Large deviations in transport and twist field correlation functions from hydrodynamics

Benjamin Doyon

Royal Society Leverhulme Trust Senior Research Fellow

Department of Mathematics, King's College London

with:

M. Joe Bhaseen, King's College London

Rosemary Harris, Queen Mary University London

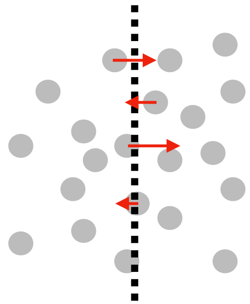
Jason Myers, CANES doctoral program

Osaka, 11 September 2019

What this is about

⋆ **One goal is to characterise nonequilibrium states**, where quantities are being transported across a surface.

⋆ **Fluctuations of the total amount of a quantity passing through an interface in a time t.** At $t \to \infty$, universal laws, analogues of equilibrium entropy and free energy.



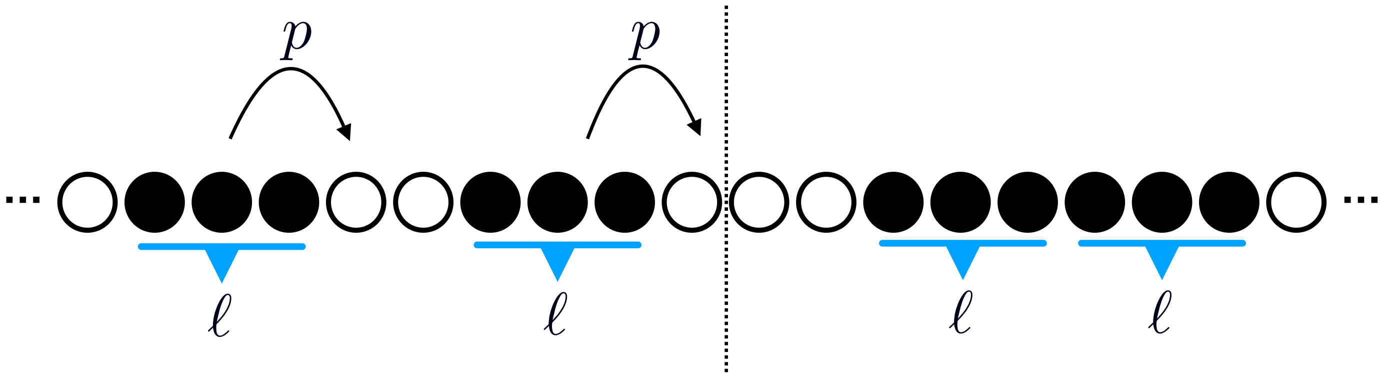
⋆ These are related to **two-point functions of twist fields in thermal states / GGEs**.

⋆ **Theory based on Euler hydrodynamics**, widely applicable: quantum and classical field theories, chains, gases, integrable or not, stochastic or deterministic. In integrable systems, we use **generalised hydrodynamics**.

⋆ Extension to **interacting models** of Lesovik-Levitov free-fermion formula, of Bernard-Doyon 1dCFT formula, of exponential behaviour of dynamical correlation functions as in XX chain.

Examples of transport fluctuation problems

The ℓ-TASEP [MacDonald, Gibbs, Pipkin 1968; Schönherr, Schütz 2004]



Particles have length ℓ and jump by one site to the right, if there is space, with probability p.

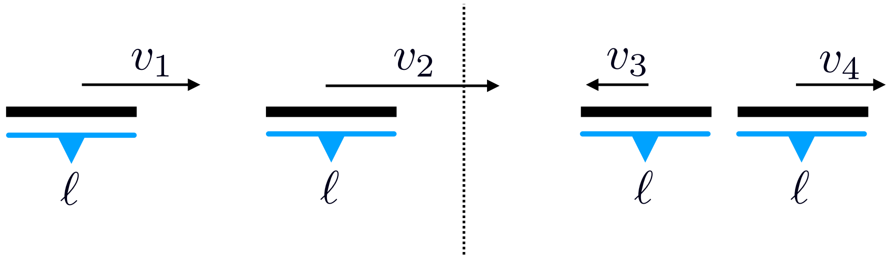
A **steady state is a distribution of positions invariant under this stochastic dynamics**.

Homogeneous, and fully characterised by the density of particles $\rho \in [0, 1/\ell]$.

random variable $= J^{(t)} =$ # particles that crossed the interface in time t

Examples of transport fluctuation problems

The hard rod gas [Lebowitz, Spohn 1970's; Boldrighini, Dobrushin, Sukhov 1983]



Rods have length ℓ and move at velocities $v_i \in \mathbb{R}$, with elastic collisions.

A **steady state is a distribution of positions and velocities invariant under this deterministic evolution**. Homogeneous, and fully characterised by the distribution of velocities $\rho(v) \in \mathbb{R}^+$ (constrained to $\int \mathrm{d}v\, \rho(v) \in [0, 1/\ell]$).

$$\text{measure on initial condition} = \chi\{|x_i - x_j| \geq \ell\} \Big[\prod_i \mathrm{d}x_i \mathrm{d}v_i \rho(v_i) \Big]$$

$$\text{random variable} = J^{(t)} = \text{total amount of the quantity } h(v) \text{ transferred from left to right}$$

$$= \sum_{i: x_i(t) > 0} h(v_i(t)) - \sum_{i: x_i > 0} h(v_i)$$

Examples of transport fluctuation problems

The classical sinh-Gordon field (or any other field theory)

$$H = H[\Phi, \Pi] = \frac{1}{2}\int \mathrm{d}x \left[\Pi^2 + (\partial_x \Phi)^2 + \frac{m^2}{g^2}(\cosh(g\Phi) - 1)\right]$$

A **steady state is a distribution of field and canonical momentum invariant under dynamics** $\{H, \cdot\}$. Homogeneous, and characterised (at least formally) by the Lagrange parameters β_i for all conserved quantities: **boosted / generalised Gibbs ensemble**.

$$\text{measure on initial condition} = \mathcal{D}\Phi\mathcal{D}\Pi\, e^{-\sum_j \beta_j Q_j}$$

$$\text{random variable} = J^{(t)} = \text{total amount of the quantity } Q_i \text{ transferred from left to right}$$

