Coloured vertex models and non-symmetric Macdonald polynomials

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References

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- **Alexei Borodin** and **MW**, "*Nonsymmetric Macdonald polynomials via integrable vertex models*", `arXiv:1904.06804`.

Outline

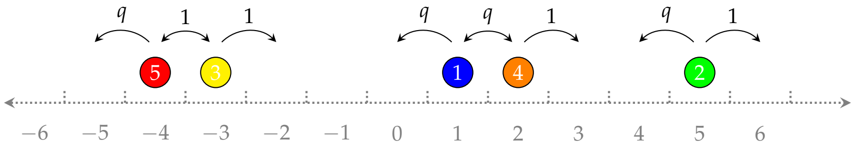
- Motivational examples (multi-species ASEP and coloured bosons).

- 1: Non-symmetric rational functions and their properties.
  - Fundamental $U_q(\widehat{\mathfrak{sl}_{n+1}})$ vertex model.
  - Non-symmetric rational functions f_μ and g_μ.
  - Transfer matrix $G_{\mu/\nu}$.
  - Summation identities involving f_μ, g_μ and $G_{\mu/\nu}$.
  - Orthogonality relations.
  - An integral formula for $G_{\mu/\nu}$ and the ASEP limit.

- 2: Non-symmetric Hall–Littlewood polynomials and distribution matches.
  - Fusion of the $U_q(\widehat{\mathfrak{sl}_{n+1}})$ vertex model.
  - Non-symmetric Hall–Littlewood polynomials, E_μ.
  - Hall–Littlewood processes.
  - A remarkable distribution match.

- 3: Non-symmetric Macdonald polynomials; no time today!
  - Cherednik–Dunkl operators and their eigenaction.
  - A deformation of the partition function for non-symmetric Hall–Littlewood polynomials.
  - Lattice model construction of non-symmetric Macdonald polynomials.

Motivation: the multi-species ASEP



- A configuration of the multi-species ASEP is given by

$$\eta(t) = \{\ldots, \eta_{-1}(t), \eta_0(t), \eta_1(t), \ldots\}, \qquad \eta_i(t) \in \mathbb{N}.$$

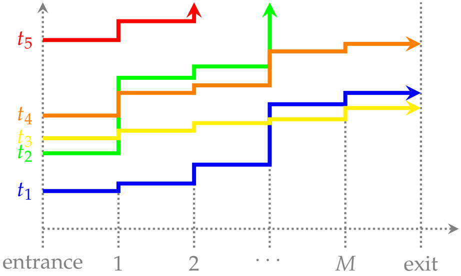
- 0 indicates an unoccupied site, and an integer $j \geqslant 1$ indicates a particle of colour j.
- Let $\{\eta_{i-1}, \eta_i, \eta_{i+1}\}$ be the occupation data for the site i and its two neighbours, at some point in time. This site is assigned two exponential clocks: a left clock of rate $q \cdot \mathbf{1}_{\eta_i > \eta_{i-1}}$ and a right clock of rate $\mathbf{1}_{\eta_i > \eta_{i+1}}$.
- When the left clock rings, the occupation data is updated as

$$\{\eta_{i-1}, \eta_i, \eta_{i+1}\} \mapsto \{\eta_i, \eta_{i-1}, \eta_{i+1}\}.$$

- When the right clock rings, the occupation data is updated as

$$\{\eta_{i-1}, \eta_i, \eta_{i+1}\} \mapsto \{\eta_{i-1}, \eta_{i+1}, \eta_i\}.$$

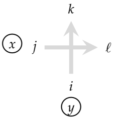
Motivation: coloured bosons



- Particles arrive at site 1 of the lattice according to a Poisson process of rate r_0. The i-th particle to enter the system is assigned colour $t_i \in \mathbb{R}_{>0}$, where t_i is its time of entrance.
- There is an exponential clock of rate $r_i \in \mathbb{R}_{>0}$ attached to the i-th site of the lattice, for each $1 \leqslant i \leqslant M$.
- When the i-th clock rings, $1 \leqslant i \leqslant M - 1$, the particle with maximal colour at position i hops to position $i + 1$.
- When the M-th clock rings, the particle with maximal colour at position M exits the system.

