Exact results for the simplest four point function in $N=4$ SYM theory

D. Serban

IPhT SACLAY

w/ I. Kostov, V. Petkova, arXiv:1903.05038, 1905.11467
F. Coronado, arXiv:1811.00467, 1811.033282
T. Bargheer, F. Coronado, P. Vieira, 1909.04077
A. Belitsky, G. Korchemsky, 1907.13131

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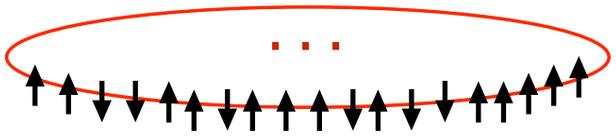
Correlation functions in planar N=4 SYM from integrability

- Integrability of planar N=4 SYM theory opened a window into exploring in a detailed and precise manner the gauge/string, or the AdS/CFT correspondence.

- Local, gauge-invariant operators with definite conformal dimension correspond to eigenstates of a long-range interacting (super)spin chain - and to a string propagating in the AdS5 x S5 background (described by a sigma model).

$$\mathrm{Tr}_{\,\mathrm{SU(N)}} ZZXXZZXZZZ...(x) \longleftrightarrow$$

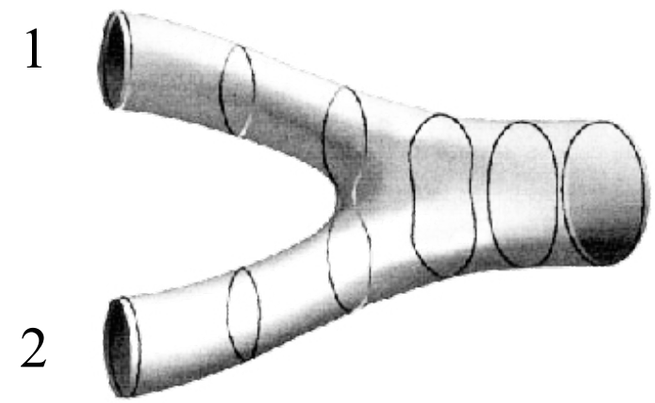


- Correlation functions of such operators can be computed with techniques inspired from 2d integrable field theories (non-local form factors).

- A practical motivation is to replace the tedious Feynman graph computations in gauge theories with more powerful techniques based on symmetries, and eventually to get a non-perturbative description for the gauge theory observables.

Three point functions in $\mathcal{N}$=4 SYM

the three point function
dual to three-string interaction
is the basic building block for correlation function



$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}(\lambda)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$

initial data: three states with definite conformal dimensions and psu(2,2|4) charges

$$\mathcal{O}_\alpha(x)\ , \quad \alpha = 1,2,3$$

each characterised by a set of rapidities $\mathbf{u}_\alpha$

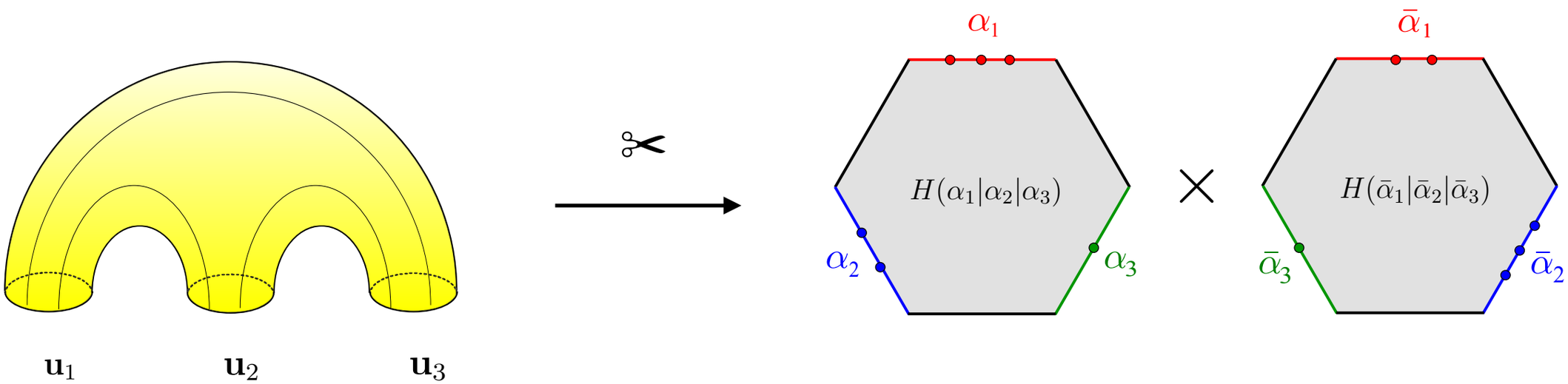
and polarisations (or global rotations with respect to some reference BPS state, e.g Tr Z^L)