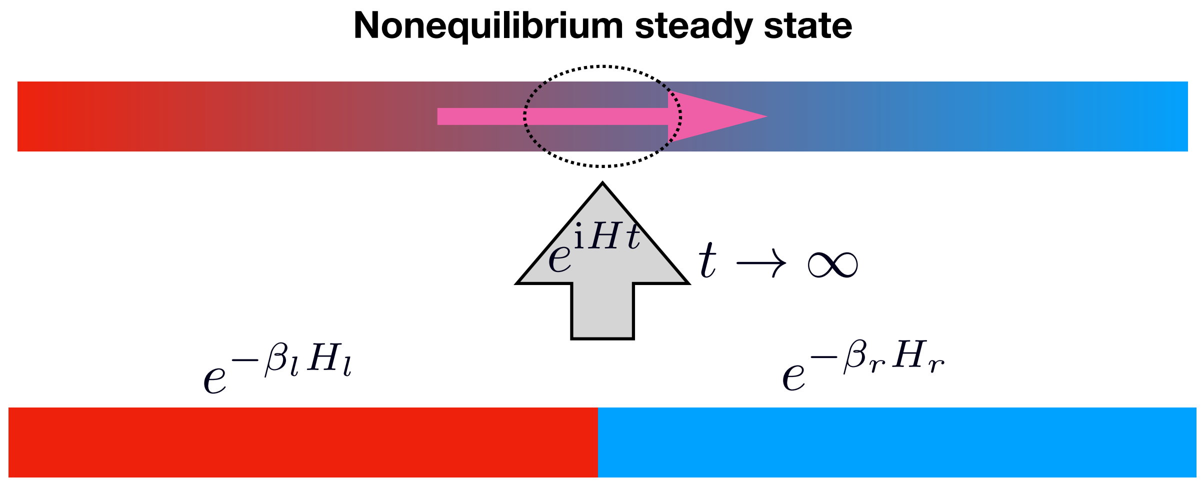
$$= \int_0^\infty \mathrm{d}x\, q_i(x,t) - \int_0^\infty \mathrm{d}x\, q_i(x,0)$$

where

$$Q_i = Q_i[\Phi, \Pi] = \int \mathrm{d}x\, q_i(x)$$

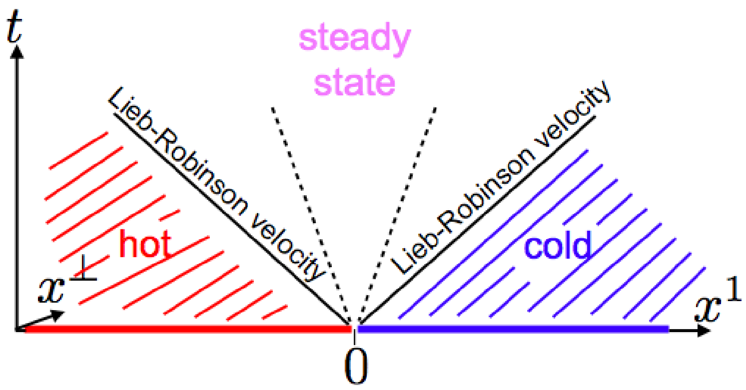
Nonequilibrium steady states

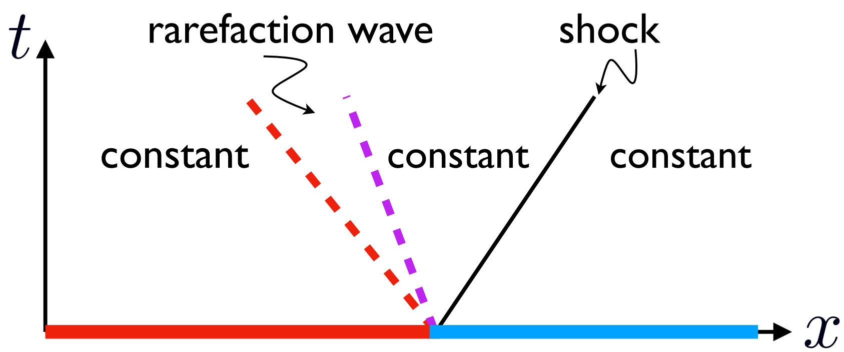
We can evaluate the above in nonequilibrium steady states. For instance, those generated by the partitioning protocol, or **Riemann problem**, where nontrivial **currents emerge in steady-state region** if there is ballistic transport.



Nonequilibrium steady states

In quantum chains there is the Lieb-Robinson bound. Phenomenology from conventional hydrodynamics involves **shocks and rarefaction waves**.

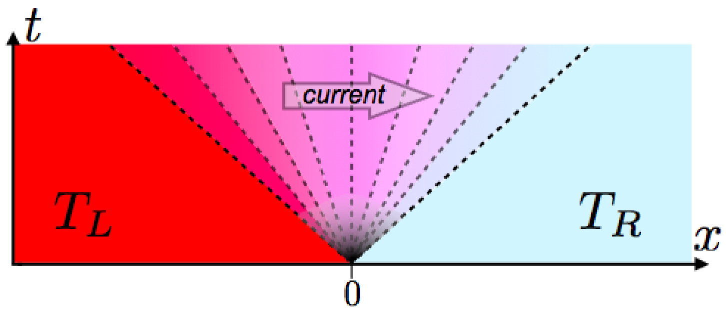




Our theory will work generically in steady state regions away from shocks and rarefaction waves. In rarefaction waves: nonlinear fluctuating hydrodynamics instead (e.g. TASEP from step initial condition).

Nonequilibrium steady states

In integrable systems, including e.g. the hard rods, the phenomenology is different: **generically smooth profiles**, because of the continuum of normal mode's velocities [Castro Alvaredo, BD, Yohimura 2016; Bertini, Collura, De Nardis, Fagotti 2016]. In fluid dynamic terms, solution is a continuum of **contact singularities** [Castro Alvaredo, BD, Yohimura 2016].



Our theory will work in integrable models.