1: Fundamental $U_q(\widehat{\mathfrak{sl}_{n+1}})$ vertex model

- Our main focus will be a higher-rank extension of the six-vertex model. It is a model of vertices of the form

$$R_z(i,j;k,\ell) =$$ 

where $i,j,k,\ell \in \{0,1,\dots,n\}, \quad z = y/x.$

- The model has a basic conservation identity:

$$R_z(i,j;k,\ell) = 0, \qquad \text{unless} \quad \{i\} \cup \{j\} = \{k\} \cup \{\ell\}.$$

- The non-zero weights are simple rational functions of the spectral parameter z and quantum parameter q:

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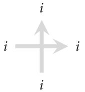
<td>

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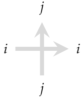
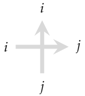
<td>

$\frac{q(1-z)}{1-qz}$</td>
<td>

$\frac{1-q}{1-qz}$</td>
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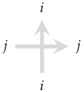
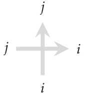
<tr>
<td>$0 \leqslant i < j \leqslant n$</td>
<td>

$\frac{1-z}{1-qz}$</td>
<td>

$\frac{(1-q)z}{1-qz}$</td>
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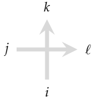
1: Fundamental $U_q(\widehat{\mathfrak{sl}_{n+1}})$ vertex model

- The model has a sum-to-unity property:

$$\sum_{0 \leqslant k, \ell \leqslant n}$$

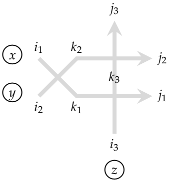


$$= 1, \quad \forall\, i, j \in \{0, 1, \ldots, n\}.$$

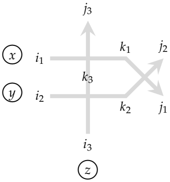
With mild assumptions on the parameters z and q (that ensure positivity), this allows us to construct discrete-time Markov processes from the model.

- The model satisfies the Yang–Baxter equation:

$$\sum_{0 \leqslant k_1, k_2, k_3 \leqslant n}$$



$$=\sum_{0 \leqslant k_1, k_2, k_3 \leqslant n}$$



for all fixed $i_1, \ldots, i_3, j_1, \ldots, j_3 \in \{0, 1, \ldots, n\}$.

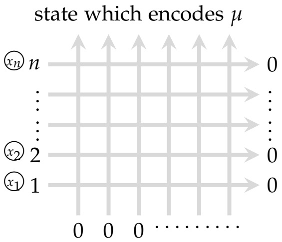
1: Non-symmetric rational functions f_μ and g_μ

- In what follows, all vertical lines carry a trivial rapidity variable $(=1)$:

1	$\dfrac{q(x-1)}{x-q}$	$\dfrac{(1-q)x}{x-q}$
$0 \leqslant i < j \leqslant n$	$\dfrac{x-1}{x-q}$	$\dfrac{(1-q)}{x-q}$

- Let $\mu = (\mu_1, \ldots, \mu_n)$ be a vector of pairwise-distinct natural numbers (a strict composition). We define f_μ as a partition function in our vertex model:

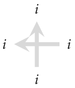
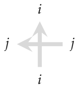
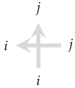
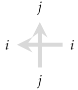
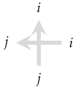
$f_\mu(x_1, \ldots, x_n) :=$



Colour i exits at position μ_i.

