Generalized ODE/IM Correspondence

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Introduction

Introduction

We consider the second order differential equation:

$$-\hbar^2 \frac{d^2}{dq^2}\psi(q) + (V(q) - E)\psi(q) = 0.$$

where $V(q)$ is a polynomial in q.

This ODE appears in

- quantum mechanics: 1D Schrödinger equation
- 4d SUSY gauge theory [Mironov-Morozov]
  quantum SW curve in $N = 2$ gauge theory
  ($\hbar$: Omega deformation parameter in the Nekrasov-Shatashvili limit)

Quantum mechanics and resurgence

exact WKB method [Voros 1981, Sato-Aoki-Kawai-Takei, ...]

- the WKB expansion is asymptotic expansion in $\hbar$
- Borel and Laplace transformations

$$\phi(z) = \sum_{n=0}^{\infty} c_n z^{-n-1} \to \tilde{\phi}(\xi) = \sum_{n=0}^{\infty} c_n \frac{\xi^n}{n!}$$
$$\to \mathcal{L}[\tilde{\phi}](z) = \int_0^{\infty} d\xi e^{-\xi z} \tilde{\phi}(\xi)$$

- resurgence (perturbative $\longleftrightarrow$ non-perturbative) [Ecalle]
- the exact WKB periods show discontinuity across the Stokes lines

  Discontinuity formula [Delabaere-Dillinger-Pham 1993]

Analytic bootstrap

- Voros' idea: The classical periods and their discontinuity structure determine the exact WKB periods.
  analytic bootstrap or the Riemann-Hilbert problem
- The exact WKB periods and the exact quantization condition determine the exact spectrum of QM.
- explicitly worked out for cubic and quartic potentials

ODE/IM correspondence

The ODE/IM correspondence [Dorey-Tateo 9812211]

- a relation between spectral analysis approach of ordinary differential equation (ODE), and the "functional relations" approach to 2d quantum integrable model (IM).
- Stokes coefficients of the solutions satisfy functional relations
- Baxter's T-Q relation, T-system, Y-system
  $\Longrightarrow$ NLIE, Thermodynamic Bethe Ansatz (TBA) equations
- TBA equations solve the spectral determinants and the exact WKB periods of QM with the monic potential $V(x) = x^{2M}$.

KI-Mariño-Shu 2018

We present the TBA equations governing the exact WKB periods for general polynomial potential. These TBAs also provide a generalization of the ODE/IM correspondence.

We will discuss

- the relation between the discontinuity formula in Quantum Mechanics and the TBA equations
- generalization of the ODE/IM correspondence

for general polynomial potential.

We then apply the TBA equations to solve the spectral problem in QM.

Introduction

Exact WKB and resurgent quantum mechanics

Generalized ODE/IM correspondence

Example: cubic potential

Conclusions and outlook

Exact WKB and resurgent quantum mechanics

Exact WKB method

the Schrödinger equation

$$-\hbar^2\psi''(q) + (V(q) - E)\psi(q) = 0.$$

the WKB solution:

$$\psi(q) = \exp\left[\frac{i}{\hbar}\int^q Q(q')dq'\right].$$

$$Q^2(q) - i\hbar\frac{dQ(q)}{dq} = p^2(q), \quad p(q) = (E - V(q))^{1/2},$$

$$Q(q) = \sum_{k=0}^{\infty} Q_k(q)\hbar^k = Q_{\rm even} + Q_{\rm odd} = P(q) + \frac{i\hbar}{2}\frac{d}{dq}\log P(q).$$

$$P(q) = Q_{\rm even} = \sum_{n\geq 0} p_n(q)\hbar^{2n}, \quad Q_{\rm odd} = \text{total derivative}$$

$p_0(q) = p(q)$ and $p_n(q)$ are determined recursively.