Dynamical two-point functions of twist fields

In all the examples mentioned, the variable can be re-expressed as an integral over the flux,

$$J^{(t)} = \int_0^t \mathrm{d}s\, j_i(0,s)$$

which is related to the charge density by the continuity equation

$$\partial_t q_i + \partial_x j_i = 0$$

Consider the **generating function for the cumulants of** $J^{(t)}$:

$$F(\lambda;0,t) = \log\langle e^{\lambda J^{(t)}}\rangle = \log \sum_J \mathbf{P}(J^{(t)} = J) e^{\lambda J}$$

Solve the continuity equation by defining φ_i such that

$$q_i = \partial_x \varphi_i, \quad j_i = -\partial_t \varphi_i$$

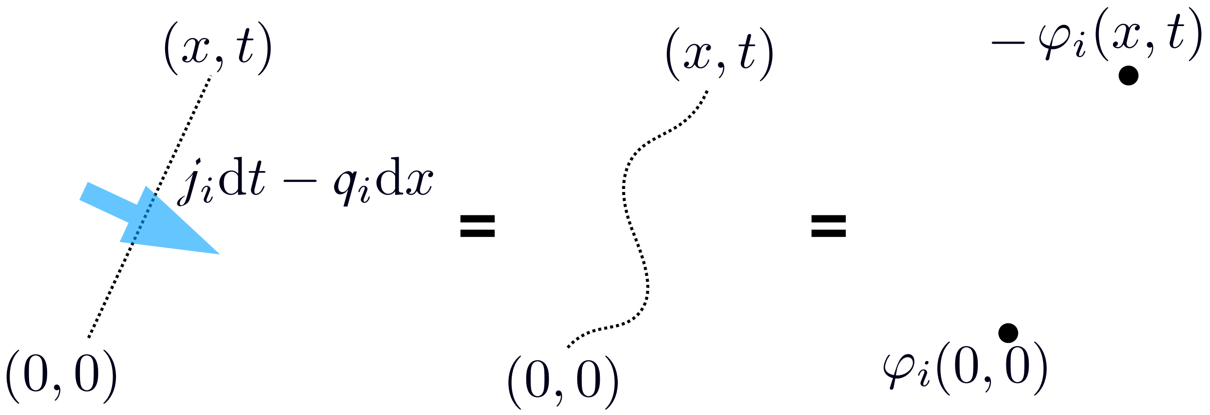
Then

$$F(\lambda;0,t) = \langle e^{\lambda \varphi_i(0,0)} e^{-\lambda \varphi_i(0,t)}\rangle$$

Dynamical two-point functions of twist fields

We can consider the more general problem of evaluating

$$F(\lambda;x,t) = \log\langle e^{\lambda\varphi_i(0,0)}e^{-\lambda\varphi_i(x,t)}\rangle = \log\langle \exp\left[\lambda\int_{(0,0)}^{(x,t)}(j_i\mathrm{d}t - q_i\mathrm{d}x)\right]\rangle$$



The variables $e^{\varphi_i(x,t)}$ are twist fields associated to the symmetry generated by Q_i. By the continuity relation the integral is independent of the shape of the path. A typical example is the vertex operator in the sine-Gordon model, the twist field associated to its topological charge (equivalently, the $U(1)$-twist field in the massive Thirring model); or the $\langle\sigma^+(0,0)\sigma^-(x,t)\rangle$ correlation function in the XX chain.

Quantum

Subtleties in **quantum models**:

- The observable $J^{(t)} = \int_0^t \mathrm{d}s\, j_i(0,s)$ does not represent well the transport fluctuations, as it is not a natural observable on which we can make measurements. Instead, could use **two projective measurement protocol** for the half-charge $Q_+ = \int_0^\infty \mathrm{d}x\, q_i(x)$:

$$\mathbf{P}(J^{(t)} = J) = \sum_{J_2 - J_1 = J} \mathrm{Tr}\Big[\mathbb{P}_{J_2}^{Q_+}\, e^{-\mathrm{i}Ht}\, \mathbb{P}_{J_1}^{Q_+}\, \rho_0\, \mathbb{P}_{J_1}^{Q_+}\, e^{\mathrm{i}Ht}\, \mathbb{P}_{J_2}^{Q_+}\Big]$$

  where $\rho_0 = e^{-\sum_j \beta_j Q_j}$. For instance: quantum field theories (UV regularisation needed), quantum chains, etc.

- By non-commutativity,

$$\log\langle e^{\lambda\varphi_i(0,0)} e^{-\lambda\varphi_i(x,t)}\rangle \neq \log\langle \exp\left[\lambda \int_{(0,0)}^{(x,t)} (j_i \mathrm{d}t - q_i \mathrm{d}x)\right]\rangle$$

But in the limit of large time / space-time, these problems are expected to disappear...

Universal behaviours: large deviations / large space-time distances

Consider the large-time behaviour. $J^{(t)}$ is extensive, and it concentrates on some value $t\bar{\mathrm{j}}$.

More precisely, we expect a **large deviation principle** to hold [e.g. Touchette, 2009]:

$$\mathbf{P}(J^{(t)} = tj) \asymp e^{-tI(j)}, \quad t \to \infty$$

$I(j)$ is the **large-deviation rate function**. It describes rare but significant fluctuations of the extensive state characteristics $J^{(t)}$. It is the analogue of an **entropy**. It satisfies

$$I(\bar{\mathrm{j}}) = 0, \qquad I(j) > 0 \quad (j \neq \bar{\mathrm{j}})$$

Universal behaviours: large deviations / large space-time distances

We will argue that $I(j)$ **is universal**:

- $I(j)$ **is determined by the Euler hydrodynamics of the system**, independently from its microscopic details.
- In the quantum case, $I(j)$ appears to be largely **independent of the measurement protocol**, and we can use the distribution of the operator $J^{(t)} = \int_0^t \mathrm{d}s\, j_i(0,s)$.
- In quantum field theory, $I(j)$ appears to be **free from UV divergencies**.

Universal behaviours: large deviations / large space-time separations

More useful to work with is the **Legendre-Frenchel** transform of $I(j)$: the **scaled cumulant generating function**

$$F(\lambda) = \lim_{t\to\infty} t^{-1} \log\langle e^{\lambda J^{(t)}}\rangle = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} c_n.$$

$F(\lambda)$ is (up to sign) a nonequilibrium analogue of the specific free energy. We have the scaling

$$\langle e^{\lambda J^{(t)}}\rangle \asymp e^{tF(\lambda)}$$

c_n are the cumulants of $J^{(t)}$ scaled by $1/t$. $c_1 = \bar{\text{J}}$ is the average current. In mesoscopic physics, c_2 is the zero-frequency noise.

