New quantum toroidal algebras from supersymmetric gauge theories

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New trends in Integrable Systems

Osaka City University

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JEB, Saebyeok Jeong [arXiv:1906.01625]

What are quantum toroidal algebras?

In a nutshell

Toroidal algebras are central extensions of 2-loop algebras (= double affine algebras).

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Main applications:

- Construction of integrable systems using the Hopf algebra structure
  [Feigin, Jimbo, Miwa, Mukhin 2015]
- Non-perturbative symmetries of supersymmetric (SUSY) gauge theories
  $\rightsquigarrow$ From string theory realizations: (p,q)-branes web or topological strings
  [Awata, Feigin, Shiraishi 2011]
  ✎ Main motivation for introducing our deformation of quantum toroidal $\mathfrak{gl}(p)$.
- Correspondence with W-algebras (AGT correspondence)
  [Awata, Feigin, Hoshino, Kanai, Shiraishi, Yanagida 2011]

Outline

Toroidal algebras

Consider a simple Lie algebra $\mathfrak{g}$ with Chevalley basis x^+_ω, x^-_ω, h_ω ($\omega = 1 \cdots \text{rank}$).

⚠ Notations: $x^+_\omega \equiv e_\omega \equiv e_{\alpha_\omega}$, $x^-_\omega \equiv f_\omega \equiv e_{-\alpha_\omega}$.

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   Example $\widehat{\mathfrak{sl}(2)}$: $x_k^\pm = t^k \otimes x^\pm$, $h_k = t^k \otimes h$,
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   $\rightsquigarrow$ Introduce the currents $h(z) = \sum_{k\in\mathbb{Z}} z^{-k} h_k$, $x^\pm(z) = \sum_{k\in\mathbb{Z}} z^{-k} x_k^\pm$.
   $[h(z), x^\pm(w)] = \pm 2\delta(z/w) x^\pm(z)$, with $\delta(z) = \sum_{k\in\mathbb{Z}} z^k$.

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$\rightsquigarrow$ Toroidal algebras were formulated by combining these two methods. **[Moody, Rao, Yokonuma 1990]**

Quantization

✎ Quantization is used to define a non-trivial coalgebraic structure.

Reminder: coalgebras have a coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, a counit $\epsilon : \mathcal{A} \to \mathbb{C}$

Hopf algebra: antipode $\mathcal{S} : \mathcal{A} \to \mathcal{A}$ such that $\nabla(\mathcal{S} \otimes 1)\Delta = \nabla(1 \otimes \mathcal{S})\Delta = \epsilon$.

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✎ Replace the Cartan sector with operators $\psi^{\pm}_{\omega}$ in the universal envelopping algebra. A parameter $q \in \mathbb{C}^{\times}$ is also introduced.

Example $U_q(\mathfrak{sl}(2))$: generators $x^{\pm}$, $\psi^{\pm} = q^{\pm h}$,

$$\psi^+ x^{\pm} = q^{\pm 2} x^{\pm} \psi^+, \quad \psi^- x^{\pm} = q^{\mp 2} x^{\pm} \psi^-, \quad [x^+, x^-] = \frac{\psi^+ - \psi^-}{q - q^{-1}}$$

$$\Delta(x^+) = x^+ \otimes 1 + \psi^- \otimes x^+, \quad \Delta(x^-) = 1 \otimes x^- + x^- \otimes \psi^+, \quad \Delta(\psi^{\pm}) = \psi^{\pm} \otimes \psi^{\pm}$$

$$\mathcal{S}(x^+) = -(\psi^-)^{-1} x^+, \quad \mathcal{S}(x^-) = -x^- (\psi^+)^{-1}, \quad \mathcal{S}(\psi^{\pm}) = (\psi^{\pm})^{-1},$$

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$\rightsquigarrow$ Obtain quantum toroidal algebras **[Ginzburg, Kapranov, Vasserot 1995]**

Quantum toroidal $\mathfrak{gl}(p)$: definition

When $\mathfrak{g} = \mathfrak{gl}(p)$, an extra parameter $\kappa \in \mathbb{C}^\times$ can be introduced. We use:

$$q_1 = q^{-1}\kappa, \quad q_2 = q^{-1}\kappa^{-1}, \quad q_3 = q^2 \quad \Rightarrow \quad q_1 q_2 q_3 = 1.$$

The algebra is formulated in terms of a central element c and $4p$ Drinfeld currents

$$x_\omega^\pm(z) = \sum_{k\in\mathbb{Z}} z^{-k} x_{\omega,k}^\pm, \quad \psi_\omega^\pm(z) = \sum_{k\geq 0} z^{\mp k} \psi_{\omega,\pm k}^\pm.$$