$$g_\alpha = e^{\zeta_\alpha^A J^A}$$

The hexagon decomposition of the correlation functions

[Basso, Komatsu, Vieira, 15]

the asymptotic part of the three point function can be written as a sum over partitions for the three groups of rapidities $\mathbf{u}_1 = \alpha_1 \cup \bar{\alpha}_1, \mathbf{u}_2 = \alpha_2 \cup \bar{\alpha}_2, \mathbf{u}_3 = \alpha_3 \cup \bar{\alpha}_3$



- contribution of virtual particles exponentially suppressed if the bridges $\ell_{12},\ \ell_{23},\ \ell_{31} >> 0$

$$\ell_{ij} = \frac{1}{2}(L_i + L_j - L_k)$$

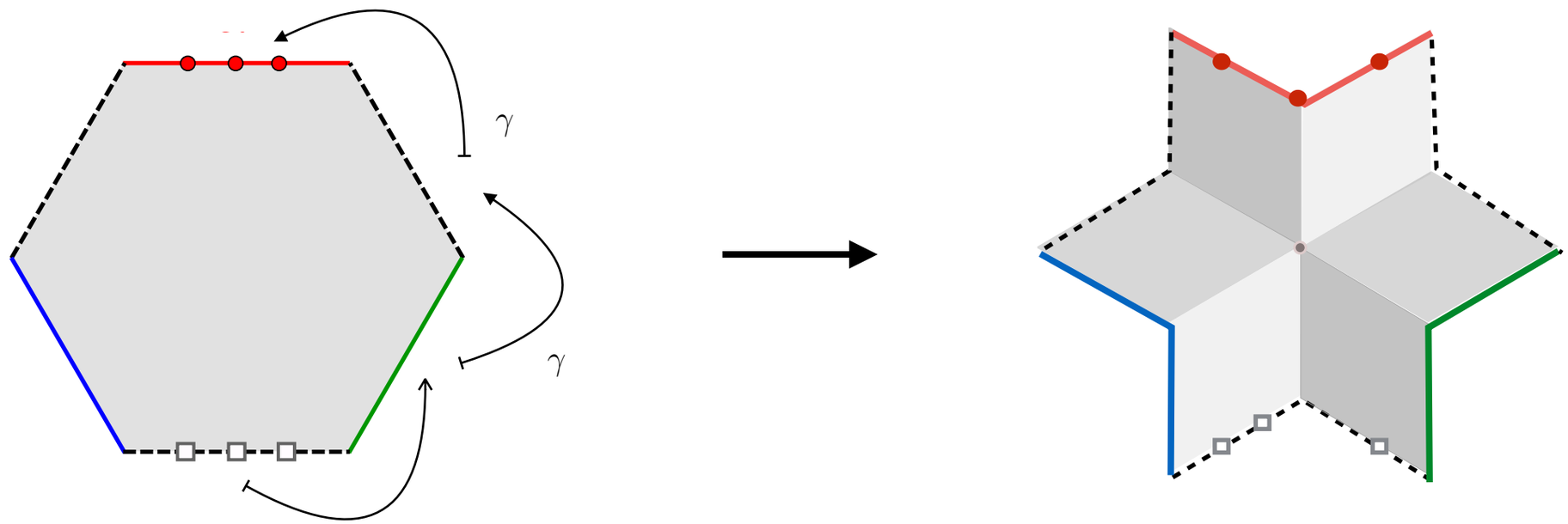
$$[\mathcal{C}_{123}^{\bullet\bullet\bullet}]^{\text{asympt}} = \sum_{\alpha_i \cup \bar{\alpha}_i = \mathbf{u}_i} (-1)^{|\alpha_1|+|\alpha_2|+|\alpha_3|}\, w_{\ell_{31}}(\alpha_1, \bar{\alpha}_1)\, w_{\ell_{12}}(\alpha_2, \bar{\alpha}_2)\, w_{\ell_{23}}(\alpha_3, \bar{\alpha}_3)$$
$$\times \quad \mathrm{H}(\alpha_1|\alpha_3|\alpha_2)\mathrm{H}(\bar{\alpha}_2|\bar{\alpha}_3|\bar{\alpha}_1)\ .$$

