The Rule 54: Completely Solvable Statistical Mechanics Model of Deterministic Interacting Dynamics

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- Can matrix product ansatz be useful for encoding (time-dependent, or steady) states of deterministic reversible interacting systems?
- Find minimal interacting deterministic $(1+1)$d model about which we can 'know everything' (without approximations and assumptions)
- Check if the model has generic physical (say transport) properties!

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Outline

1. The model: Integrable reversible interacting cellular automaton (*Rule 54*)
   A.Bobenko, M.Bordemann, C.Gunn, U.Pinkall, CMP **158**,127(1993)

2. Rule 54 chain between stochastic soliton baths – steady state problem
   TP and C. Mejia-Monasterio, J. Phys. A **49**, 185003 (2016)
   see also: A. Inoue, S. Takesue, arXiv:1806.07099

3. Matrix product form of eigenvectors and diagionalization of Liouvillian
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4. Explicit matrix product form of time-dependent observables and
   analytical evaluation of dynamic structure factor,
   as well as the solution of inhomogeneous quench problem
   K.Klobas, M.Medenjak, TP, M.Vanicat, Commun.Math.Phys.(2019)

5. Exact large deviations for space-time extensive obervables in terms of an
   inhomogeneous martrix product ansatz
   B. Buča, J. P. Garrahan, TP, M. Vanicat, arXiv:1901.00845

6. **Extra:** New(?) integrable SO(3) symmetric classical-spin dynamics on
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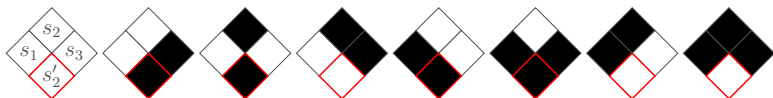
$$0 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 0 \times 2^3 + 1 \times 2^4 + 1 \times 2^5 + 0 \times 2^6 + 0 \times 2^7 = 54$$

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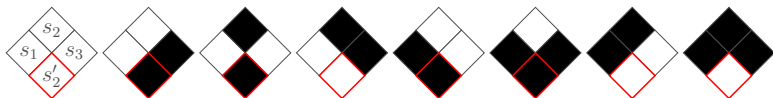