WKB periods and Voros symbols

potential: polynomial in q

$$V(q) = q^{r+1} + u_1 q^r + \cdots + u_r q$$

the WKB curve: hyperelliptic Riemann surface of genus $g = \left[\frac{r+1}{2}\right]$

$$\Sigma_{\mathrm{WKB}} : y^2 = 2(E - V(q)).$$

WKB periods (*quantum periods*)

$$\Pi_\gamma(\hbar) = \oint_\gamma P(q)dq, \qquad \gamma \in H_1(\Sigma_{\mathrm{WKB}}).$$

$$\Pi_\gamma(\hbar) = \sum_{n \geq 0} \Pi_\gamma^{(n)} \hbar^{2n}, \qquad \Pi_\gamma^{(n)} = \oint_\gamma p_n(q)dq.$$

Voros symbol

$$\mathcal{V}_\gamma = \exp\left(\frac{i}{\hbar}\Pi_\gamma\right).$$

Borel resummation

The WKB expansion of the quantum periods:

$$\Pi_\gamma(\hbar) = \sum_{n\geq 0} \Pi_\gamma^{(n)} \hbar^{2n}, \qquad \Pi_\gamma^{(n)} = \oint_\gamma p_n(q) dq.$$

- asymptotic expansion: $\Pi_\gamma^{(n)} \sim (2n)!$.
- The Borel transformation

$$\widehat{\Pi}_\gamma(\xi) = \sum_{n\geq 0} \frac{1}{(2n)!} \Pi_\gamma^{(n)} \xi^{2n},$$

Laplace transformation

The Borel resummation (Laplace transformation)

$$s\left(\Pi_\gamma\right)(\hbar) = \frac{1}{\hbar}\int_0^\infty e^{-\xi/\hbar}\widehat{\Pi}_\gamma(\xi)d\xi, \qquad \hbar \in \mathbf{R}_{>0}.$$

Borel resummation along a direction specified by an angle φ:

$$s_\varphi\left(\Pi_\gamma\right)(\hbar) = \frac{1}{\hbar}\int_0^{e^{i\varphi}\infty} e^{-\xi/\hbar}\widehat{\Pi}_\gamma(\xi)d\xi.$$

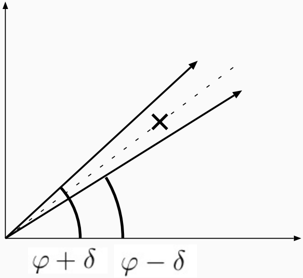
lateral Laplace tranformation

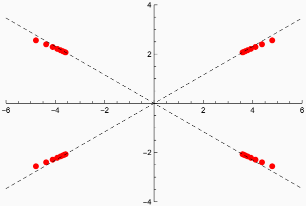
- singularity in the φ-direction in the ξ-plane
- the *lateral Borel resummations* along a direction φ:

$$s_{\varphi\pm}(\Pi_\gamma)\left(e^{i\varphi}\hbar\right) = \lim_{\delta\to 0+} s\left(\Pi_\gamma\right)\left(e^{i\varphi\pm i\delta}\hbar\right), \qquad \hbar \in \mathbf{R}_{>0},$$

- The discontinuity across the direction φ

$$\mathrm{disc}_\varphi\ (\Pi_\gamma) = s_{\varphi+}(\Pi_\gamma) - s_{\varphi-}(\Pi_\gamma).$$



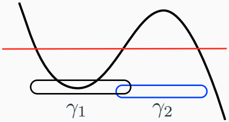


cubic potential

Delabaere-Pham formula

WKB periods

- cycle γ_1: a classically allowed interval $E > V(x)$
- cycle γ_2: a classically forbidden interval $E < V(x)$



- Π_{γ_1}: discontinuous along the real positive axis
- Π_{γ_2}: continuous along the real positive axis

$$\mathrm{disc}_{\varphi=0}\left(\Pi_{\gamma_1}\right)(\hbar) = -i\hbar\log\left(1+\exp\left(-\frac{i}{\hbar}\Pi_{\gamma_2}(\hbar)\right)\right).$$

[Delabaere-Pham 1999, Iwaki-Nakanishi]

Exact quantization condition

Bohr-Sommerfeld quantization condition

$$s(\Pi_p)(\hbar) = 2\pi\hbar\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

$\Longrightarrow E = E_n(\hbar)$: perturbative spectrum

Exact quantization condition [V, DP, Zinn-Justin,..., Grassi-Marino]

$$2\cos(\frac{1}{2\hbar}s(\Pi_p)(\hbar)) + e^{-\frac{1}{2\hbar}s(\Pi_{np})(\hbar)} = 0 \quad \text{(cubic potential)}$$

