-Vonk]
- Two-cut $\mathbb{Z}_2$-symmetric $y^2 = \frac{1}{36}\lambda^2 z^2 \left(z^2 - a^2\right)\left(z^2 - b^2\right)$. [RS-Vaz]

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Resurgent Solution around One-Cut Background

- Transseries solution to (quartic) string equation:

$$\mathcal{R}(x)\left\{1-\frac{\lambda}{6}\Big(\mathcal{R}(x-g_{\mathsf{s}})+\mathcal{R}(x)+\mathcal{R}(x+g_{\mathsf{s}})\Big)\right\}=x.$$

- Requires both "instanton" actions $+A$ and $-A$, leading to transseries:

$$\mathcal{R}(x,\sigma_1,\sigma_2)=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\sigma_1^n\sigma_2^m\,\mathrm{e}^{-(n-m)A(x)/g_{\mathsf{s}}}\sum_{g=\beta_{nm}}^{+\infty}g_{\mathsf{s}}^g\,R_g^{(n|m)}(x).$$

- Fully nonperturbative solution ⇒ Via Stokes transitions can move anywhere in (multi-cut) phase diagram.
- Extensive resurgent checks of large-order asymptotics on both perturbative and multi-instantonic sectors! **[Aniceto-RS-Vonk]**

Double-Scaling Limit and the Painlevé I Equation

- DSL yields Painlevé I equation for $u(z) = -F''_{\text{ds}}(z)$

$$u^2(z) - \frac{1}{6}u''(z) = z.$$

- Perturbative solution

$$u(z) \simeq \sqrt{z} \sum_{g=0}^{+\infty} \frac{u_g}{z^{\frac{5}{2}g}},$$

yields recursion equation; obtain asymptotic expansion

$$u(z) \simeq \sqrt{z} \left(1 - \frac{1}{48} z^{-\frac{5}{2}} - \frac{49}{4608} z^{-5} + \cdots \right).$$

- A second order differential equation ⇒ Yields two instanton actions

$$A = \pm \frac{8\sqrt{3}}{5}.$$

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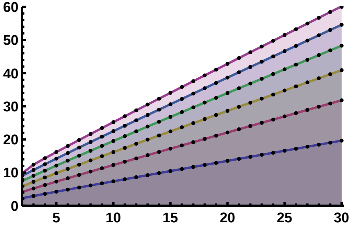
Two-Parameter Transseries Solution

- General two-parameter transseries solution is ($g_s = z^{-5/4}$):

$$u(g_s,\sigma_1,\sigma_2) = \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\sigma_1^n\sigma_2^m\, \mathrm{e}^{-(n-m)\frac{A}{g_s}}\left(\sum_{k=0}^{\min(n,m)}\log^k(g_s)\cdot\Phi_{(n|m)}^{[k]}(g_s)\right).$$

- Checked nonperturbative sectors via resurgent large-order analysis.
- Resurgence allows extremely accurate tests: at genus $g = 30$, including six instantons corrections, results correct up to 60 decimal places!



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Holomorphic Effective Potential

- What exactly controls saddle-points/asymptotics of matrix integral?
- In diagonal gauge, $M = \text{diag}\,(\lambda_1, \ldots, \lambda_N)$, holomorphic effective potential

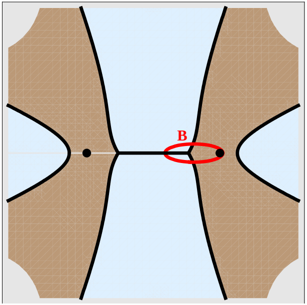
$$V_{\text{h;eff}}(\lambda) = \int_a^\lambda \mathrm{d}z\, y(z)$$

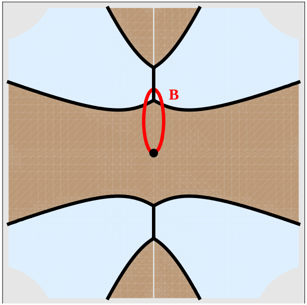
appears at leading order in large N expansion of the matrix integral

$$Z \sim \int \prod_{i=1}^{N} \mathrm{d}\lambda_i\, \exp\left(-\frac{1}{g_{\mathsf{s}}} \sum_{i=1}^{N} V_{\text{h;eff}}(\lambda_i) + \cdots\right).$$

- Zero-locus $\mathcal{V}_0 = \left\{ z \in \mathbb{C} \,\middle|\, \mathbb{R}\text{e}\, V_{\text{h;eff}}(z) = 0 \right\}$ constructs "spectral network" suitable for (non-linear) steepest-descent analysis.