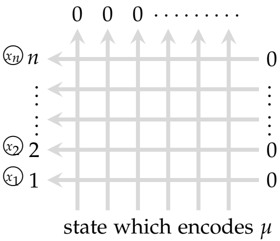
1: Non-symmetric rational functions f_μ and g_μ

- We can also consider 90° rotations of the vertices:

 1	 $\frac{q(1-x)}{1-qx}$	 $\frac{(1-q)}{1-qx}$
$0 \leqslant i < j \leqslant n$	 $\frac{1-x}{1-qx}$	 $\frac{(1-q)x}{1-qx}$

- We define g_μ as a partition function in our rotated vertex model:

$g_\mu(x_1, \ldots, x_n) :=$



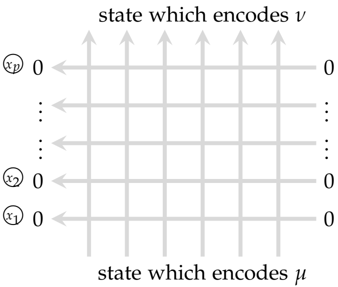
Colour i enters at position μ_i.

1: Transfer matrix $G_{\mu/\nu}$

- We will define one more type of partition function, which plays the role of a multivariate transfer matrix.

- Let $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$ be two strict, natural-number compositions, with $\mu_i \geqslant \nu_i$ for all $i \in \{1, \ldots, n\}$. We define $G_{\mu/\nu}$ as follows:

$G_{\mu/\nu}(x_1, \ldots, x_p) :=$



Colour i enters at position μ_i, and exits at position ν_i.

- The value of p should be sufficiently large, but is otherwise unrelated to n.

1: Summation identities

- Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_p)$ be two alphabets which satisfy the technical constraint

$$\left| \frac{x_i - 1}{x_i - q} \cdot q \cdot \frac{1 - y_j}{1 - q y_j} \right| < 1, \qquad \forall\, i \in \{1, \dots, n\}, \quad j \in \{1, \dots, p\}.$$

- We have the following skew Cauchy summation identity:

$$\sum_{\mu} f_{\mu}(x_1, \dots, x_n) G_{\mu/\nu}(y_1, \dots, y_p) = \prod_{i=1}^{n} \prod_{j=1}^{p} \frac{x_i - y_j}{x_i - q y_j} \cdot f_{\nu}(x_1, \dots, x_n),$$

where the summation is taken over all length-n, strict compositions $\mu = (\mu_1, \dots, \mu_n)$.

- The proof makes use of the Yang–Baxter equation of the model.

1: Summation identities

- Introduce a function g^*_μ which differs from g_μ by an overall multiplicative factor:

$$g^*_\mu(x_1,\dots,x_n) := \frac{q^{n(n+1)/2}}{(q-1)^n} \cdot g_\mu(x_1,\dots,x_n).$$

- Let $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ be two alphabets which satisfy the technical constraint

$$\left| \frac{x_i-1}{x_i-q} \cdot q \cdot \frac{1-y_j}{1-qy_j} \right| < 1, \qquad \forall\, i,j \in \{1,\dots,n\}.$$

- We have the following summation identity of Mimachi–Noumi type:

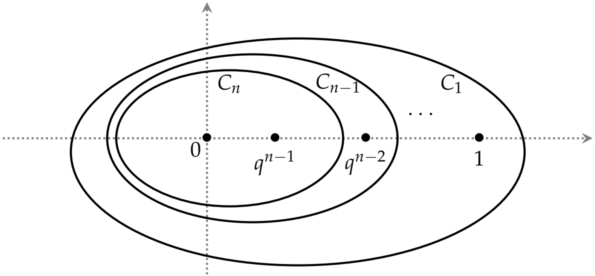
$$\sum_\mu f_\mu(x_1,\dots,x_n) g^*_\mu(y_1,\dots,y_n) = \prod_{i=1}^{n} \frac{qy_i}{qy_i-x_i} \prod_{n\geqslant i>j\geqslant 1} \frac{x_i-y_j}{x_i-qy_j},$$

  where the summation is over all length-n, strict compositions $\mu = (\mu_1,\dots,\mu_n)$.

- The proof makes use of the Yang–Baxter equation of the model.