$\Longrightarrow E = E_n(\hbar)$: energy spectrum

$\frac{1}{\hbar_n} = x_n(E)$: Voros spectrum

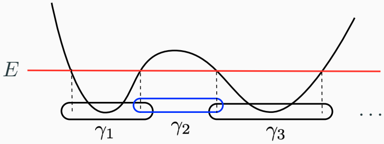
Generalized ODE/IM correspondence

TBA equations from DP formula

- $V_{r+1}(q)$: a polynomial potential of degree $r+1$
- all the turning points q_i, $i = 1, \cdots, r+1$, are real and different $(q_1 < q_2 < \cdots < q_{r+1}.)$
- γ_{2i-1}: $[q_{2i}, q_{2i}]$ classically allowed
  γ_{2i}: $[q_{2i}, q_{2i+1}]$ classically forbidden
- period integrals

$$m_{2i-1} = \Pi^{(0)}_{\gamma_{2i-1}} = 2\int_{q_{2i-1}}^{q_{2i}} p(q)dq,\ m_{2i} = i\,\Pi^{(0)}_{\gamma_{2i}} = 2i\int_{q_{2i}}^{q_{2i+1}} p(q)dq,$$

are real and positive.



TBA equations

- discontinuity of $\Pi_{\gamma_{2i-1}}$ is determined by the DP formula.

$$\mathrm{disc}\,\Pi_{\gamma_{2i-1}} = -i\hbar\log\left(1+\exp\left(-\frac{i}{\hbar}\Pi_{\gamma_{2i-2}}(\hbar)\right)\right) - i\hbar\log\left(1+\exp\left(-\frac{i}{\hbar}\Pi_{\gamma_{2i}}(\hbar)\right)\right),$$

- discontinuity of $\Pi_{\gamma_{2i}}$: $\hbar \to \pm i\hbar$ ($E - V(x) \to V(x) - E$)

Introduce the spectral parameter $\theta = -\log\hbar$ ($e^{\theta} = \frac{1}{\hbar}$)

$$-i\epsilon_{2i-1}\left(\theta + \frac{i\pi}{2} \pm i\delta\right) = \frac{1}{\hbar}s_{\pm}\left(\Pi_{\gamma_{2i-1}}\right)(\hbar), \quad -i\epsilon_{2i}\left(\theta\right) \quad = \frac{1}{\hbar}s\left(\Pi_{\gamma_{2i}}\right)(\hbar),$$

The DP-formula reads

$$\mathrm{disc}_{\frac{\pi}{2}}\epsilon_a(\theta) = L_{a-1}(\theta) + L_{a+1}(\theta), \qquad a = 1, \cdots, r,$$

$$L_a(\theta) = \log\left(1 + e^{-\epsilon_a(\theta)}\right),$$

$L_0 = L_{r+1} = 0$

Discontinuity and integral equations

Integral transformation

$$F(\zeta) = \int_0^\infty \frac{dx}{x}\frac{x+\zeta}{x-\zeta} f(x) = \int_{\mathbf{R}} d\theta' \coth\frac{\theta'-\theta}{2} f(e^{\theta'})$$

$\zeta = e^\theta$, $x = e^{\theta'}$

The discontinuity of $F(\zeta)$ along the positive real axis

$$F(\zeta+i\epsilon) - F(\zeta-i\epsilon) = \int_0^\infty \frac{dx}{x}\frac{4i\epsilon x}{(x-\zeta)^2+\epsilon^2} f(x) \to 2\pi i f(\zeta)$$

discontinuity at $\varphi = \pi/2$ and $-\pi/2 \Longrightarrow$ TBA equations

$$\epsilon_a(\theta) = m_a e^\theta - \int_{-\infty}^{\infty} \frac{L_{a-1}(\theta')}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi} - \int_{\infty}^{\infty} \frac{L_{a+1}(\theta')}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi},$$

asymptotics $\epsilon_a(\theta) \to m_a e^\theta$ $(\theta \to \infty)$

generalized ODE/IM correspondence

ODE

$$\left(-\partial_z^2 + z^{r+1} + \sum_{a=1}^{r} b_a z^{r-a}\right)\psi(z,b_a) = 0 \quad z \in \mathbf{C}$$