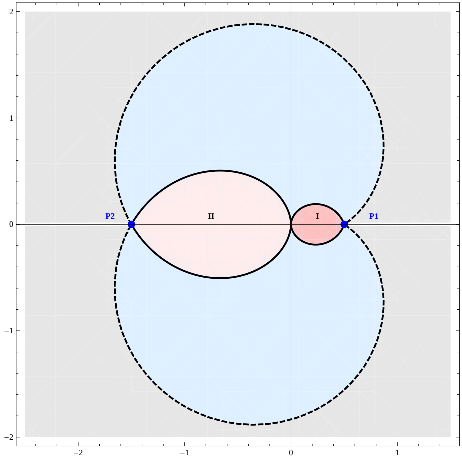
Multi-Instanton Sectors from B-Cycles





- Instantons from B-cycles [David, Seiberg-Shih, Mariño-RS-Weiss, RS-Vaz].
- Stokes lines ("jumps" in Borel plane): $\mathbb{I}\mathrm{m}\left(\frac{A(t)}{g_s}\right) = 0$.
- Anti-Stokes lines (phase boundaries): $\mathbb{R}\mathrm{e}\left(\frac{A(t)}{g_s}\right) = 0$.

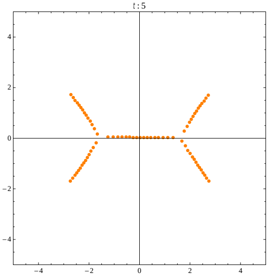
Quartic Phase Diagram at Complex 't Hooft Coupling



- Anti-Stokes phase [Bonnet-David-Eynard, Mariño-Pasquetti-Putrov, Aniceto-RS-Vonk].
- Trivalent-tree phase [David, Bertola, Couso-RS-Vaz].

Phases and their Transseries Asymptotics

- Stokes phases (either 1 or $\mathbb{Z}_2$-symmetric 2 cuts): usual $\frac{1}{N}$ perturbative expansion, alongside $\exp(-N)$ instanton corrections...
- Anti-Stokes phase (with 3 cuts): unusual oscillatory asymptotics in N, given by theta functions $\Rightarrow$ No clear string dual!
- Trivalent phase (here numerical solution $N = 75$, $t = 5$):



- More intricate theta function asymptotics...
- Accesses strong 't Hooft coupling region...

The Modularity of Resurgent Transseries

- Resummation methods help describing different phases of our systems.
- Can reorganize the transseries double-sum

$$u\left(z,\sigma\right) = \sum_{n=0}^{+\infty} \sigma^n \, \mathrm{e}^{-nAz} \, z^{-n\beta} \sum_{g=1}^{+\infty} \frac{u_g^{(n)}}{z^g},$$

summing first over all instanton numbers (transasymptotics [Costin]).

- Introducing variable $\xi = \sigma\, z^{-\beta}\, \mathrm{e}^{-Az}$ this yields

$$u(z,\xi) = \sum_{g=0}^{+\infty} \frac{\mathbb{U}_g(\xi)}{z^g} \simeq \mathbb{U}_0(\xi) + \cdots, \qquad \mathbb{U}_g(\xi) = \sum_{n=0}^{+\infty} \xi^n u_g^{(n)}.$$

Surprisingly, now each $\mathbb{U}_g(\xi)$ results in a convergent series!

- For Painlevé I, find Weierstrass elliptic function in variable $s = \log\xi$,

$$\left(\mathbb{U}_0'(s)\right)^2 = 4\,\mathbb{U}_0^3(s) - g_2\,\mathbb{U}_0(s) - g_3.$$

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Finding Poles of Painlevé Solutions

- Painlevé solutions generically have double poles,

$$u(z)\Big|_{z=z_0} \simeq \frac{1}{(z-z_0)^2} + \cdots,$$

translating to simple zeroes of partition function $Z(z) \approx (z-z_0) + \cdots$.