It has a second central element $\bar{c}$ obtained as

$$q_3^{\mp\frac{1}{2}\bar{c}} = \prod_{\omega=0}^{p-1} \psi_{\omega,0}^\pm.$$

Quantum toroidal $\mathfrak{gl}(p)$: definition

The algebraic relations read

$$\psi^+_\omega(z)x^\pm_{\omega'}(w) = g_{\omega\omega'}(q_3^{\pm c/4}z/w)^{\pm 1}x^\pm_{\omega'}(w)\psi^\pm_\omega(z),$$
$$\psi^-_\omega(z)x^\pm_{\omega'}(w) = g_{\omega\omega'}(q_3^{\mp c/4}z/w)^{\pm 1}x^\pm_{\omega'}(w)\psi^-_\omega(z),$$
$$[\psi^\pm_\omega(z),\psi^\pm_{\omega'}(w)] = 0, \quad \psi^+_{\omega,0}\psi^-_{\omega,0} = \psi^-_{\omega,0}\psi^+_{\omega,0} = 1$$
$$\psi^+_\omega(z)\psi^-_{\omega'}(w) = \frac{g_{\omega\omega'}(q_3^{c/2}z/w)}{g_{\omega\omega'}(q_3^{-c/2}z/w)}\psi^-_{\omega'}(w)\psi^+_\omega(z),$$
$$x^\pm_\omega(z)x^\pm_{\omega'}(w) = g_{\omega\omega'}(z/w)^{\pm 1}x^\pm_{\omega'}(w)x^\pm_\omega(z),$$
$$[x^+_\omega(z),x^-_{\omega'}(w)] = \frac{\delta_{\omega,\omega'}}{q_3^{1/2}-q_3^{-1/2}}\left[\delta(q_3^{-c/2}z/w)\psi^+_\omega(q_3^{-c/4}z) - \delta(q_3^{c/2}z/w)\psi^-_\omega(q_3^{c/4}z)\right],$$

together with the Serre relations

$$\sum_{\sigma\in S_2}\left[x^\pm_\omega(z_{\sigma(1)})x^\pm_\omega(z_{\sigma(2)})x^\pm_{\omega\pm1}(w) - (q_3^{1/2}+q_3^{-1/2})x^\pm_\omega(z_{\sigma(1)})x^\pm_{\omega\pm1}(w)x^\pm_\omega(z_{\sigma(2)}) + x^\pm_{\omega\pm1}(w)x^\pm_\omega(z_{\sigma(1)})x^\pm_\omega(z_{\sigma(2)})\right] = 0.$$

Quantum toroidal $\mathfrak{gl}(p)$: definition

The structure functions $g_{\omega\omega'}(z)$ are defined as ($\delta_{\omega,\omega'}$ is the Kronecker delta mod p)

$$g_{\omega\omega'}(z) = \left(q_3^{-1}\frac{1-q_3z}{1-q_3^{-1}z}\right)^{\delta_{\omega,\omega'}} \left(q_3^{1/2}\frac{1-q_1z}{1-q_2^{-1}z}\right)^{\delta_{\omega,\omega'-1}} \left(q_3^{1/2}\frac{1-q_2z}{1-q_1^{-1}z}\right)^{\delta_{\omega,\omega'+1}}$$

Example: For $p=6$,

$$\begin{pmatrix}
q_3^{-1}\frac{1-q_3z}{1-q_3^{-1}z} & q_3^{1/2}\frac{1-q_1z}{1-q_2^{-1}z} & 0 & 0 & 0 & q_3^{1/2}\frac{1-q_2z}{1-q_1^{-1}z} \\
q_3^{1/2}\frac{1-q_2z}{1-q_1^{-1}z} & q_3^{-1}\frac{1-q_3z}{1-q_3^{-1}z} & q_3^{1/2}\frac{1-q_1z}{1-q_2^{-1}z} & 0 & 0 & 0 \\
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0 & 0 & 0 & q_3^{1/2}\frac{1-q_2z}{1-q_1^{-1}z} & q_3^{-1}\frac{1-q_3z}{1-q_3^{-1}z} & q_3^{1/2}\frac{1-q_1z}{1-q_2^{-1}z} \\
q_3^{1/2}\frac{1-q_1z}{1-q_2^{-1}z} & 0 & 0 & 0 & q_3^{1/2}\frac{1-q_2z}{1-q_1^{-1}z} & q_3^{-1}\frac{1-q_3z}{1-q_3^{-1}z}
\end{pmatrix}$$