- invariant under the rotation (Symanzik rotation)

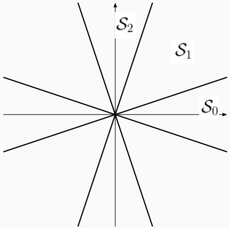
$$(z,b_a) \to (\omega z, \omega^{a+1} b_a), \qquad \omega = e^{\frac{2\pi i}{r+3}}.$$

- asymptotically decaying solution along the positive real axis

$$y(z,b_a) \sim \frac{1}{\sqrt{2i}} z^{n_r} \exp\Big(-\frac{2}{r+3} z^{\frac{r+3}{2}}\Big),$$

Stokes sector
$\mathcal{S}_k = \Big\{ z \in \mathbf{C} : \Big|\arg(z) - \frac{2k\pi}{r+3}\Big| < \frac{\pi}{r+3} \Big\}.$
decaying solutions in $\mathcal{S}_k$:

$$y_k(z,b_a) = \omega^{\frac{k}{2}} y(\omega^{-k} z, \omega^{-(a+1)k} b_a).$$



Y-functions and Wronskian

Wronskian

$$W_{k_1,k_2}(b_a) \equiv y_{k_1}(z,b_a)\partial_z y_{k_2}(z,b_a) - y_{k_2}(z,b_a)\partial_z y_{k_1}(z,b_a)$$

- independent of z, $W_{0,1}(=1)$.
- periodicity

$$W_{k_1+1,k_2+1}(b_a) = W^{[2]}_{k_1,k_2}(b_a).$$

$$f^{[j]}(z,b_a) := f(\omega^{-j/2}z, \omega^{-j(a+1)/2}b_a).$$

Y-function

$$\mathcal{Y}_{2j}(b_a) = \frac{W_{-j,j}(b_a)W_{-j-1,j+1}(b_a)}{W_{-j-1,-j}(b_a)W_{j,j+1}(b_a)},$$

$$\mathcal{Y}_{2j+1}(b_a) = \left[\frac{W_{-j-1,j}(b_a)W_{-j-2,j+1}(b_a)}{W_{-j-2,-j-1}(b_a)W_{j,j+1}(b_a)}\right]^{[+1]},$$

Y-system

Y-system ($\leftarrow$ Plücker identities for 2×2 determinants)

$$\mathcal{Y}_s^{[+1]}(b_a)\mathcal{Y}_s^{[-1]}(b_a) = \Big(1 + \mathcal{Y}_{s-1}(b_a)\Big)\Big(1 + \mathcal{Y}_{s+1}(b_a)\Big).$$

boundary conditions: $\mathcal{Y}_0 = \mathcal{Y}_{r+1} = 0$

A_r-type Y-system

- $b_1 = \cdots = b_{r-1} = 0$, $b_r = E$ (monomial pot.) [Dorey-Tateo]
- multiple spectral parameters $b_1, \cdots, b_r$
- It is difficult to write down TBA equations
  6-term TBA for cubic potential Masoero 1005.1046
  NLIE for $x^6 + \alpha x^2$-potential Suzuki 0003066

generalized ODE/IM

Introduce scaled variables

$$q = \zeta^{\frac{2}{r+3}} z, \quad u_a = -\zeta^{\frac{2(a+1)}{r+3}} b_a, \qquad a = 1, 2, \cdots, r.$$

$$\left(-\zeta^2 \partial_q^2 + q^{r+1} - \sum_{a=1}^{r} u_a q^{r-a} \right) \hat{\psi}(x, u_a, \zeta) = 0.$$

- regard ζ as a new spectral parameter ($\zeta = \hbar$ in QM)
- Symanzik rotation $\leftrightarrow$ rotation of ζ

$$\hat{y}(q, u_a, \zeta) = y(z, b_a) = y(\zeta^{-\frac{2}{r+3}} q, -\zeta^{-\frac{2(a+1)}{r+3}} u_a).$$

$$(\omega^{-k} z, \omega^{-(a+1)k} b_a) = \left((e^{i\pi k}\zeta)^{-\frac{2}{r+3}} q, -(e^{i\pi k}\zeta)^{-\frac{2(a+1)}{r+3}} u_a \right).$$