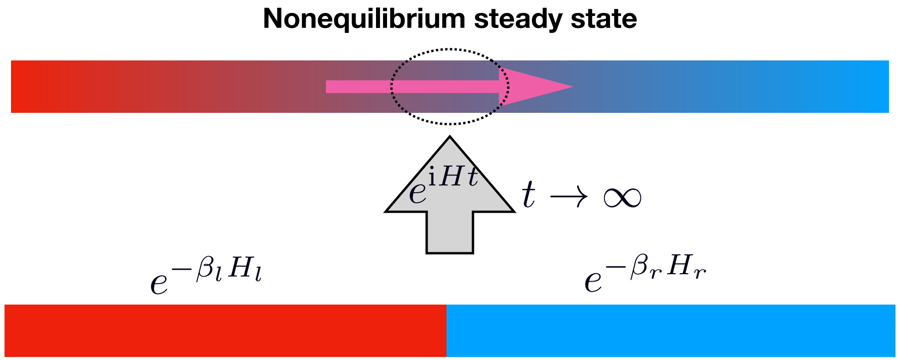
Universal behaviours: large deviations / large space-time separations

On more general rays, the large deviation principle implies the **exponential behaviour of twist field correlation functions at large space-time separations**

$$\langle e^{\lambda\varphi_i(0,0)}e^{-\lambda\varphi_i(\xi t,t)}\rangle \asymp e^{tF(\lambda;\xi)},\quad t\to\infty$$

This large-t behaviour holds also in the quantum case, despite the lack of commutativity.

Nonequilibrium steady states and fluctuation relations



In nonequilibrium steady states, **nonequilibrium fluctuation relations** relate probabilities for rare events where stuff goes in the "wrong" direction to those where it goes in the "right" direction

$$\frac{\mathbf{P}_{\text{ness}}(J^{(t)} = tj)}{\mathbf{P}_{\text{ness}}(J^{(t)} = -tj)} \asymp e^{t(\beta_{\mathrm{r}}-\beta_{\mathrm{l}})j} \qquad \Leftrightarrow \qquad F_{\text{ness}(\beta_{\mathrm{r}},\beta_{\mathrm{l}})}(\beta_{\mathrm{l}} - \beta_{\mathrm{r}} - \lambda) = F_{\text{ness}(\beta_{\mathrm{r}},\beta_{\mathrm{l}})}(\lambda)$$

In 1dCFT and in free models, the stronger "extended fluctuation relation" holds [Bernard, Doyon 2012, 2013]

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} F_{\text{ness}(\beta_{\mathrm{r}},\beta_{\mathrm{l}})}(\lambda) = \langle j_i \rangle_{\text{ness}(\beta_{\mathrm{r}}+\lambda/2,\beta_{\mathrm{l}}-\lambda/2)}$$

Main result: ballistic fluctuation framework

[Myers, Bhaseen, Harris, Doyon 2019; Doyon, Myers 2019]

A general framework giving an **exact result** for $F(\lambda)$ (scaled cumulant generating function) and more generally $F(\lambda;\xi)$ (exponential behaviour of twist field two-point function), in any homogeneous, stationary **maximal-entropy state**, such as (G)GEs, NESSs and all states discussed in the examples.

Main result: ballistic fluctuation framework

Maximal entropy states (MES): a Lagrange parameter β^j for every conserved quantity Q_j,

$$e^{-\sum_j \beta^j Q_j} \quad \leftrightarrow \quad \langle \cdots \rangle_{\underline{\beta}}$$

$\underline{\beta}$ is a coordinate on the MES manifold.

But there's no need for this explicit distribution. Can define β^j as coordinates associated to conserved charges Q_j, which form vector in the tangent space of the MES manifold. The tangent space is the Hilbert space of pseudolocal charges [Doyon 2017]

$$-\frac{\partial}{\partial \beta^j} \langle \mathcal{O}(0,0) \rangle_{\underline{\beta}} = (q_j, \mathcal{O})_{\underline{\beta}}$$

where

$$(\mathcal{O}, \mathcal{O}')_{\underline{\beta}} = \int_{-\infty}^{\infty} \mathrm{d}x \, \langle \mathcal{O}(x,0) \mathcal{O}'(0,0) \rangle_{\underline{\beta}}^{\mathrm{c}}$$

Main results: ballistic fluctuation framework

Consider set of averages of all local or quasi-local densities and associated fluxes

$$\mathsf{q}_j = \mathsf{q}_j(\underline{\beta}) = \langle q_j(0,0)\rangle_{\underline{\beta}}, \qquad \mathsf{j}_j = \mathsf{j}_j(\underline{\beta}) = \langle j_j(0,0)\rangle_{\underline{\beta}}$$

Seeing fluxes as functions of densities $\mathsf{j}_j = \mathsf{j}_j(\underline{\mathsf{q}})$, calculate the so-called **flux Jacobian**

$$\mathsf{A}_j^{\ k} = \mathsf{A}_j^{\ k}(\underline{\beta}) = \frac{\partial \mathsf{j}_j}{\partial \mathsf{q}_k} = \sum_l \frac{\partial \mathsf{j}_j}{\partial \beta_l}\frac{\partial \beta_l}{\partial \mathsf{q}_k}$$

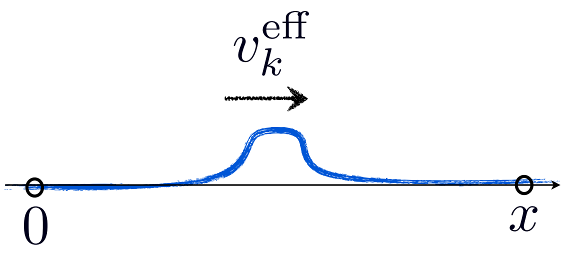
Main results: ballistic fluctuation framework