- sewing back over the black lines: insertion of an arbitrary number of virtual particles

The hexagon as a non-local form factor

[Basso, Komatsu, Vieira, 15]

the hexagon can be seen as the form factor of a twist-like operator inducing a curvature excess of 180 degrees **[Cardy, Castro-Alvaredo, Doyon, 06]**



solution from bootstrap (form factor axioms)

$$\mathrm{H}^{A_1\dot{A}_1\cdots} = (-1)^{\mathfrak{f}} \prod_{i<j} h_{ij} \langle \chi_N^{\dot{A}_N} \ldots \chi_1^{\dot{A}_1} | \mathcal{S} | \chi_1^{A_1} \ldots \chi_N^{A_N} \rangle$$

$|\chi_i^{A_i}\rangle$: psu(2|2) state

dynamical part

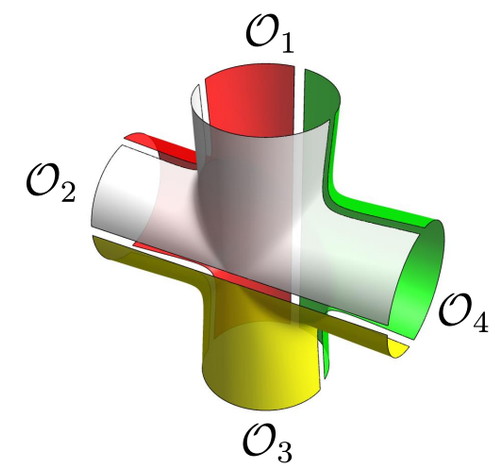
matrix part **[Beisert, 06]**

The hexagon as building block for correlation functions

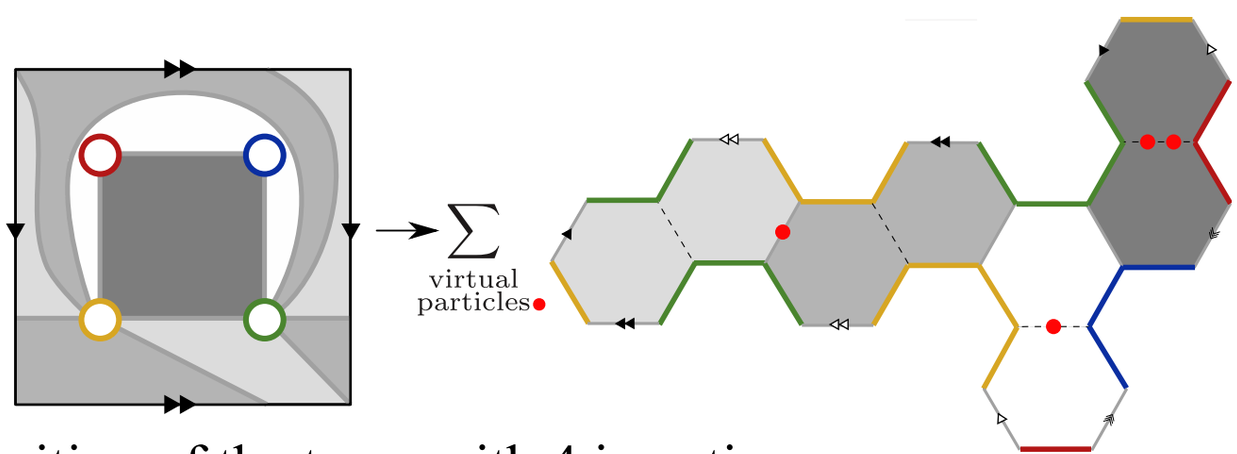
four point function by hexagon decomposition:

[Fleury, Komatsu, 16; also Eden, Sfondrini, 16]

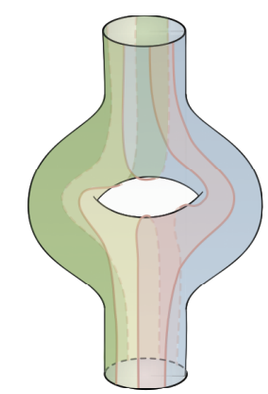
$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle =$



1/N corrections **[Bargeer, Caetano, Fleury, Komatsu, Vieira, 18]**



decomposition of the torus with 4 insertions



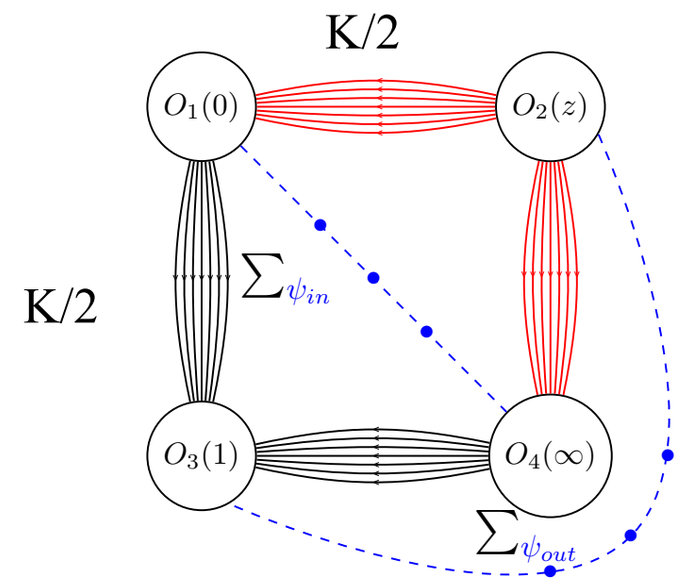
torus with 2 insertions

& non-planar resummation **[Bargeer, Coronado, Vieira, 19]**

Four point functions: the “simplest” correlator

four point function: dependence on two cross ratios:

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$



for BPS operators with large R-charges and particular polarisations: factorisation into two octagons **[Coronado, 18]**

$$\langle O_1 O_2 O_3 O_4 \rangle = \left[\frac{1}{x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2}\right]^{\frac{K}{2}} \times \mathbb{O}^2(z,\bar{z}) \qquad K \to \infty$$

compute the octagon analytically by resumming the virtual particle contribution

$\longrightarrow$ Fredholm determinant **[Kostov, Petkova, D.S., 19]**

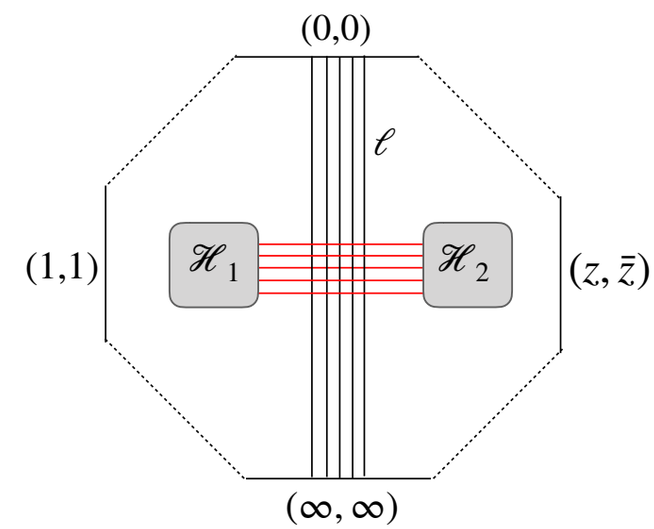
analysis of the Fredholm determinant in the light-like limit **[Belitsky, Korchemsky, 19]**