Quantum toroidal $\mathfrak{gl}(p)$: coalgebraic structure

The algebra has the structure of a Hopf algebra with the Drinfeld coproduct

$$\Delta(x_\omega^+(z)) = x_\omega^+(z)\otimes 1 + \psi_\omega^-(q_3^{c_{(1)}/4}z)\otimes x_\omega^+(q_3^{c_{(1)}/2}z),$$
$$\Delta(x_\omega^-(z)) = x_\omega^-(q_3^{c_{(2)}/2}z)\otimes \psi_\omega^+(q_3^{c_{(2)}/4}z) + 1\otimes x_\omega^-(z),$$
$$\Delta(\psi_\omega^\pm(z)) = \psi_\omega^\pm(q_3^{\pm c_{(2)}/4}z)\otimes \psi_\omega^\pm(q_3^{\mp c_{(1)}/4}z),$$

the counit $\epsilon(x_\omega^\pm(z)) = 0$, $\epsilon(\psi_\omega^\pm(z)) = 1$, and the antipode

$$\mathcal{S}(x_\omega^+(z)) = -\psi_\omega^-(q_3^{-c/4}z)^{-1}x_\omega^+(q_3^{-c/2}z), \quad \mathcal{S}(x_\omega^-(z)) = -x_\omega^-(q_3^{-c/2}z)\psi_\omega^+(q_3^{-c/4}z)^{-1},$$
$$\mathcal{S}(\psi_\omega^\pm(z)) = \psi_\omega^\pm(z)^{-1}.$$

Remarks:

★ We denoted $c_{(1)} = c\otimes 1$, $c_{(2)} = 1\otimes c$, and $\Delta(c) = c_{(1)} + c_{(2)}$, $\epsilon(c) = 0$, $\mathcal{S}(c) = -c$.

★ We recover the Ding-Iohara-Miki algebra when $p = 1$.

[Ding, Iohara 1997 - Miki 2007]

Outline

Motivation

• In applications to SUSY gauge theories, two types of representations are used.
In both representations, the following functions play a central role:

$$S_{\omega\omega'}(z)=\frac{(1-q_1z)^{\delta_{\omega,\omega'-1}}(1-q_2z)^{\delta_{\omega,\omega'+1}}}{(1-z)^{\delta_{\omega,\omega'}}(1-q_1q_2z)^{\delta_{\omega,\omega'}}},\quad g_{\omega\omega'}(z)=q_3^{-\frac{1}{2}c_{\omega\omega'}}\frac{S_{\omega\omega'}(z)}{S_{\omega\omega'}(q_3z)}$$

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- Some SUSY gauge theories observables can be constructed purely algebraically.

For quantum toroidal $\mathfrak{gl}(p)$, theories are defined on the spacetime $S^1 \times (\mathbb{C}\times\mathbb{C})/\mathbb{Z}_p$.
The $\mathbb{Z}_p$-action $(\theta, z_1, z_2) \in S^1 \times \mathbb{C} \times \mathbb{C} \to (\theta, e^{2i\pi/p} z_1, e^{-2i\pi/p} z_2)$ defines an orbifold.
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• The $\mathbb{Z}_p$-action can be generalized with two integers $(\nu_1,\nu_2)\in\mathbb{Z}_p\times\mathbb{Z}_p$,

$$(\theta,z_1,z_2)\in S^1\times\mathbb{C}\times\mathbb{C}\to(\theta,e^{2i\pi\nu_1/p}z_1,e^{2i\pi\nu_2/p}z_2).$$

This deformation of the orbifold leads to (ν_1,ν_2)-dependent functions

$$S_{\omega\omega'}(z)=\frac{(1-q_1z)^{\delta_{\omega,\omega'-\nu_1}}(1-q_2z)^{\delta_{\omega,\omega'-\nu_2}}}{(1-z)^{\delta_{\omega,\omega'}}(1-q_1q_2z)^{\delta_{\omega,\omega'-\nu_1-\nu_2}}}.$$

Definition of the (ν_1, ν_2)-deformed algebra

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The structure functions for the deformed algebra are defined as

$$g_{\omega\omega'}(z) = \frac{S_{\omega\omega'}(z)}{S_{\omega'\omega}(z^{-1})}, \quad (\omega, \omega' \in \mathbb{Z}_p).$$

The algebraic relations between currents needs to be deformed into

$$\begin{aligned}
&\psi^+_\omega(z) x^\pm_{\omega'}(w) = g_{\omega\omega'}(z/w)^{\pm 1} x^\pm_{\omega'}(w) \psi^+_\omega(z),\\
&\psi^-_\omega(z) x^-_{\omega'}(w) = g_{\omega\omega'}(z/w)^{-1} x^-_{\omega'}(w) \psi^-_\omega(z),\\
&\psi^-_\omega(z) x^+_{\omega'}(w) = g_{\omega-\nu_3 c\ \omega'}(q_3^{-c} z/w) x^+_{\omega'}(w) \psi^-_\omega(z),\\
&\psi^+_\omega(z) \psi^-_{\omega'}(w) = \frac{g_{\omega\omega'-\nu_3 c}(q_3^c z/w)}{g_{\omega\omega'}(z/w)} \psi^-_{\omega'}(w) \psi^+_\omega(z), \quad [\psi^\pm_\omega(z), \psi^\pm_{\omega'}(w)] = 0,\\
&x^\pm_\omega(z) x^\pm_{\omega'}(w) = g_{\omega\omega'}(z/w)^{\pm 1} x^\pm_{\omega'}(w) x^\pm_\omega(z),\\
&[x^+_\omega(z), x^-_{\omega'}(w)] = \Omega \left[\delta_{\omega,\omega'} \delta(z/w) \psi^+_\omega(z) - \delta_{\omega,\omega'-\nu_3 c} \delta(q_3^c z/w) \psi^-_{\omega+\nu_3 c}(q_3^c z) \right].
\end{aligned}$$