- One-parameter transseries, $u_0^{(n)} = \frac{n}{12^{n-1}}$, implying ($\tau = \frac{1}{12}\sigma z^{-\beta} e^{-Az}$)

$$\mathbb{U}_0(\tau) = \frac{1+10\tau+\tau^2}{(1-\tau)^2}.$$

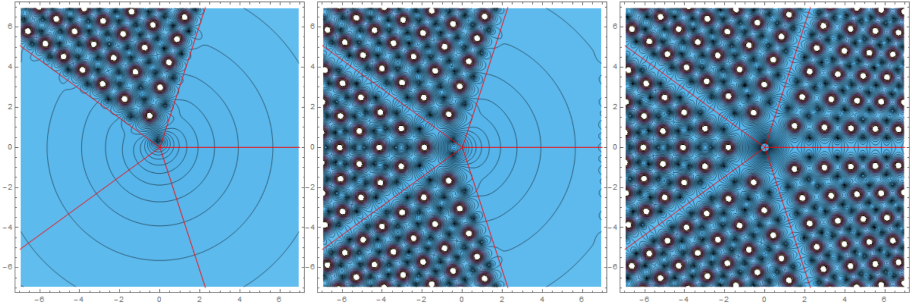
- Two-parameter transseries harder, now find [RS-Vonk]

$$\mathbb{U}_0(T) = \frac{1+10T+T^2}{(1-T)^2},$$

with

$$T = \frac{\sigma_1}{12\sqrt{z}} \exp\left(-\frac{47\sqrt{3}}{72\,z}\sigma_1^2\sigma_2^2 + \frac{2\sqrt{3}}{3}\sigma_1\sigma_2 \log z - Az\right)$$

All Painlevé Solutions from Transseries



- Tritronquée solution: perturbative series alone.
- Tronquée solution: one-parameter transseries.
- General solution: full two-parameter transseries [RS-Vonk].

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Finding Zeroes of Matrix-Model Partition Function

- Solutions to string equation also have double poles,

$$\mathcal{R}(t) \simeq \frac{g_s^2}{(t-t_0)^2} + \cdots,$$

leading to simple zeroes (Lee–Yang zeroes) of the partition function

$$Z(t) \approx (t-t_0) + \cdots.$$

- For one-parameter transseries, find [RS-Vaz]

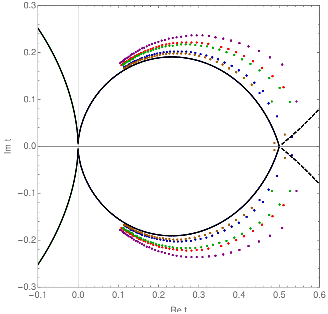
$$\mathbb{R}_0(\zeta) = \frac{\zeta}{(1-\zeta)\left(1-X^2\zeta\right)} + \cdots,$$

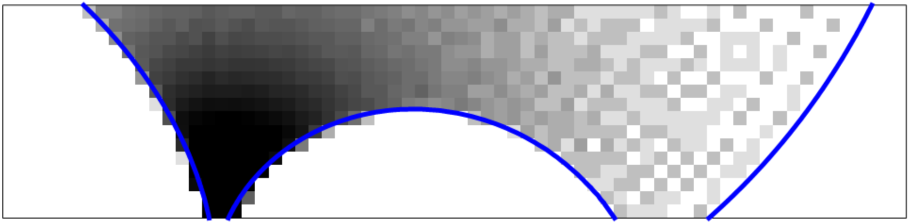
where we have defined

$$\zeta = \frac{1}{18}\,\sigma^2\lambda^2\,\frac{X\left(1+4X+X^2\right)^2}{(1-X)^5\,(1+X)^3} \qquad \text{and} \qquad X = \mathrm{e}^{A'}.$$

- Harder than previous Lambert W-function but still doable...

Matrix Model Zeroes of the Partition Function





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Summary and Future Directions

- Wrap-up:
  - Resurgence and transseries definitely powerful tools in string theory! ⇒ Play a critical role in nonperturbative constructions.
  - Observables described by resurgent functions/transseries ⇒ Define observables nonperturbatively starting out from perturbation theory!
  - Nonperturbative structure fully encoded in Borel surface.
  - Constructed and rigorously tested resurgent solutions in many models...
- Upcoming:
  - Deal with trivalent-tree phase: resurgent transseries & asymptotics?
  - Uncover new phases of spacetime via large N duality?
  - Extend to other toric or non-toric geometries and multi-matrix models... Need fully general classification of resurgent asymptotics?
  - Systematics as a tool to access finite and complex coupling!

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