- subdominant solution in the sector $\mathcal{S}_k$

$$\hat{y}_k(q, u_a, \zeta) = \omega^{\frac{k}{2}} \hat{y}(q, u_a, e^{i\pi k}\zeta).$$

Y-system

Wronskian

$$\hat{W}_{k_1,k_2}(\zeta, u_a) = \zeta^{\frac{2}{r+3}} \Big(\hat{y}_{k_1} \partial_q \hat{y}_{k_2} - \hat{y}_{k_2} \partial_q \hat{y}_{k_1} \Big)(q, u_a, \zeta) = W_{k_1,k_2}(b_a).$$

Y functions

$$Y_{2j}(\zeta, u_a) = \frac{\hat{W}_{-j,j}\hat{W}_{-j-1,j+1}}{\hat{W}_{-j-1,-j}\hat{W}_{j,j+1}}(\zeta, u_a),$$

$$Y_{2j+1}(e^{-\frac{\pi i}{2}}\zeta, u_a) = \frac{\hat{W}_{-j-1,j}\hat{W}_{-j-2,j+1}}{\hat{W}_{-j-2,-j-1}\hat{W}_{j,j+1}}(\zeta, u_a).$$

Y-system:

$$Y_s(\zeta e^{\frac{\pi i}{2}}, u_a) Y_s(\zeta e^{-\frac{\pi i}{2}}, u_a) = \Big(1 + Y_{s+1}(\zeta, u_a)\Big)\Big(1 + Y_{s-1}(\zeta, u_a)\Big),$$

- u_a fixed, single spectral parameter ζ
- massless limit of Y-system in AdS_3 minimal surface with light-like boundary [Alday-Maldacena-Sever-Vieira, Hatsuda-KI-Sakai-Satoh]

TBA equation

asymptotics:

$$\hat{y}_k(q, u_a, \zeta) \sim (-1)^{\frac{k}{2}} c(\zeta) \exp\left(-\frac{i\delta_k}{\zeta} \int_{q^{(k)}}^{q} P(q')dq'\right).$$

asymptotic behavior of the Y-function $\zeta \to \infty$

$$\log Y_{2k+1}(\zeta, u_a) \sim -\frac{1}{\zeta} \oint_{\gamma_{r-2k}} p(q)dq = -\frac{m_{r-2k}}{\zeta},$$

$$\log Y_{2k}(\zeta, u_a) \sim -\frac{i}{\zeta} \oint_{\gamma_{r+1-2k}} p(q)dq = -\frac{m_{r+1-2k},}{\zeta},$$

$$\zeta = e^{-\theta}, \qquad Y_a(\zeta) = e^{-\epsilon_a(\theta)}$$

TBA equations ($\Longleftarrow$ Y-system+asymptotics [Al. Zamolodchikov])

$$\epsilon_a(\theta) = m_a e^{\theta} - \int_{\mathbf{R}} \frac{L_{a-1}(\theta')}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi} - \int_{\mathbf{R}} \frac{L_{a+1}(\theta')}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi},$$

TBA in the UV limit

- constant solution $\epsilon_a(\theta) \to \epsilon_a^\star$ $(\theta \to -\infty)$

$$Y_a^\star = \frac{\sin\left(\frac{\pi a}{r+3}\right)\sin\left(\frac{\pi(a+2)}{r+3}\right)}{\sin^2\left(\frac{\pi}{r+3}\right)}.$$

- effective central charge [Kirillov,Kuniba-Nakanishi-Suzuki]

$$c_{\text{eff}} = \frac{6}{\pi^2}\sum_{a=1}^{r} m_a \int_{\mathbf{R}} e^\theta L_a(\theta) d\theta = \frac{r(r+1)}{r+3}.$$

- generalized paramermion $SU(r+1)_2/U(1)^r$ [HISS]
- TBA system for Homogeneous sine-Gordon model $SU(r+1)_2/U(1)^r$ [Fernandez-Pousa-Gallas-Hollowood-Miramontes]
- PNP relation (quantum Matone relation) $\sum_{i=1}^{g} \nu_i^{(0)}\nu_{Di}^{(1)} - \nu_i^{(0)}\nu_{Dj}^{(1)} = -\frac{c_{\text{eff}}}{12}$