Four point functions: the "simplest" correlator

$$\mathbb{O}_l(z,\bar{z},\alpha,\bar{\alpha}) \;=\; 1+\sum_{n=1}^{\infty}\mathcal{X}_n(z,\bar{z},\alpha,\bar{\alpha})\times\mathcal{I}_{n,l}(z,\bar{z})$$

simple kinematical factor

more general setting: octagon with a bridge of length l



multiple integral over virtual particles:

$$\mathcal{I}_{n,l}(z,\bar{z}) = \frac{1}{n!}\sum_{a_1=1}^{\infty}\cdots\sum_{a_n=1}^{\infty}\int du_1\cdots\int du_n\,\prod_{j=1}^{n}\bar{\mu}_{a_j}(u_j,l,z,\bar{z})\times\prod_{j<k}^{n}P_{a_j a_k}(u_j,u_k)$$

one-particle measure: $\bar{\mu}_a(u,l,z,\bar{z}) = \dfrac{1}{\sqrt{z\bar{z}}}\,\dfrac{\sin a\phi}{\sin\phi}\times\;\mu_a(u)\times e^{-E_a(u)\,l}\times(z\bar{z})^{-i\,p_a(u)}$

two-particle interaction: $P_{ab}(u,v) = \mathcal{K}_{ab}^{++}(u,v)\mathcal{K}_{ab}^{+-}(u,v)\mathcal{K}_{ab}^{-+}(u,v)\,\mathcal{K}_{ab}^{--}(u,v)$

$$\mathcal{K}_{ab}^{\pm\pm}(u,v) = \frac{x^{[\pm a]}(u)-x^{[\pm b]}(v)}{1-x^{[\pm a]}(u)\,x^{[\pm b]}(v)} \qquad x^{[\pm a]}+\frac{1}{x^{[\pm a]}} = \frac{u\pm ia/2}{g} \qquad g=\frac{\sqrt{\lambda}}{4\pi}$$

't Hooft coupling constant

Four point functions: the "simplest" correlator

$$\mathcal{I}_{n,l}(z,\bar{z}) = \frac{1}{n!}\sum_{a_1=1}^{\infty}\cdots\sum_{a_n=1}^{\infty}\int du_1\cdots\int du_n \prod_{j=1}^{n}\bar{\mu}_{a_j}(u_j,l,z,\bar{z}) \times \prod_{j<k}^{n} P_{a_j a_k}(u_j,u_k)$$

- at **weak coupling**

$$x^{[\pm a]} = \frac{u \pm \frac{i}{2}a}{g} - \frac{g}{u \pm \frac{i}{2}a} - \frac{g^3}{(u \pm \frac{i}{2}a)^3} \quad +\ldots$$

the two-particle measure separates $\longrightarrow$ sum of factorised ladder integrals

$$\sum_{J=n^2}^{\infty}\sum_{\vec{j}\in Z_n^+(J)} g^{2J} \times d_{\vec{j}} \times f_{j_1}\cdots f_{j_n}$$

$$f_p = -v\sum_{j=p}^{2p}\frac{j!(p-1)!\,[-\log(z\bar{z})]^{2p-j}}{(j-p)!(2p-j)!}\left[\frac{\mathrm{Li}_j(z)-\mathrm{Li}_j(\bar{z})}{z-\bar{z}}\right] \qquad v=(1-z)(1-\bar{z})$$

find a systematic way to determine the coefficients d **[Coronado, 18]**

- at **strong coupling** the two-particle measure does not separate but simplifies

[Jiang, Komatsu, Kostov, D.S., 16; Bargeer, Coronado, Vieira, 19]

Exact results for the octagon

convenient parametrisation for the cross ratios: $z = e^{-\xi+i\phi}, \qquad \bar{z} = e^{-\xi-i\phi},$