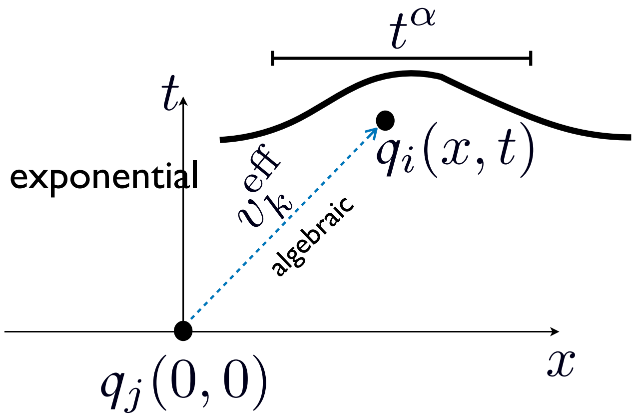
This is a (model-specific) matrix that is a function of the MES, hence **a function of $\underline{\beta}$**. It enters into the Euler equations of the model, and controls propagation of perturbations and Euler-scale two-point functions:

$$\partial_t \langle q_j(x,t)\mathcal{O}(0,0)\rangle + \mathsf{A}_j^{\,k}\partial_x\langle q_k(x,t)\mathcal{O}(0,0)\rangle = 0$$

The fluid's normal modes and their velocities are the obtained by diagonalising A:

$$\mathsf{A} = \mathsf{M} v^{\text{eff}} \mathsf{M}^{-1}$$





Main results: ballistic fluctuation framework

Define a **flow $\lambda \mapsto \underline{\beta}(\lambda)$ on the manifold of MES** by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\beta^j(\lambda) = -\,\mathrm{sgn}\left[\mathsf{A}(\underline{\beta}(\lambda))\right]_i^{\ j}$$

where

$$\mathrm{sgn}\left[\mathsf{A}\right] = \mathsf{M}\,\mathrm{sgn}(v^{\mathrm{eff}})\mathsf{M}^{-1}$$

Main result: We identify the flow parameter λ with the conjugate parameter generating the cumulants. Recall

$$F(\lambda) = \lim_{t\to\infty} t^{-1}\log\langle e^{\lambda J^{(t)}}\rangle_{\underline{\beta}}$$

We set the initial condition of the flow as $\underline{\beta}(0) = \underline{\beta}$. Then we have

$$F(\lambda) = \int_0^{\lambda} \mathrm{d}\mu\, \mathsf{j}_i(\underline{\beta}(\mu))$$

Main results: ballistic fluctuation framework

Putting all together, the main formula is

$$\left\langle \exp\left[\lambda \int_0^t \mathrm{d}s\, j_i(0,s)\right]\right\rangle_{\underline{\beta}} \asymp \exp\left[t \int_0^\lambda \mathrm{d}\mu\, \langle j_i(0,0)\rangle_{\underline{\beta}(\mu)}\right]$$

with

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\beta^j(\lambda) = -\,\mathrm{sgn}\left[\mathsf{A}(\underline{\beta}(\lambda))\right]_i^{\ j}, \qquad \underline{\beta}(0) = \underline{\beta}$$

Main results: ballistic fluctuation framework

This can be extended to other rays. Written in terms of twist fields:

$$\langle e^{\lambda\varphi_i(0,0)} e^{-\lambda\varphi_i(\xi t,t)}\rangle_{\underline{\beta}} \asymp \exp\left[t\int_0^\lambda \mathrm{d}\mu\,\langle j_i(0,0) - \xi q_i(0,0)\rangle_{\underline{\beta}(\mu;\xi)}\right]$$

with

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\beta^j(\lambda;\xi) = -\,\mathrm{sgn}\left[\mathsf{A}(\underline{\beta}(\lambda;\xi)) - \xi\mathbf{1}\right]_i^{\ j}, \qquad \underline{\beta}(0;\xi) = \underline{\beta}$$

Application to ℓ-TASEP

A single local conserved density: the density of particles ρ. The current can be obtained in terms of this [Schoenherr, Schuetz 2004]:

$$\mathsf{j} = p\frac{\rho(1-\ell\rho)}{1-(\ell-1)\rho}$$

The Lagrange parameter β can also be related to the density:

$$e^{\beta} = \frac{(1-\ell\rho)^{\ell}}{\rho(1-(\ell-1)\rho)^{\ell-1}}$$

We calculate $\mathsf{A} = \partial\mathsf{j}/\partial\rho$ and then solve the flow equation. It is trivial here as $\mathrm{sgn}(\mathsf{A})$ is constant, except at the value of ρ where A changes sign (the normal-mode velocity is zero), which separates phases [Erdmann-Pham, Duc, Song 2018].

At $\ell = 1$ the obtained $F(\lambda)$ agrees with previous calculations [de Gier, Essler 2011; Lazarescu, Mallick 2011]. For $\ell > 1$, new resuts, for instance, second cumulant is

$$c_2 = p\frac{\rho(\ell\rho-1)(\ell\rho((\ell-1)\rho-2)+1)}{(\ell-1)\rho-1}$$

“Free” systems and constant flux Jacobian

If A is **independent of the state**, then it is a simple matter to solve for the flow:

$$\beta^j(\lambda) = \beta^j - \lambda\,\mathrm{sgn}(\mathsf{A})_i^{\ j}$$

That is, the flow corresponds to a shift of the Lagrange parameters proportional to λ. In particular

$$F(\lambda) = \int_0^\lambda \mathrm{d}\mu\, \langle j_i \rangle_{\{\beta^\bullet - \mu\,\mathrm{sgn}(\mathsf{A})_i^{\ \bullet}\}}$$

When applied to NESSs, this leads to the **extended fluctuation relations**

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} F_{\mathrm{ness}(\beta_{\mathrm{r}},\beta_{\mathrm{l}})}(\lambda) = \langle j_i \rangle_{\mathrm{ness}(\beta_{\mathrm{r}}+\lambda/2,\beta_{\mathrm{l}}-\lambda/2)}$$

Application to integrable systems

The exact Euler hydrodynamics of the hard rod system was obtained in [Boldrighini, Dobrushin, Sukhov 1983], and for integrable systems in [Castro-Alvaredo, Doyon, Yoshimura 2016; Bertini, Collura, De Nardis, Fagotti 2016]. The flux Jacobian was first evaluated in [Doyon, Spohn 2017]. There is a common formalism, TBA-like, which works for quantum and classical systems.