Wall-crossing and TBA

For a general polynomial potential, turning points are complex.
$(V(q) = q^3 = E)$
complex periods (complex mass)

$$m_a = |m_a| e^{i\phi_a}, \qquad a = 1, \cdots, r.$$

$$\widetilde{\epsilon}_a(\theta) = \epsilon_a(\theta - i\phi_a), \qquad \widetilde{L}_a(\theta) = L_a(\theta - i\phi_a).$$

TBA system:

$$\widetilde{\epsilon}_a = |m_a| e^{\theta} - K_{a,a-1} \star \widetilde{L}_{a-1} - K_{a,a+1} \star \widetilde{L}_{a+1}$$

$$K_{r,s} = \frac{1}{2\pi} \frac{1}{\cosh\left(\theta + i(\phi_s - \phi_r)\right)}$$

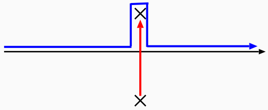
$$(K \star f)(\theta) = \int_{\mathbf{R}} K(\theta - \theta') f(\theta') d\theta'$$

This TBA formula is valid for ϕ_a satisfying $|\phi_a - \phi_{a\pm 1}| < \frac{\pi}{2}$

When

$$\phi_2 - \phi_1 > \frac{\pi}{2},$$

the kernel picks the pole at $\theta' = \theta - i\phi_2 + i\phi_1 - \frac{\pi i}{2}$.



TBA equations are modified to

$$\widetilde{\epsilon}_1(\theta) = |m_1| e^{\theta} - K_{1,2} \star \widetilde{L}_2 - L_2\left(\theta - i\phi_1 - \frac{i\pi}{2} + i\delta\right),$$

$$\widetilde{\epsilon}_2(\theta) = |m_2| e^{\theta} - K_{2,1} \star \widetilde{L}_1 - L_1\left(\hat{\theta} - i\phi_2 + \frac{i\pi}{2} - i\delta\right).$$

Wall crossing phnenomena of the TBA equations

[Gaiotto-Moore-Neitzke, AMSV, Toledo unpublished]

3-term TBA

We can transform this 2-term TBA into 3-term TBA equations.

Y-functions $\tilde{\epsilon}_a^{\mathrm{n}}(\theta - i\phi_a) = -\log Y_a^{(\mathrm{n})}(\theta)$ $(a = 1, 2, 12)$

cycles γ_1, γ_2 and $\gamma_{12} = \gamma_1 + \gamma_2$

$$Y_1^{\mathrm{n}}(\theta) = \frac{Y_1(\theta)}{1 + Y_2\left(\theta - \frac{i\pi}{2}\right)}, \qquad Y_2^{\mathrm{n}}(\theta) = \frac{Y_2(\theta)}{1 + Y_1\left(\theta + \frac{i\pi}{2}\right)},$$

$$Y_{12}^{\mathrm{n}}(\theta) = \frac{Y_1(\theta) Y_2\left(\theta - \frac{i\pi}{2}\right)}{1 + Y_1(\theta) + Y_2\left(\theta - \frac{i\pi}{2}\right)}.$$

3-term TBA equations: [GMN, HISS]

$$\widetilde{\epsilon}_1(\theta) = |m_1| e^{\theta} - K_{1,2} \star \widetilde{L}_2 - K^{+}_{1,12} \star \widetilde{L}_{12},$$

$$\widetilde{\epsilon}_2(\theta) = |m_2| e^{\theta} - K_{2,1} \star \widetilde{L}_1 - K_{2,12} \star \widetilde{L}_{12},$$

$$\widetilde{\epsilon}_{12}(\theta) = |m_{12}| e^{\theta} - K^{-}_{12,1} \star \widetilde{L}_1 - K_{12,2} \star \widetilde{L}_2.$$

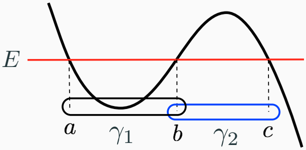
$m_{12} = m_1 - i m_2 = |m_{12}| e^{i\phi_{12}}$, $f^{\pm}(\theta) := f(\theta \pm i\frac{\pi}{2})$

In general r-term TBA $\to \cdots \to r(r+1)/2$-term TBA

Example: cubic potential

cubic potential

$$V(x) = \frac{\kappa x^2}{2} - x^3$$



m_1, m_2 are represented by the elliptic integral.