(here the R-charge cross ratios $\alpha = \bar{\alpha} = 1$) $\alpha = e^{\varphi-\xi+i\theta}, \quad \bar{\alpha} = e^{\varphi-\xi-i\theta}.$

$$\mathbb{O}_\ell(z,\bar{z},\alpha,\bar{\alpha}) \quad = \tfrac{1}{2}\sum_{\pm}\sqrt{\mathrm{Det}\,[\mathbf{I} - \lambda^{\pm}\ \mathbf{CK}]}.$$

$$\lambda^{\pm}(\phi,\varphi,\alpha) = 2\,[\cos\phi - \cosh(\varphi \pm i\theta)]$$

$$\mathrm{I}_{m,n} = \delta_{m,n}\ , \quad \mathrm{C}_{nm} = \delta_{n+1,m} - \delta_{n,m+1}, \qquad m,n \geq 0,$$

$$\mathrm{K}_{mn} = \frac{g}{2i}\int_{|\xi|}^{\infty} dt \frac{\left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{m-n} - \left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{n-m}}{\cos\phi - \cosh t}\ J_{m+\ell}(2g\sqrt{t^2-\xi^2})\ J_{n+\ell}(2g\sqrt{t^2-\xi^2})$$

The octagon as a Fredholm Pfaffian

$$\mathcal{I}_{n,l}(z,\bar{z}) = \frac{1}{n!}\sum_{a_1=1}^{\infty}\cdots\sum_{a_n=1}^{\infty}\int du_1\cdots\int du_n \prod_{j=1}^{n}\bar{\mu}_{a_j}(u_j,l,z,\bar{z}) \times \prod_{j<k}^{n} P_{a_j a_k}(u_j,u_k)$$

- octagon at **any coupling:** Fredholm Pfaffian **[Basso, Coronado, Komatsu, Lam, Zhong, 17]**

- bilocal measure:

$$P_{ab}(u,v) = \mathcal{K}_{ab}^{++}(u,v)\mathcal{K}_{ab}^{+-}(u,v)\mathcal{K}_{ab}^{-+}(u,v)\,\mathcal{K}_{ab}^{--}(u,v)$$

$$\mathcal{K}_{ab}^{\pm\pm}(u,v) = \frac{x^{[\pm a]}(u) - x^{[\pm b]}(v)}{1 - x^{[\pm a]}(u)\,x^{[\pm b]}(v)}$$

- fermionic dipole representation:

$$P_{1,1}(u,v) \;= \langle : e^{\phi(u^+)-\phi(u^-)} :: e^{\phi(v^+)-\phi(v^-)} : \rangle$$

fermions on the Zhukovsky surface; Majorana-like
[Kostov, Petkova, D.S., 19]

- Cauchy-like identity:

$$\mathrm{Pf}\left(\left[\frac{x_j - x_k}{x_j x_k - 1}\right]_{i,j=1}^{2n}\right) = \prod_{j<k}^{2n}\frac{x_j - x_k}{x_j x_k - 1}$$

The octagon as a Fredholm Pfaffian

$$\mathbf{K}(u,v) = \begin{pmatrix} K^{++}(u,v) & K^{+-}(u,v) \\ K^{-+}(u,v) & K^{--}(u,v) \end{pmatrix}, \quad K^{\varepsilon\varepsilon'}(u,v) = -K^{\varepsilon'\varepsilon}(v,u)$$

$$\mathbf{I}(u,v) = \mathbf{I}\,\delta(u-v), \quad \mathbf{J}(u,v) = \mathbf{J}\,\delta(u-v),$$
$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

- definition of a Fredholm Pfaffian:

$$\mathrm{Pf}[\mathbf{J}+\mathbf{K}] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} \prod_{j=1}^{n} d\mu(u_j)\, \mathrm{Pf} \begin{bmatrix} K^{++}(u_j,u_k) & K^{+-}(u_j,v_k) \\ K^{-+}(u_j,v_k) & K^{--}(u_j,u_k) \end{bmatrix}_{1\le j,k\le n}$$