Application to integrable systems

In this formalism, we know how to **diagonalise the flux Jacobian**. For any "spanning set" of functions $h_k(\theta)$ of the "rapidity" θ,

$$\mathsf{A}_j^{\;k} h_k^{\rm dr}(\theta) = v^{\rm eff}(\theta) h_j^{\rm dr}(\theta)$$

where $v^{\rm eff}(\theta) = (E')^{\rm dr}(\theta)/(p')^{\rm dr}(\theta)$, where $p(\theta)$ and $E(\theta)$ are momentum and energy of a quasiparticle $h^{\rm dr}(\theta) = h(\theta) + \int \frac{{\rm d}\alpha}{2\pi}\varphi(\theta,\alpha)n(\alpha)h^{\rm dr}(\alpha)$, where $n = {\rm d}\mathcal{F}(\epsilon)/{\rm d}\epsilon$ is the occupation, where $\mathcal{F}(\epsilon)$ is the non-interacting free energy of a mode of energy ϵ, and $\epsilon(\theta) = \sum_j \beta^j h_j(\theta) + \int \frac{{\rm d}\alpha}{2\pi}\,\varphi(\theta,\alpha)F(\epsilon(\alpha))$ where $\varphi(\theta,\alpha)$ is the differential scattering phase.

For the hard rods: set $\theta = v$ the velocity, and $\varphi(\theta,\alpha) = -\ell$ and $\mathcal{F}(\epsilon) = -e^{-\epsilon}$ and velocity distribution $\rho(v) = (1-\ell\rho)/2\pi\, n(v)$ where ρ is the total density per unit length.

For the quantum sinh-Gordon model (e.g. at the self-dual point): set $\varphi(\theta,\alpha) = 1/\cosh(\theta-\alpha)$ and $\mathcal{F}(\epsilon) = -\log(1+e^{-\epsilon})$, and $h_j(\theta)$ are one-particle eigenvalues of a spanning set of conserved charges.

Application to integrable systems

Consider transport problem ($\xi = 0$) for simplicity. We get the **flow equation for the pseudoenergy**

$$\partial_\lambda \epsilon(\theta;\lambda) = -\,\mathrm{sgn}(v^{\mathrm{eff}}(\theta;\lambda))h_i^{\mathrm{dr}}(\theta;\lambda).$$

An expression for $F(\lambda)$ can be obtained:

$$F(\lambda) = -\int \frac{\mathrm{d}\theta}{2\pi}\, E'(\theta) \Big(\mathrm{sgn}\left(v^{\mathrm{eff}}(\theta;\lambda)\right) \left(\mathscr{F}(\epsilon(\theta;\lambda)) - \mathscr{F}(\epsilon(\theta;0))\right) - \sum_{\sigma\in\{\pm\}} \sum_{\tilde\lambda \in \lambda_\star^\sigma(\theta)\cap[0,\lambda]} \sigma \big(\mathscr{F}(\epsilon(\theta;\tilde\lambda)) - \mathscr{F}(\epsilon(\theta;0))\big)\Big),$$

where the sets $\lambda_\star^\pm(\theta)$ are the turning points of the sign of the effective velocity:

$$\lambda_\star^\pm(\theta) = \{\lambda : v^{\mathrm{eff}}(\theta;\lambda) = 0,\ \partial_\lambda v^{\mathrm{eff}}(\theta;\lambda) \gtrless 0\}.$$

- This reduces to the Levitov-Lesovik formula in free-particle models, where $\varphi(\theta,\alpha) = 0$.
- This reproduces the known exponential behaviour, both in time-like and space-like regimes, of $\langle \sigma_+(0,0)\sigma_-(x,t)\rangle$ in XX chain (again $\varphi(\theta,\alpha) = 0$) [Itz, Izergin, Korepin, Slavnov 1993].

Application to integrable systems

From this the cumulants can be obtained, for instance

$$c_2 = \int \mathrm{d}\theta\, \rho_{\mathrm{p}} |v^{\mathrm{eff}}| (h_i^{\mathrm{dr}})^2 f,$$

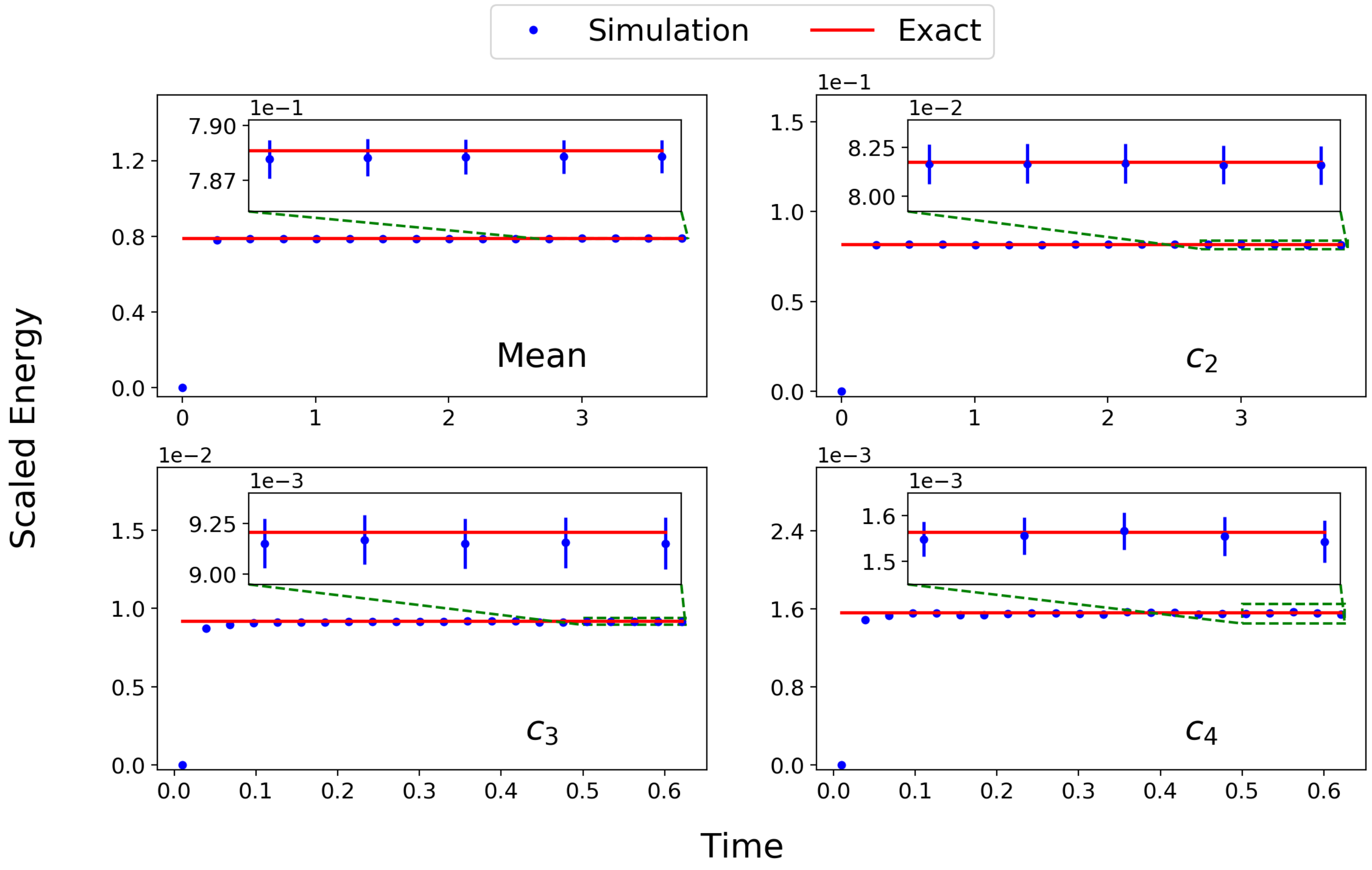
where $f = -\mathrm{d}\log n/\mathrm{d}\epsilon$ is the statistical factor, in agreement with [Doyon, Spohn 2017], and