where we have introduced $\nu_3 = -\nu_1 - \nu_2$.

Important remarks

- The Cartan currents now contain some *zero modes* $a^{\pm}_{\omega,0}$,

$$\psi^{\pm}_{\omega}(z) = z^{\mp a^{\pm}_{\omega,0}} \sum_{k\geq 0} z^{\mp k} \psi^{\pm}_{\omega,\pm k}.$$

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- The coincidence between shifts in the arguments $q_3^{\pm c} z$ and indices $\omega \pm \nu_3 c$ follows from the definition of the $\mathbb{Z}_p$-action in the gauge theory.
- Well defined only if $\rho(c) \in \mathbb{Z}$.
- The comparison with quantum toroidal $\mathfrak{gl}(p)$ leads to the following conjecture: this algebra is equivalent to the quantum toroidal algebra built upon a Kac-Moody algebra with the non-symmetrizable Cartan matrix

$$C_{\omega\omega'} = \delta_{\omega\omega'} + \delta_{\omega\omega'+\nu_1+\nu_2} - \delta_{\omega\omega'+\nu_1} - \delta_{\omega\omega'+\nu_2}.$$

Typically,

$$C_{\omega\omega'} = \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

Coalgebraic structure

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$$\Delta(\psi_\omega^+(z)) = \psi_\omega^+(z) \otimes \psi^+_{\omega-\nu_3 c_{(1)}}(q_3^{-c_{(1)}} z),$$
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the counit $\epsilon(x_\omega^\pm(z)) = 0$, $\epsilon(\psi_\omega^\pm(z)) = 1$, and the antipode

$$\mathcal{S}(x_\omega^+(z)) = -\psi^-_{\omega+\nu_3 c}(q_3^c z)^{-1} x_\omega^+(z), \quad \mathcal{S}(x_\omega^-(z)) = -x^-_{\omega+\nu_3 c}(q_3^c z)\psi^+_{\omega+\nu_3 c}(q_3^c z)^{-1},$$
$$\mathcal{S}(\psi_\omega^+(z)) = \psi^+_{\omega+\nu_3 c}(q_3^c z)^{-1}, \quad \mathcal{S}(\psi_\omega^-(z)) = \psi^-_{\omega+2\nu_3 c}(q_3^{2c} z)^{-1},$$

The coproduct has been twisted to make manifest the coincidence between shifts in arguments and indices.

Outline

Vertical representations

- Deform the highest weight modules of quantum toroidal $\mathfrak{gl}(p)$

(analogous to finite dimensional representations of quantum affine algebras)

[Feigin, Jimbo, Miwa, Mukhin 2012]

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- The action of the Cartan reads

$$\psi_\omega^\pm(z^{-1})\,|\emptyset\rangle\rangle = \left[\frac{p_{\omega-\nu_3}(q_3^{-1/2}z)}{p_\omega(q_3^{1/2}z)}\right]_\mp |\emptyset\rangle\rangle, \quad p_\omega(z) = \prod_{\alpha=1}^n (1 - zq_3^{-1/2}v_\alpha)^{\delta_{c_\alpha,\omega}},$$

where $p_\omega(z)$ is the Drinfeld polynomial and $[f(z)]_\pm$ denotes an expansion of $f(z)$ in powers of $z^{\mp 1}$.

Vertical representations

- Vertical modules have a basis of states $|\boldsymbol{\lambda}\rangle\rangle$ labelled by *n*-tuples Young diagrams $\boldsymbol{\lambda}$.