TBA equations

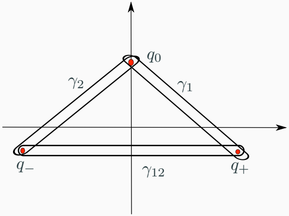
$$\epsilon_1(\theta) = m_1 e^{\theta} - \int_{\mathbf{R}} \frac{\log\left(1 + e^{-\epsilon_2(\theta')}\right)}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi},$$
$$\epsilon_2(\theta) = m_2 e^{\theta} - \int_{\mathbf{R}} \frac{\log\left(1 + e^{-\epsilon_1(\theta')}\right)}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi}.$$

- $c_{\text{eff}} = \frac{6}{5}$: $SU(3)_2/U(1)^2$

PT-symmetric Hamiltonian

PT-symmetric Hamiltonian [Bender-Boettcher 1997]

$$\hat{H} = \frac{1}{2}\hat{p}^2 + i\hat{q}^3 - i\lambda\hat{q}.$$



- PT-symmetry: parity $q \to -q+$ complex conjugation
- three complex turning points
- $\lambda \leq 0$ real, positive discrete spectrum
- $\lambda > 0$ and large, PT-symmetry is broken due to non-perturbative effect

$\lambda = 1$ case

$$\Pi_{\gamma_1} = \frac{1}{2}\Pi_{\mathrm{p}} - \frac{i}{2}\Pi_{\mathrm{np}},\ \Pi_{\gamma_2} = \Pi_{\gamma_1}^*,\ \Pi_{\gamma_{12}} = \Pi_{\gamma_1} + \Pi_{\gamma_2}$$

3-term TBA

$$\begin{aligned}\widetilde{\epsilon}_1(\theta) &= |m_1|e^{\theta} - K_{1,2}\star\widetilde{L}_2 - K_{1,12}^{+}\star\widetilde{L}_{12},\\ \widetilde{\epsilon}_2(\theta) &= |m_2|e^{\theta} - K_{2,1}\star\widetilde{L}_1 - K_{2,12}\star\widetilde{L}_{12},\\ \widetilde{\epsilon}_{12}(\theta) &= |m_{12}|e^{\theta} - K_{12,1}^{-}\star\widetilde{L}_1 - K_{12,2}\star\widetilde{L}_2.\end{aligned}$$

with $\phi_1 = -\alpha$, $\phi_2 = \frac{\pi}{2} + \alpha$, $\phi_{12} = 0$.

quantum periods:

$$\frac{1}{\hbar}s(\Pi_{\gamma_1})(\hbar) = -i\widetilde{\epsilon}_1(\theta + i\frac{\pi}{2} - i\alpha), \quad \frac{1}{\hbar}s(\Pi_{\gamma_1})(\hbar) = -i\widetilde{\epsilon}_2(\theta - i\frac{\pi}{2} + i\alpha)$$

BS vs exact quantization condition

Bohr-Sommerfeld quantization

$$s(\Pi_p)(\hbar) = 2\pi\hbar(n + \frac{1}{2}), \quad n = 0, 1, 2, \ldots$$

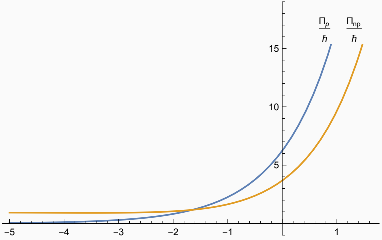
exact quantization condition

$$2\cos\left(\frac{1}{2\hbar}s(\Pi_{\mathrm{p}})(\hbar)\right) + e^{-\frac{1}{2\hbar}s(\Pi_{\mathrm{np}})(\hbar)} = 0.$$