- relation to Fredholm determinants: **[Rains, 00; Borodin, Kanzieper, 07]**

$$\mathrm{Pf}[\mathbf{J}+\mathbf{K}] = \sqrt{\det\,[\mathbf{I}-\mathbf{JK}]}.$$

- formal result: $\mathbb{O}_\ell(z,\bar{z},\alpha,\bar{\alpha}) = \frac{1}{2}\sum_{\pm} \mathrm{Pf}\,[\boldsymbol{J}+\lambda^{\pm}\mathbb{K}]$

$$\mathbb{K}^{\varepsilon\delta}(u,v) = -\frac{1}{ig} \sum_{m,n\ge 0} \Omega_{\ell+n}(u+\varepsilon i\epsilon)\, \frac{\mathrm{C}_{nm}}{2\cos\phi - 2\cos\partial_v} \Omega_{\ell+m}(v+\delta i\epsilon) \qquad \Omega_\ell(u) \equiv \frac{e^{ig\xi\,[x(u)-1/x(u)]}}{x(u)-1/x(u)}\, x(u)^{-\ell}$$

The octagon at weak coupling

- when expanded in powers of g, the octagon depends on $\xi = -\log\sqrt{z\bar{z}}$ only through the ladder integrals **[Coronado, 18]**

- this is not true on individual matrix elements, since

$$\mathrm{K}_{m+r,m} = e^{-\xi}\sum_{p=0}^{\infty}\binom{2\ell+2m+2p+r}{p}\frac{(-1)^{l+m}(ig)^{2\ell+2m+2p+r+1}}{(\ell+m+p)!(\ell+m+p+r)!}$$
$$\times\sum_{k=1}^{[\frac{r+1}{2}]}\binom{r-k}{k-1}|2\xi|^{r-2k+1}f_{\ell+m+p+k}(z,\bar{z}) \qquad (r\geq 1).$$

- nevertheless we have checked "experimentally" that the traces $\mathrm{tr}[(\mathbf{CK})^n]$ do not depend on ξ explicitly, so we can take

$$\mathrm{K}^{\circ}_{m+2s,m} = 0.$$

$$\mathrm{K}^{\circ}_{m+2s-1,m} = e^{-\xi}\sum_{j=l+m+s}^{\infty}\binom{2j-1}{j-l-m-s}\frac{(-1)^{j-l-m}}{(j-s)!(j+s-1)!}g^{2j}f_j(z,\bar{z})$$

and conclude that $\mathrm{tr}[(\mathbf{CK})^n] = \mathrm{tr}[(\mathbf{CK}^{\circ})^n]$

The octagon at strong coupling

[Bargheer, Coronado, Vieira, 19]

- hard to extract the strong coupling behaviour directly from the determinant; restart from scratch and use the method from **[Jiang, Komatsu, Kostov, D.S., 16]**

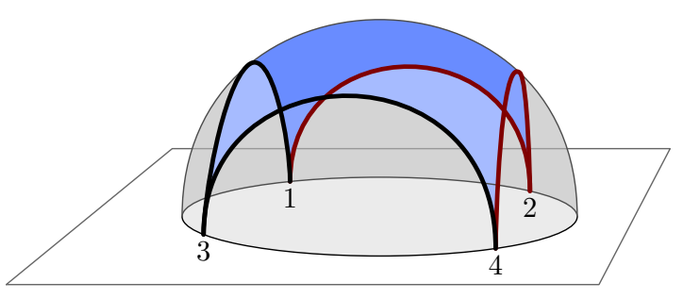
$$\log \mathbb{O} \simeq 2g\xi \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh\theta \log(1+Y(\theta)) \qquad u = 2g\tanh(\theta)$$

with

$$Y(\theta) = -\frac{\sin\left(\frac{\phi}{2}+i\frac{\xi}{2}\right)\sin\left(\frac{\phi}{2}-i\frac{\xi}{2}\right)}{\sin\left(\frac{\phi}{2}+i\frac{\xi}{2}\cosh\theta\right)\sin\left(\frac{\phi}{2}-i\frac{\xi}{2}\cosh\theta\right)}$$

TBA-like answer!

[Alday, Gaiotto, Maldacena, 09]



area delimited by geodesics in AdS5 (x sphere)

get the answer from semiclassics *à la* **[Kazama, Komatsu, Nishimura, 13-16]**?