To each box $\square = (\alpha, i, j) \in \boldsymbol{\lambda}$ of coordinates $(i, j) \in \lambda^{(\alpha)}$, we associate:

- a *position* $\chi_\square = v_\alpha q_1^{i-1} q_2^{j-1} \in \mathbb{C}^\times$,
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- The action of the Drinfeld currents on this basis read

$$\rho_V(x_\omega^+(z))\,|\boldsymbol{\lambda}\rangle\rangle = \sum_{\square \in A_\omega(\boldsymbol{\lambda})} \delta(z/\chi_\square)\mathcal{Y}_\omega^{[\boldsymbol{\lambda}+\square]}(\chi_\square)\,|\boldsymbol{\lambda}+\square\rangle\rangle,$$

$$\rho_V(x_\omega^-(z))\,|\boldsymbol{\lambda}\rangle\rangle = \sum_{\square \in R_\omega(\boldsymbol{\lambda})} \delta(z/\chi_\square)\mathcal{Y}_{\omega+\nu_1+\nu_2}^{*[\boldsymbol{\lambda}-\square]}(q_3^{-1}\chi_\square)\,|\boldsymbol{\lambda}-\square\rangle\rangle,$$

$$\rho_V(\psi_\omega^\pm(z))\,|\boldsymbol{\lambda}\rangle\rangle = \left[\frac{\mathcal{Y}_{\omega+\nu_1+\nu_2}^{*[\boldsymbol{\lambda}]}(q_3^{-1}z)}{\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)}\right]_\pm |\boldsymbol{\lambda}\rangle\rangle.$$

- $A_\omega(\boldsymbol{\lambda})/R_\omega(\boldsymbol{\lambda})$ = set of boxes of color $c(\square) = \omega$ that can be added/removed to $\boldsymbol{\lambda}$.
- Matrix elements are written in terms of $\mathcal{Y}$-observables $\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)$, $\mathcal{Y}_\omega^{*[\boldsymbol{\lambda}]}(z)$.
- The highest state $|\emptyset\rangle\rangle$ corresponds to empty Young diagrams.

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$$[\alpha_{\omega,k}, \alpha_{\omega',l}] = k\delta_{k+l} q_3^{k/2} \left[\delta_{\omega\omega'} + q_3^{-k}\delta_{\omega\ \omega'-\nu_3} - q_1^k \delta_{\omega\ \omega'+\nu_1} - q_2^k \delta_{\omega\ \omega'+\nu_2}\right], \quad (k > 0).$$

and the zero modes $P_\omega(z)$, $Q_\omega(z)$ (written with $2p$ finite Heisenberg algebras),

$$P_\omega(z) Q_{\omega'}(w) = F_{\omega\omega'} w^{C_{\omega\omega'}} z^{-C_{\omega\omega'}} Q_{\omega'}(w) P_\omega(z).$$

with $F_{\omega\omega'} = (-1)^{\delta_{\omega\omega'}} (-q_3)^{-\delta_{\omega,\omega'-\nu_3}} (-q_1)^{-\delta_{\omega\omega'+\nu_1}} (-q_2)^{-\delta_{\omega\omega'+\nu_2}}$.

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- Define the vacuum $|\emptyset\rangle$ such that $\alpha_{\omega,k>0} |\emptyset\rangle = 0$, $P_\omega(z) |\emptyset\rangle = |\emptyset\rangle$.

$\rightsquigarrow$ Standard PBW basis obtained by acting with $\alpha_{\omega,k<0}$ and $Q_\omega(z)$.

Horizontal representations

Drinfeld currents are represented in terms of vertex operators,

$$\rho_H(x_\omega^+(z)) = u_\omega z^{-n_\omega} Q_\omega(z) \exp\left(\sum_{k>0} \frac{z^k}{k}\alpha_{\omega,-k}\right) \exp\left(-\sum_{k>0} \frac{z^{-k}}{k} q_3^{-k/2}\alpha_{\omega,k}\right),$$

$$\rho_H(x_\omega^-(z)) = u_\omega^{-1} z^{n_\omega} Q_\omega(z)^{-1} P_{\omega-\nu_3}(q_3^{-1}z) \exp\left(-\sum_{k>0} \frac{z^k}{k}\alpha_{\omega,-k}\right) \exp\left(\sum_{k>0} \frac{z^{-k}}{k} q_3^{k/2}\alpha_{\omega-\nu_3,k}\right)$$

$$\rho_H(\psi_\omega^+(z)) = F^{-1/2} P_{\omega-\nu_3}(q_3^{-1}z) \exp\left(-\sum_{k>0} \frac{z^{-k}}{k}(q_3^{-k/2}\alpha_{\omega,k} - q_3^{k/2}\alpha_{\omega-\nu_3,k})\right),$$

$$\rho_H(\psi_\omega^-(z)) = F^{1/2} \frac{u_{\omega-\nu_3}}{u_\omega} q_3^{n_{\omega-\nu_3}} z^{n_\omega - n_{\omega-\nu_3}} \frac{Q_{\omega-\nu_3}(q_3^{-1}z)}{Q_\omega(z)} P_{\omega-\nu_3}(q_3^{-1}z)$$
$$\times \exp\left(\sum_{k>0} \frac{z^k}{k}(q_3^{-k}\alpha_{\omega-\nu_3,-k} - \alpha_{\omega,-k})\right).$$

Intertwining operators

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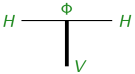
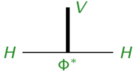
- These operators are obtained by solving the following equation:

$$\rho_H(e)\Phi = \Phi\left(\rho_V \otimes \rho_H\ \Delta(e)\right), \quad \text{or} \quad \left(\rho_V \otimes \rho_H\ \Delta'(e)\right)\Phi^* = \Phi^* \rho_H(e),$$

for every element $e = x^{\pm}_{\omega}(z), \psi^{\pm}_{\omega}(z), c$ of the algebra.
(Δ' is the opposite coproduct obtained by permutation.)