1: Orthogonality

- Consider n positively-oriented, q-nested integration contours as shown below:



- The rational functions f_μ and g^*_ν satisfy the following orthogonality relation:

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \oint_{C_1} \frac{dx_1}{x_1} \cdots \oint_{C_n} \frac{dx_n}{x_n} \prod_{1 \leqslant i < j \leqslant n} \left(\frac{x_i - x_j}{x_i - q x_j}\right) f_\mu(qx_1, \dots, qx_n) g^*_\nu(x_1, \dots, x_n) = \mathbf{1}_{\mu=\nu}.$$

- The proof is long and complicated!

1: Spectral decomposition of $G_{\mu/\nu}$

- It is now very easy to combine the skew Cauchy identity and the orthogonality relation to obtain the following formula for our transfer matrix:

$$G_{\mu/\nu}(y_1,\dots,y_p) = \frac{q^{-np}}{(2\pi\sqrt{-1})^n}\oint_{C_1}\frac{dx_1}{x_1}\cdots\oint_{C_n}\frac{dx_n}{x_n}$$
$$\prod_{1\leqslant i<j\leqslant n}\left(\frac{x_i-x_j}{x_i-qx_j}\right)\prod_{i=1}^{n}\prod_{j=1}^{p}\frac{qx_i-y_j}{x_i-y_j}f_\nu(qx_1,\dots,qx_n)g^*_\mu(x_1,\dots,x_n),$$

  with the same integration contours as previously.

- So far all rational functions were indexed by natural-number compositions. It is easy to extend to integer compositions by noting a simple shift property of the functions:

  For all $\mu \in \mathbb{N}^n$ and $k > 0$ there holds

$$f_{\mu+k^n}(x_1,\dots,x_n) = \prod_{i=1}^{n}\left(\frac{x_i-1}{x_i-q}\right)^k f_\mu(x_1,\dots,x_n),$$
$$g^*_{\mu+k^n}(x_1,\dots,x_n) = \prod_{i=1}^{n}\left(q\cdot\frac{1-x_i}{1-qx_i}\right)^k g^*_\mu(x_1,\dots,x_n).$$

1: Spectral decomposition of $G_{\mu/\nu}$

- For strict compositions $\mu \in \mathbb{Z}^n$, we then define

$$f_\mu(x_1,\dots,x_n) := \prod_{i=1}^{n} \left(\frac{x_i - q}{x_i - 1}\right)^k f_{\mu+k^n}(x_1,\dots,x_n),$$

$$g^*_\mu(x_1,\dots,x_n) := \prod_{i=1}^{n} \left(q^{-1}\cdot\frac{1-qx_i}{1-x_i}\right)^k g^*_{\mu+k^n}(x_1,\dots,x_n),$$

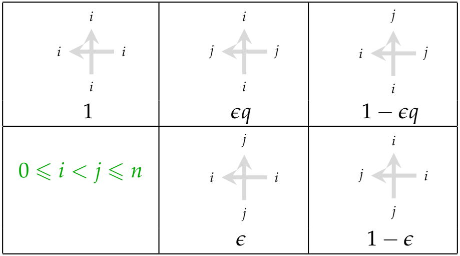
  where k is sufficiently large such that $\mu + k^n \in \mathbb{N}^n$.

- The stated formula for $G_{\mu/\nu}$ continues to hold for arbitrary strict compositions $\mu,\nu \in \mathbb{Z}^n$:

$$G_{\mu/\nu}(y_1,\dots,y_p) = \frac{q^{-np}}{(2\pi\sqrt{-1})^n} \oint_{C_1} \frac{dx_1}{x_1} \cdots \oint_{C_n} \frac{dx_n}{x_n} \prod_{1\leqslant i<j\leqslant n} \left(\frac{x_i - x_j}{x_i - qx_j}\right) \prod_{i=1}^{n}\prod_{j=1}^{p} \frac{qx_i - y_j}{x_i - y_j} f_\nu(qx_1,\dots,qx_n) g^*_\mu(x_1,\dots,x_n).$$

1: Reduction to ASEP

- Let us study the 90°-rotated vertex weights in the case when all rapidity variables take the form $x = 1 - \epsilon(1-q)$, where $\epsilon > 0$ is small. The weights become



  where we only show $O(\epsilon)$ dependence.