$$c_3 = \int \mathrm{d}\theta\, \rho_{\mathrm{p}} f |v^{\mathrm{eff}}| h_i^{\mathrm{dr}} \left[(h_i^{\mathrm{dr}})^2 \tilde{f} s + 3 \left((h_i^{\mathrm{dr}})^2 f s \right)^{\mathrm{dr}} \right]$$

where $\tilde{f} = -(\mathrm{d}\log f/\mathrm{d}\epsilon + 2f)$ and $s = \mathrm{sgn}(v^{\mathrm{eff}})$

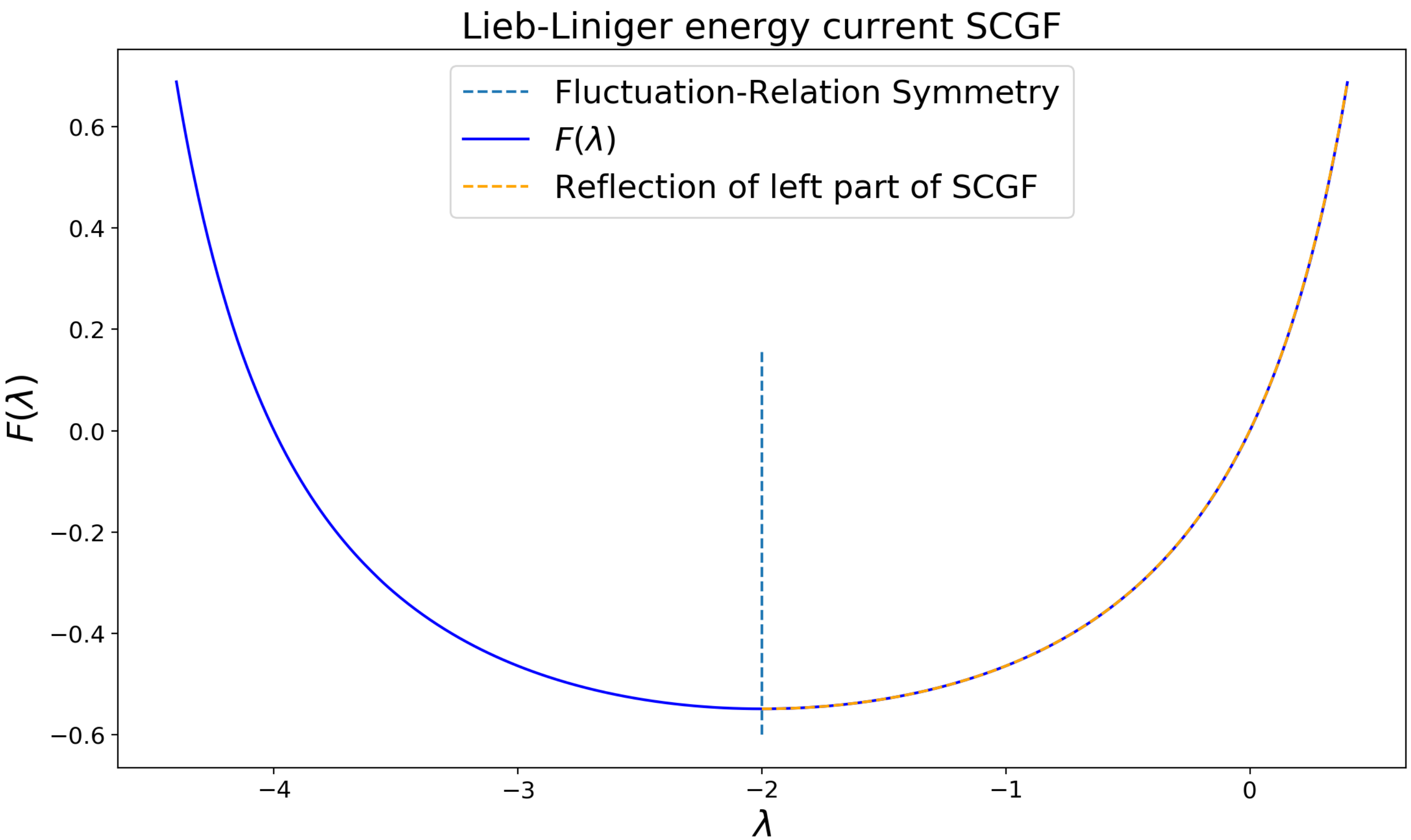
Application to integrable systems

We have performed numerical checks by simulating the hard rods



Application to integrable systems

We have checked numerically that the nonequilibrium fluctuation relations are satisfied in the NESSs found in [Castro-Alvaredo, Doyon, Yoshimura 2016; Bertini, Collura, De Nardis, Fagotti 2016]



Derivation: biasing the measure

Main assumption: clustering at large time differences. This is natural, but its breaking is a signal of a dynamical phase transition.