Application to gauge theories

- The solutions have been found, they decompose on the vertical basis

$$\Phi = \sum_{\lambda} \Phi_{\lambda} \langle\langle \lambda | \,, \quad \Phi^* = \sum_{\lambda} \Phi^*_{\lambda} | \lambda \rangle\rangle$$

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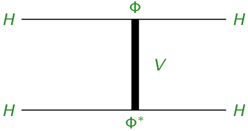

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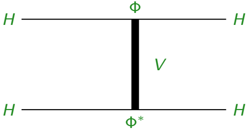

- Along the vertical direction, gluing is done by a scalar product:

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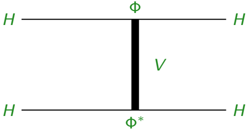

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The vacuum expectation value reproduces the gauge theory partition function:

$$\mathcal{Z} = \left(\langle \emptyset | \otimes \langle \emptyset | \right) \mathcal{T} \left(| \emptyset \rangle \otimes | \emptyset \rangle \right) .$$

Outline

Summary of the results

From the mathematics perspective:

- Define a new algebra that *deforms* the quantum toroidal $\mathfrak{gl}(p)$ algebra.
- Show that it has the structure of a Hopf algebra.
- Provide a highest weight representation on Young diagrams.
- Provide the vertex representation (or level one representation).
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From the physics perspective:

- Define a colored topological vertex pertaining to the spacetime orbifold.
- Extend the algebraic construction of SUSY gauge theories' partitions functions.
- Include the case $\nu_1 = 1$, $\nu_2 = 0$ corresponding to the insertion of a surface defect.
- Construct the qq-characters of the gauge theories (other type of observable).
  - $\rightsquigarrow$ Observe two inequivalent fundamental qq-characters.

Open questions

From the mathematics perspective:

- Show that Serre relations are obeyed in both horizontal and vertical representations.
- Prove the conjectured equivalence with a quantum toroidal algebra defined upon the Cartan matrix $C_{\omega\omega'}$. ($\rightsquigarrow$ quantum affine algebra?)
- Look for Miki's automorphism mapping $(c,\bar{c}) \to (-\bar{c},c)$ (S-duality).
- Find more representations (MacMahon modules?) and classify them.

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From the physics perspective:

- Find a duality with q-deformed W-algebras $\rightsquigarrow$ AGT-correspondence!
- Construct the associated quantum integrable systems.

Outline

Main idea

Combine different ingredients to derive a general approach to SUSY gauge theories:

I. ADHM construction

Description of the non-perturbative sector of SUSY gauge theories.

(instanton moduli space)

[Atiyah-Drinfeld-Hitchin-Manin, Kronheimer, Nakajima,...]

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$\rightsquigarrow$ Associate an algebra to the quivers.

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$\rightsquigarrow$ Construct the intertwining operator from a Hopf algebra.
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$\rightsquigarrow$ Compute amplitudes via a diagrammatic technique.

$\rightsquigarrow$ Identify amplitudes with gauge theory observables.