- Fix a parameter $t \in \mathbb{R}_{>0}$ and two strict compositions $\mu, \nu \in \mathbb{Z}^n$. We find that

$$\lim_{\epsilon \to 0} G_{\mu/(\nu - p^n)}(y_1, \dots, y_p)\Big|_{p \mapsto t/\epsilon}\Big|_{y_i \mapsto 1-\epsilon(1-q)} = \mathbb{P}_t^{\text{ASEP}}(\mu \to \nu),$$

  where the right hand side is the probability of being in state ν at time t, given that the system was initialized in state μ.

1: Reduction to ASEP

- Performing this limit in the explicit formula for $G_{\mu/(\nu-p^n)}$, we obtain all ASEP transition probabilities:

$$\mathbb{P}_t^{\mathrm{ASEP}}(\mu \to \nu) = \frac{1}{(2\pi\sqrt{-1})^n} \oint_{C_1} \frac{dx_1}{x_1} \cdots \oint_{C_n} \frac{dx_n}{x_n}$$

$$\prod_{1 \leqslant i < j \leqslant n} \left(\frac{x_i - x_j}{x_i - q x_j} \right) \prod_{i=1}^{n} \exp\left[\frac{(1-q)^2 x_i t}{(1-x_i)(1-q x_i)} \right] f_\nu(q x_1, \ldots, q x_n) g^*_\mu(x_1, \ldots, x_n).$$

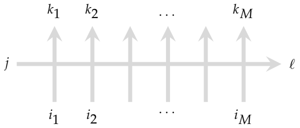
- Explicit formulae for f_ν and g^*_μ (as sums over the symmetric group) exist, but are rather complicated.

  If we assume that $\mu_1 > \cdots > \mu_n$ and $\nu_1 < \cdots < \nu_n$ (so that the particles have totally reversed their order after time t), then both f_ν and g^*_μ factorize.

2: Fusion and higher-spin vertex models

- The fusion procedure (in the sense of Kulish–Reshetikhin–Sklyanin) makes use of rows of vertices in the $U_q(\widehat{\mathfrak{sl}_{n+1}})$ model:

$$R_{y/x}\Big((i_1,\ldots,i_M),j;(k_1,\ldots,k_M),\ell\Big) :=$$



where the spectral parameters form a geometric progression in q:

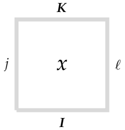
$$\left(q^{M-1}\frac{y}{x},\ldots,q\frac{y}{x},\frac{y}{x}\right).$$

- From this, one defines

$$\mathcal{L}^{(M)}_{y/x}(\boldsymbol{I},j;\boldsymbol{K},\ell) := \frac{1}{Z_q(M;\boldsymbol{I})}\sum_{\substack{\mathcal{C}(i_1,\ldots,i_M)=\boldsymbol{I}\\ \mathcal{C}(k_1,\ldots,k_M)=\boldsymbol{K}}} q^{\mathrm{inv}(i_1,\ldots,i_M)} R_{y/x}\Big((i_1,\ldots,i_M),j;(k_1,\ldots,k_M),\ell\Big).$$

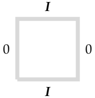
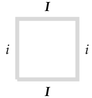
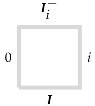
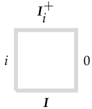
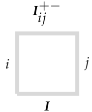
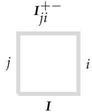
2: Fusion and higher-spin vertex models

- The vertex weights that one obtains in this way depend on M only via q^M. One may analytically continue q^M to $\mathbb{C}$, writing $q^M = s^{-2}$.
- We then consider the following sequence of limits:



$$:= (-s)^{-\mathbf{1}_{\ell>0}} \mathcal{L}^{(M)}_{y/x}(\boldsymbol{I}, j; \boldsymbol{K}, \ell)\Big|_{y\to s}\Big|_{q^M \to s^{-2}}\Big|_{s\to 0}.$$