Modify state $\langle \cdots \rangle_{\underline{\beta}}$ by a time-integral of the current $j_i(0,t) \equiv j(0,t)$:

$$\langle \mathcal{O}(x,t) \rangle^{(\lambda)} = \frac{\langle e^{\lambda \int_{-\infty}^{\infty} \mathrm{d}t\, j(0,t)} \mathcal{O}(x,t) \rangle_{\underline{\beta}}}{\langle e^{\lambda \int_{-\infty}^{\infty} \mathrm{d}t\, j(0,t)} \rangle_{\underline{\beta}}}$$

Biasing the dynamics, changing the weights of trajectories in order to make rare events "typical" and access their probabilities.

Related to generalised (classical or quantum) Doob transformation [Chetrite, Touchette 2015; Carollo, Garrahan, Lesanovsky, Pérez-Espigares, 2018]

Derivation: biasing the measure

This state is useful to calculate $F(\lambda)$

$$\frac{\mathrm{d}F(\lambda)}{\mathrm{d}\lambda} = \lim_{t\to\infty} \frac{1}{t} \int_{-t/2}^{t/2} \mathrm{d}s \, \frac{\langle e^{\lambda \int_{-t/2}^{t/2} \mathrm{d}r \, j(0,r)} j(0,s) \rangle_{\underline{\beta}}}{\langle e^{\lambda \int_{-t/2}^{t/2} \mathrm{d}r \, j(0,r)} \rangle_{\underline{\beta}}}.$$

giving, neglecting boundaries in time,

$$\frac{\mathrm{d}F(\lambda)}{\mathrm{d}\lambda} = \langle j(0,0) \rangle^{(\lambda)}$$

Derivation: biasing the measure

Crucial observation: The state $\langle\cdots\rangle^{(\lambda)}$ is a MES. Therefore, there exists $\underline{\beta}(\lambda)$ such that

$$\langle\mathcal{O}\rangle^{(\lambda)} = \langle\mathcal{O}\rangle_{\underline{\beta}(\lambda)}.$$

With this observation,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\langle\mathcal{O}\rangle_{\underline{\beta}(\lambda)} = \int_{-\infty}^{\infty}\mathrm{d}t\,\langle j(0,t)\mathcal{O}\rangle^{\mathrm{c}}_{\underline{\beta}(\lambda)}$$

Choose conserved densities $q_j(0,0)$, get a flow for the q_j-coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathsf{q}_j(\lambda) = \int_{-\infty}^{\infty}\mathrm{d}t\,\langle j(0,t)q_j(0,0)\rangle^{\mathrm{c}}_{\underline{\beta}(\lambda)}$$

The right-hand side is a function of the state, hence of the state coordinates $\underline{\mathsf{q}}(\lambda)$.

Derivation: biasing the measure

A result from **linear fluctuating hydrodynamics** is

[Spohn 1991, 2014; Mendl, Spohn 2015; Doyon, Spohn 2017]

$$\langle j_i(x,t)q_j(0,0)\rangle^{\mathrm{c}}_{\underline{\beta}} \sim (\mathsf{A}\delta(x-\mathsf{A}t)\mathsf{C})_{ij}$$

where the covariance matrix is

$$\mathsf{C}_{ij} = \int \mathrm{d}x\, \langle q_i(x,0)q_j(0,0)\rangle_{\underline{\beta}} = -\frac{\partial \mathsf{q}_j}{\partial \beta^i}$$

and $\delta(x-\mathsf{A}t) = \mathsf{M}\delta(x-v^{\mathrm{eff}}t)\mathsf{M}^{-1}$. Integrating over time, the result is independent of x:

$$\int_{-\infty}^{\infty} \mathrm{d}t\, \langle j_i(0,t)q_j(0,0)\rangle^{\mathrm{c}}_{\underline{\beta}} = (\mathrm{sgn}[\mathsf{A}]\,\mathsf{C})_{ij}$$

Also

$$\frac{\mathrm{d}\mathsf{q}_j}{\mathrm{d}\lambda} = \sum_k \frac{\mathrm{d}\beta^k}{\mathrm{d}\lambda}\frac{\mathrm{d}\mathsf{q}_j}{\mathrm{d}\beta_k} = -\sum_k \frac{\mathrm{d}\beta^k}{\mathrm{d}\lambda}\mathsf{C}_{jk}$$

This gives the result.

Dynamical phase transitions

Recall

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\beta^j(\lambda) = -\,\mathrm{sgn}\left[\mathsf{A}(\underline{\beta}(\lambda))\right]_i^{\ j}$$

where

$$\mathrm{sgn}\left[\mathsf{A}\right] = \mathsf{M}\,\mathrm{sgn}(v^{\mathrm{eff}})\mathsf{M}^{-1}$$

Cumulants are

$$c_1 = \mathsf{j}_i(0), \quad c_2 = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathsf{j}_i(\lambda)\Big|_{\lambda=0}, \quad c_3 = \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\mathsf{j}_i(\lambda)\Big|_{\lambda=0}$$

Around points where some $\mathrm{sgn}(v_j^{\mathrm{eff}})$ changes sign: **discontinuities of c_n, $n \geq 3$**, and **possible divergencies exactly at these points**.

- At $v_i^{\mathrm{eff}} = 0$, “stationary” normal modes create strong time-correlations which break the time-clustering hypothesis.
- Divergence of higher cumulants suggest that the leading Gaussianity (law of large numbers) is broken. This is where **nonlinear fluctuating hydrodynamics** [Spohn 2014], should be applied instead.

Conclusion

We have obtained a framework for evaluating large-deviation properties of current fluctuations / twist-field correlation functions along arbitrary rays in space-time in a large family of states, including GGEs and NESSs.

- More analysis near to / at phase transitions and relations to NLFHD
- Studying consequences on correlation functions of vertex operators in sine-Gordon model in GGEs and NESSs
- Can we get a full fluctuation theory for ballistic transport akin to macroscopic fluctuation theory?