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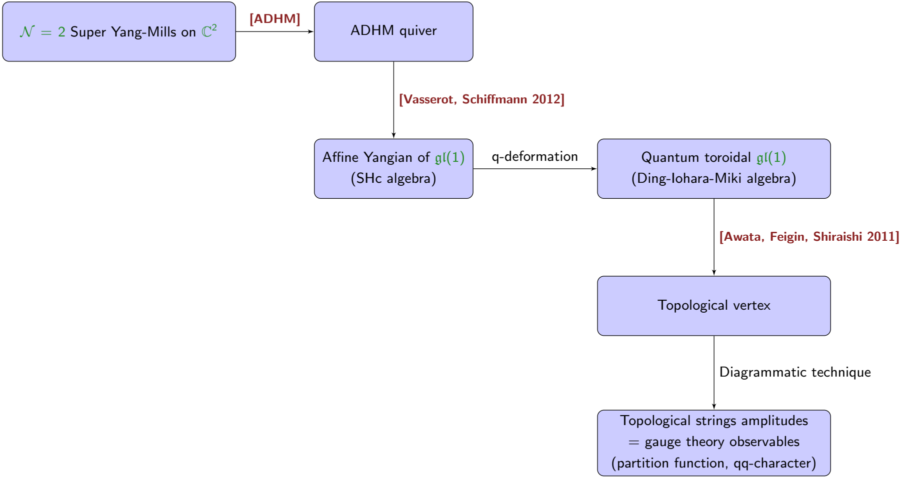
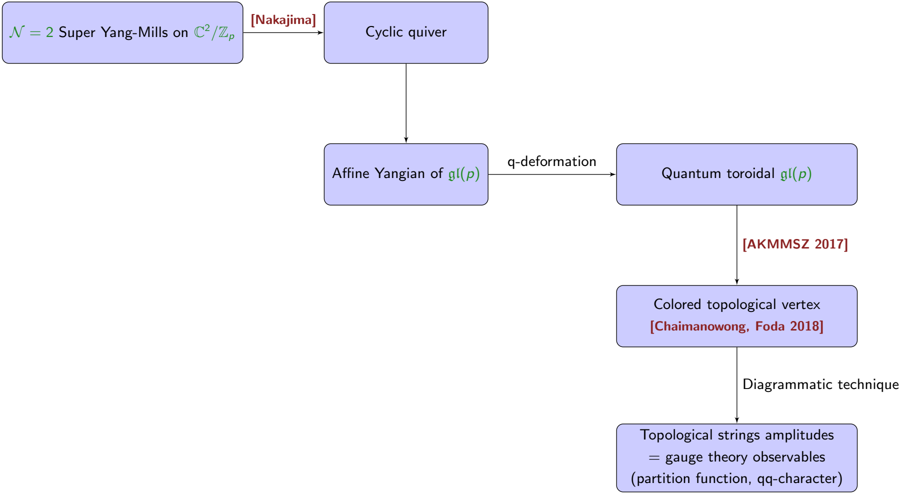
✎ The simplest context is given by $\mathcal{N} = 2$ Super Yang-Mills on $\mathbb{R}^4 \simeq \mathbb{C}^2$.

$\mathcal{N} = 2$ Super Yang-Mills on $\mathbb{C}^2$



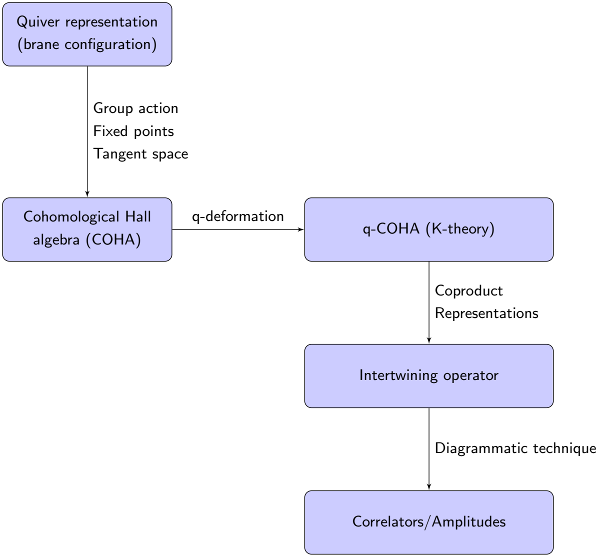
$\mathcal{N} = 2$ Super Yang-Mills on $\mathbb{C}^2/\mathbb{Z}_p$

Case $\nu_1 = -\nu_2 = 1$



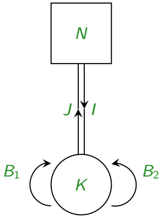
Hints for a very general construction...

General mathematical construction



✎ How is the algebra built out of the quiver representation?

ADHM quiver: definition



- Definition: N and K are vector spaces of dimension n and k.

Take $B_1, B_2 \in \mathrm{End}(K)$, $I \in \mathrm{Hom}(N, K)$, $J \in \mathrm{Hom}(K, N)$ and consider:

$$\mathcal{M}_k(n) = \{B_1, B_2, I, J \,/\, \mu_{\mathbb{C}} = 0, \mathbb{C}[B_1, B_2]IN = K\}/GL(K),$$

with the moment map $\mu_{\mathbb{C}} = [B_1, B_2] + IJ$.

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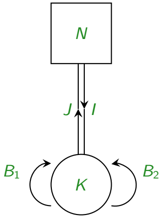

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- Consider the action of $U(1)^{n+2} \subset \mathrm{GL}(N) \times \mathrm{SO}(4)$

$$(h, t_1, t_2) \in U(1)^n \times U(1) \times U(1) : (B_1, B_2, I, J) \to (t_1 B_1, t_2 B_2, Ih, t_1 t_2 h^{-1} J).$$

$\rightsquigarrow$ The fixed points are labelled by n-tuple partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(n)})$.