- The weights of this model can be computed explicitly:

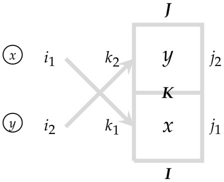
 1	 $xq^{\boldsymbol{I}_{[i+1,n]}}$	 $x(1-q^{I_i})q^{\boldsymbol{I}_{[i+1,n]}}$
 1	 $x(1-q^{I_j})q^{\boldsymbol{I}_{[j+1,n]}}$	 0

where it is assumed that $1 \leqslant i < j \leqslant n$.

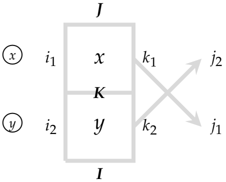
2: Fusion and higher-spin vertex models

- The model satisfies the following Yang–Baxter (RLL) equation:

$$\sum_{0\leqslant k_1,k_2\leqslant n} \sum_{\boldsymbol{K}\in\mathbb{N}^n}$$



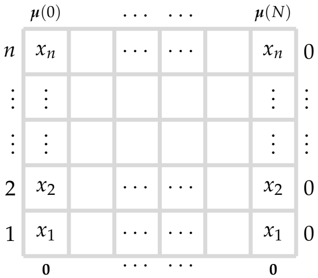
$$=\sum_{0\leqslant k_1,k_2\leqslant n} \sum_{\boldsymbol{K}\in\mathbb{N}^n}$$



2: Non-symmetric Hall–Littlewood polynomials, E_μ

- In a similar vein to before, we define

$$E_\mu(x_1,\dots,x_n) :=$$



- Here we have defined

$$N = \max_{1 \leqslant i \leqslant n} (\mu_i), \qquad \boldsymbol{\mu}(j) = \sum_{i=1}^{n} \mathbf{1}(\mu_i = j)\boldsymbol{e}_i.$$

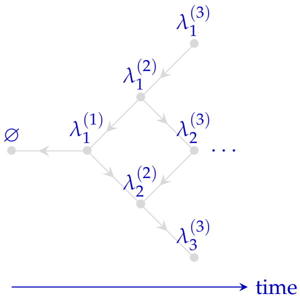
- These functions obey two (uniquely-determining) properties:

$$E_\delta(x_1,\dots,x_n) = \prod_{i=1}^{n} x_i^{\delta_i}, \qquad \delta = (\delta_1 \leqslant \cdots \leqslant \delta_n).$$

$$T_i \cdot E_\mu(x_1,\dots,x_n) = E_{\mathfrak{s}_i \cdot \mu}(x_1,\dots,x_n), \qquad \mu_i < \mu_{i+1}, \qquad T_i := q - \frac{x_i - q x_{i+1}}{x_i - x_{i+1}}(1 - \mathfrak{s}_i).$$

2: Hall–Littlewood processes

- The ascending Hall–Littlewood process is a discrete-time Markov process of growing Young diagrams:



- Alternatively, it can be viewed as a probability measure on Gelfand–Tsetlin patterns:

$$\mathbb{P}^{\mathrm{HL}}_{m,n}\left(\lambda^{(1)} \prec \cdots \prec \lambda^{(n)}\right) :=$$

$$\left(\prod_{i=1}^{n} Q_{\lambda^{(i)}/\lambda^{(i-1)}}(x_i;q)\right) P_{\lambda^{(n)}}(y_1,\dots,y_m;q) \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{1-x_i y_j}{1-q x_i y_j}.$$

2: Non-symmetric Hall–Littlewood processes

- One can extend this probability measure to the non-symmetric case, as follows:

$$\mathbb{P}^{\mathrm{ns}}_{m,n}(\mu, \boldsymbol{\lambda}) := Q_{\mu^+/\lambda^{(1)}}(x_1) \cdot \prod_{j=2}^{n} Q_{\lambda^{(j-1)}/\lambda^{(j)}}(x_j) \cdot E_\mu(y_1, \ldots, y_m) \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1 - x_i y_j}{1 - q x_i y_j}.$$