The null octagon limit

- an interesting limit is that where successive operators become light-like separated, $x_{12},\ x_{24},\ x_{34},\ x_{13} \to 0$

- this means for the cross ratios $z \to 0, 1/\bar{z} \to 0$

- useful parametrisation $z = -e^{-\xi - y}, \bar{z} = -e^{-\xi + y}$ $\qquad y \to \infty$

in this limit the octagon is amazingly simple **[Coronado, 18]**

$$\ln \mathbb{O}(z, \bar{z}) \overset{y \to \infty}{=} -\frac{y^2}{2\pi^2}\Gamma(g) + g^2\xi^2 + \frac{1}{8}C(g) + O(\mathrm{e}^{-y})$$

for $\xi = 0$ derived in **[Belitsky, Korchemsky, 19]** by methods similar to Tracy-Widom

notably: $\qquad \Gamma(g) = \log(\cosh(2\pi g))\,, \qquad C(g) = -\log\left(\frac{\sinh(4\pi g)}{4\pi g}\right)$

The null octagon limit

[Belitsky, Korchemsky, 19]

for $\xi = 0$ $y \to \infty$ the kernels simplify considerably :

$$\lambda K_{mn} = 2g \sin\left(\frac{\pi}{2}(m-n)\right) \int_0^\infty dt \frac{J_m(2gt) J_n(2gt)}{1+\mathrm{e}^{t-y}}$$

and after some massaging $\mathbb{O} = \det(\mathbb{1} - \mathbb{H})$

$$\mathbb{H} f(t) = \int_0^\infty \frac{dt'}{t'} \frac{H(2gt, 2gt')}{\mathrm{e}^{t'-y}+1} f(t')$$

$$\begin{aligned} H(2gt_1, 2gt_2) &= \sum_{m\geq 0} 2(2m+1) J_{2m+1}(2gt_1) J_{2m+1}(2gt_2) \\ &= \frac{2gt_1t_2\left(t_1 J_0\left(2gt_2\right) J_1\left(2gt_1\right) - t_2 J_0\left(2gt_1\right) J_1\left(2gt_2\right)\right)}{t_1^2 - t_2^2} \end{aligned}$$

Bessel kernel

The null octagon limit

[Belitsky, Korchemsky, 19]

natural rescaling: $x_i = (2gt_i)^2\,, \quad s = 4g^2y^2$

$$H(2gt_1, 2gt_2) = 2\sqrt{x_1x_2}\,K(x_1, x_2)$$

$$K(x_1, x_2) = \frac{\sqrt{x_1}J_1\left(\sqrt{x_1}\right)J_0\left(\sqrt{x_2}\right) - \sqrt{x_2}J_0\left(\sqrt{x_1}\right)J_1\left(\sqrt{x_2}\right)}{2(x_1 - x_2)}$$ Bessel kernel

new feature: Fermi-Dirac like distribution: $\chi(x) = \dfrac{1}{\mathrm{e}^{(\sqrt{x}-\sqrt{s})/(2g)} + 1}$

$\sqrt{s}$: chemical potential $\qquad g$: temperature

the rest follows the standard treatment of **[Its, Izergin, Korepin, Slavnov, 92; Tracy, Widom, 93]**

how to derive the (simple) ξ dependence from the complicatedly - looking kernel?

Summary and outlook

- We have given a compact, all-loop expression for a simple yet non-trivial four-point correlation function in an integrable 4-dimensional gauge theory

- Underlying the solution there is a representation of the form-factors in terms of Majorana-like fermions; the origin of these fermions is still mysterious

- Other exact solutions for the correlation functions can be written in fermionic terms (Fredholm determinants)

- Integrals associated with Feynman graphs (ladder integrals) have a nice TBA representation

- We need a better understanding of the properties of the Fredholm determinant in particular limits (strong coupling, light-like) and in general

- Five point functions have a richer structure, being associated to more complicated Feynman diagrams - but their analysis is more complicated **[Fleury, Komatsu, 17]**

- Some simplifications occur when studying non-supersymmetric deformations (fishnet theories) **[Zamolodchikov, 80, Kazakov, Gurdogan, 15]**; good laboratory to study the Feynman integrals **[Basso, Dixon, 17]**

- **…**