ADHM quiver: Nekrasov factor

- Construct the tangent space $(\delta B_1, \delta B_2, \delta I, \delta J)$

$\rightsquigarrow$ modulo $\delta\mu_{\mathbb{C}} = 0$ and $\mathfrak{gl}(K)$ transformation.

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- The Nekrasov factor is obtained by taking the plethystic exponential,

$$N(\boldsymbol{v}, \boldsymbol{\lambda}|\boldsymbol{v}', \boldsymbol{\lambda}') = \mathbb{I}[\mathcal{X}] = \prod_{\substack{\square \in \boldsymbol{\lambda} \\ \blacksquare \in \boldsymbol{\lambda}'}} S(\chi_\square/\chi_\blacksquare) \times \prod_{\alpha=1}^{n} \prod_{\square \in \boldsymbol{\lambda}'} \left(1 - \frac{v_\alpha}{\chi_\square}\right) \times \prod_{\square \in \boldsymbol{\lambda}} \prod_{\alpha=1}^{n'} \left(1 - q_1 q_2 \frac{\chi_\square}{v'_\alpha}\right),$$

with $\quad S(z) = \dfrac{(1-q_1 z)(1-q_2 z)}{(1-z)(1-q_1 q_2 z)}.$

⚠ For two different fixed points $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$, need another mathematical construction.

ADHM quiver: $\mathcal{Y}$-observables

- From the Nekrasov factor, we determine the $\mathcal{Y}$-observable

$$\mathcal{Y}^{[\boldsymbol{\lambda}]}(\chi_\square) = \frac{N(\boldsymbol{v},\boldsymbol{\lambda}|\boldsymbol{v}',\boldsymbol{\lambda}'+\square)}{N(\boldsymbol{v},\boldsymbol{\lambda}|\boldsymbol{v}',\boldsymbol{\lambda}')},\quad \mathcal{Y}^{*[\boldsymbol{\lambda}']}(q_1q_2\chi_\square) = \frac{N(\boldsymbol{v},\boldsymbol{\lambda}+\square|\boldsymbol{v}',\boldsymbol{\lambda}')}{N(\boldsymbol{v},\boldsymbol{\lambda}|\boldsymbol{v}',\boldsymbol{\lambda}')}$$

$$\Rightarrow\quad \mathcal{Y}^{[\boldsymbol{\lambda}]}(z) = \prod_{\alpha=1}^{n}\left(1-\frac{v_\alpha}{z}\right)\times\prod_{\square\in\boldsymbol{\lambda}} S(\chi_\square/z),\quad \mathcal{Y}^{*[\boldsymbol{\lambda}]}(z) = \mathcal{Y}^{[\boldsymbol{\lambda}]}(z)\times\prod_{\alpha=1}^{n}\left(-\frac{z}{v_\alpha}\right)$$

ADHM quiver: $\mathcal{Y}$-observables

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⇝ What happens in the case of a $\mathbb{Z}_p$-orbifold?

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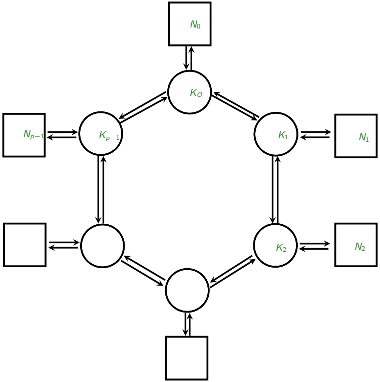

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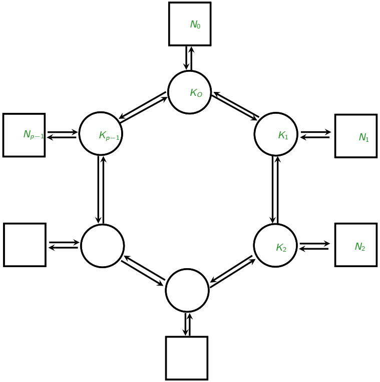

with fixed point given by colored partitions, $c(\square) = c_\alpha + i - j \in \mathbb{Z}_p$ for $\square = (i,j) \in \lambda^{(\alpha)}$.

$\rightsquigarrow$ Or we can take a shortcut and consider the $\mathbb{Z}_p$-invariant character!!!

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$\rightsquigarrow$ Now, we have all the information needed to define our algebra!!!

Construction of the algebra

✎ The algebra is determined following these few steps:

I. Define an action on states parameterized by fixed points

The matrix elements are given by the $\mathcal{Y}$-observables.

$\rightsquigarrow$ This is the vertical representation = Cohomological Hall algebra action.

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IV. Add a co-algebraic structure

$\rightsquigarrow$ Show that it defines a Hopf algebra.

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$\rightsquigarrow$ Consider more general examples...

Quiver with fixed points labelled by half-partitions? **[in progress...]**

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Thank you !!!