- The measure satisfies the sum-to-unity property:

$$\sum_{\mu} \sum_{\boldsymbol{\lambda}} \mathbb{P}^{\mathrm{ns}}_{m,n}(\mu, \boldsymbol{\lambda}) = 1.$$

- This can be proved by making use of the branching rule for symmetric Hall–Littlewood polynomials:

$$Q_\lambda(x_1, \ldots, x_n) = \sum_{\nu \prec \lambda} Q_{\lambda/\nu}(x_1) Q_\nu(x_2, \ldots, x_n),$$

  and the symmetrization property of non-symmetric Hall–Littlewood polynomials:

$$\sum_{\mu : \mu^+ = \lambda} E_\mu(y_1, \ldots, y_m) = P_\lambda(y_1, \ldots, y_m).$$

- One can proceed to compute the averages of observables by using the action of Cherednik–Dunkl operators.

2: A remarkable distribution match

- The observables that will interest us will be the zero set $z(\mu)$ of the composition $(\mu_1, \ldots, \mu_m)$, defined as

$$z(\mu) = \{1 \leqslant i \leqslant m : \mu_i = 0\},$$

and a further set

$$\zeta(\mu, \boldsymbol{\lambda}) = \{1 \leqslant j \leqslant n : \ell(\lambda^{(j-1)}) - \ell(\lambda^{(j)}) = 0\}, \qquad \lambda^{(0)} \equiv \mu^+, \qquad \lambda^{(n)} \equiv \varnothing,$$

which records the instances where neighbouring partitions in the extended Gelfand–Tsetlin pattern $\mu^+ \succ \lambda^{(1)} \succ \cdots \succ \lambda^{(n-1)} \succ \varnothing$ have the same length.

- In particular, we define

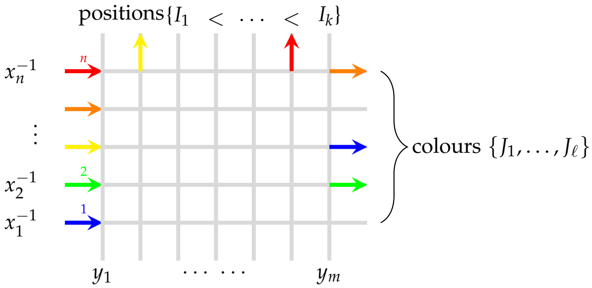
$$\mathbb{P}_{m,n}(\mathcal{I}, \mathcal{J}) := \sum_{\mu} \sum_{\boldsymbol{\lambda}} \mathbb{P}^{\mathrm{ns}}_{m,n}(\mu, \boldsymbol{\lambda}) \cdot \mathbf{1}_{z(\mu)=\bar{\mathcal{I}}} \cdot \mathbf{1}_{\zeta(\mu,\boldsymbol{\lambda})=\mathcal{J}},$$

which is the joint distribution of the random variables $z(\mu)$, $\zeta(\mu, \boldsymbol{\lambda})$ in the pair $(\mu, \boldsymbol{\lambda})$ sampled with respect to the non-symmetric Hall–Littlewood process.

2: A remarkable distribution match

- Compare this against the following probability distribution, of our starting vertex model in a quadrant:

$Z_{m,n}(\mathcal{I},\mathcal{J}) := \sum$



Theorem

Fix two integers $m, n \geqslant 1$ and two sets $\mathcal{I} = \{1 \leqslant I_1 < \cdots < I_k \leqslant m\}$ and $\mathcal{J} = \{1 \leqslant J_1 < \cdots < J_\ell \leqslant n\}$ whose cardinalities satisfy $k + \ell = n$. The following equality of distributions holds:

$$\mathbb{P}_{m,n}(\mathcal{I},\mathcal{J}) = Z_{m,n}(\mathcal{I},\mathcal{J}).$$