[Delabaere-Dillinger-Pham, Alvarez-Casares,Delabaere-Trinh]

Numerical calculation of $s(\Pi_{\rm p})$ and $s(\Pi_{\rm np})$ and Voros spectrum

$E = \lambda = 1$



horizontal axis $\theta = -\log(\hbar)$

n	x_n	$E_n(x_n^{-1})$
0	$0.560\,405\,699\,850$	$0.999\,999\,999\,855$
1	$1.496\,238\,960\,118$	$0.999\,999\,999\,980$
2	$2.509\,053\,231\,083$	$0.999\,999\,999\,993$
3	$3.507\,747\,393\,173$	$0.999\,999\,999\,997$

Voros sepctrum: $x_n = \frac{1}{\hbar_n(E)}$ computed by using EQC

$E_n(x_n^{-1})$: computed by complex dilatation technique

ODE/IM and PT-symmetric Hamiltonian

[Dorey-Dunning-Tateo, 0103051]

$\boxed{\lambda = 0}$

$H = \frac{p^2}{2} + iq^3$

turning points: $q_0 = i$, $q_+ = e^{-\frac{\pi i}{6}}$, $q_- = -e^{\frac{\pi i}{6}}$, $\alpha = \frac{\pi}{3}$

- Z_3-symmetry: $\widetilde{\epsilon}_a(\theta) = \epsilon(\theta)$
- periodicity $\epsilon(\theta + \frac{5\pi}{3}) = \epsilon(\theta)$

3-term$\Longrightarrow$ single term TBA [Dorey-Tateo 9812211]

$$\epsilon(\theta) = \frac{\sqrt{6\pi}\Gamma(1/3)}{3\Gamma(11/6)} e^{\theta} + \int_{\mathbf{R}} \Phi(\theta - \theta')L(\theta')d\theta', \quad \Phi(\theta) = \frac{\sqrt{3}}{\pi} \frac{\sinh 2\theta}{\sinh 3\theta}$$

TBA for Yang-Lee edge singularity ($c_{\text{eff}} = \frac{6}{5}$) [Al Zamolodchikov 1990]

Numerical check

PT cubic oscillator with $\hbar = \sqrt{2}$

n	E_n^{num}	E_n^{TBA}	E_n^{p}	E_n^{WKB}
0	1.156 267 071 988	1.156 267 071 988	1.134 513 239 424	1.094 269 500 533
1	4.109 228 752 810	4.109 228 752 806	4.109 367 351 095	4.089 496 119 273
2	7.562 273 854 979	7.562 273 854 971	7.562 273 170 784	7.548 980 437 586

- E_n^{num} : complex dilatation+Rayleigh-Ritz [Yaris-Bendler-Lovett-Bender-Fedders 1978]
- E_n^{TBA} : TBA+exact quantization condition
- E_n^{p}: TBA+Bohr-Sommerfeld quantization condition
- E_n^{WKB} : Bohr-Sommerfeld approximation

Conclusions and outlook

Conclusions

We have provided an efficient approach to solve the spectral problem for arbitrary polynomial potentials in one-dimensional Quantum Mechanics. The solution takes the form of a TBA system for the (resummed) quantum periods which generalizes the ODE/IM correspondence

Outlook

- including angular momentum term $\frac{\ell(\ell+1)}{x^2}$ in the potential [Dorey-Tateo, Bahzanov-Lukyanov-Zamolodchikov] (Ito-Shu to appear)
- $\hat{\mathfrak{g}}^\vee$ affine Toda and $\hat{\mathfrak{g}}$ BAE (Langlands dual) [Feigin-Frenkel, Dorey-Dunning-Masoero-Suzuki-Tateo, Sun, Masoero-Raimondo-Valeri, Ito-Locke](Ito-Kondo-Kuroda-Shu, work in progress)
- massive ODE/IM [Lukyanov-Zamolodchikov,Dorey et al. , Ito-Locke, Adamopoulou-Dunning, Negro,Ito-Shu]
- application to Argyres-Douglas theory [Ito-Shu, Grassi-Marino, Itoyama-Oota-Yano, Ito-Koizumi-Okubo, ...]
  quantum periods and free energy, 2d/4d correspondence
  Many new examples of ODEs