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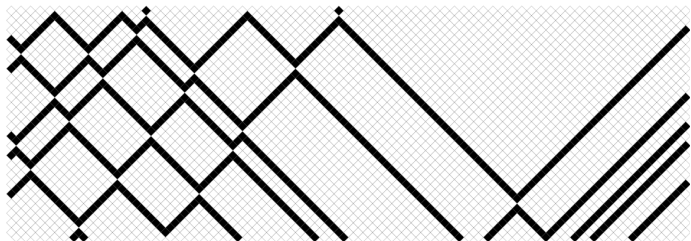Bobenko *et al.*, Commun. Math. Phys. **158,** 127 (1993)
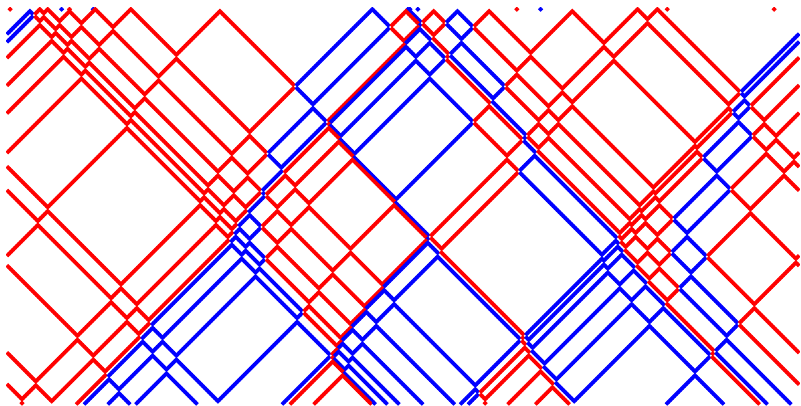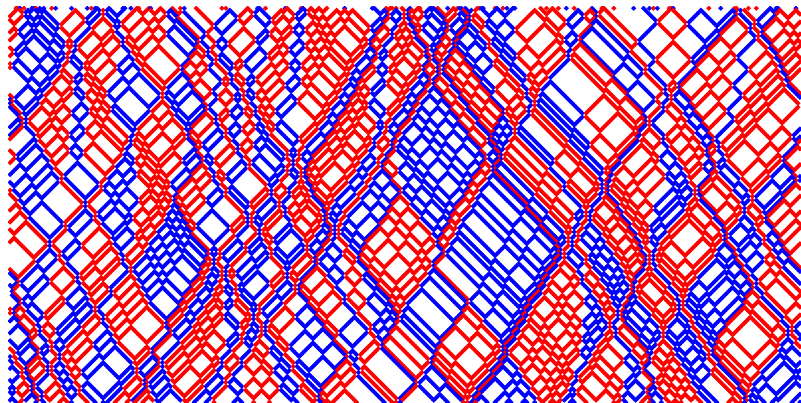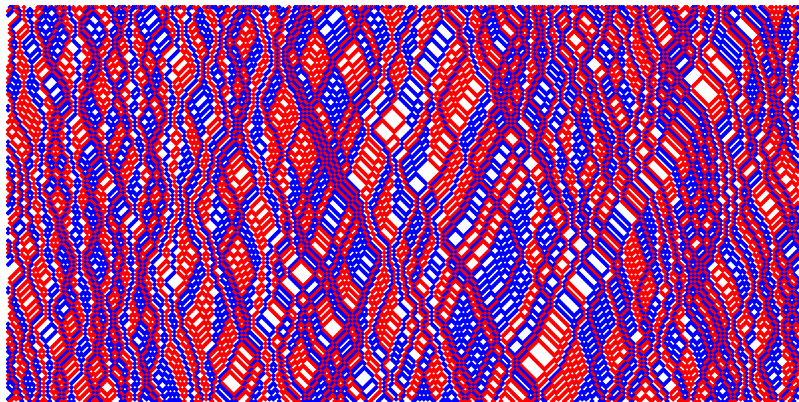
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Describe an evolution of *probability state vector* for $n-$cell automaton

$$\mathbf{p}(t) = U^t \mathbf{p}(0)$$

$$\mathbf{p} = (p_0, p_1, \ldots, p_{2^n-1}) \equiv (p_{s_1, s_2, \ldots, s_n}; s_j \in \{0, 1\})$$
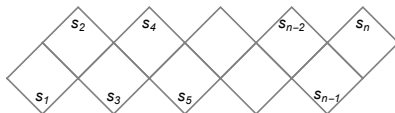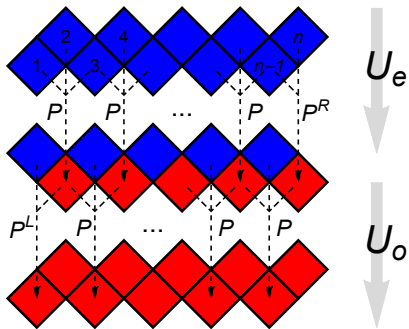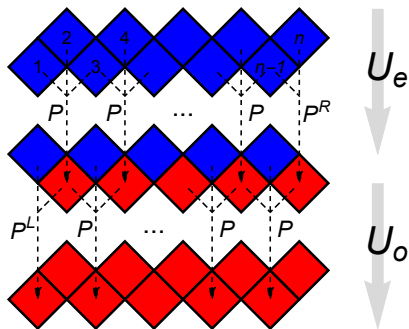
$U = U_\mathrm{o} U_\mathrm{e},$

$U_\mathrm{e} = P_{123} P_{345} \cdots P_{n-3,n-2,n-1} P^\mathrm{R}_{n-1,n},$

$U_\mathrm{o} = P_{n-2,n-1,n} \cdots P_{456} P_{234} P^\mathrm{L}_{12}.$

$$P = \begin{pmatrix} 1 & & & & & & & \\ & & & 1 & & & & \\ & & 1 & & & & & \\ & 1 & & & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & 1 & & & \\ & & & & & 1 & & \end{pmatrix}$$

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$$U_e$$

$$U_o$$

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$$P^{\mathrm{L}} = \begin{pmatrix} \alpha & 0 & \alpha & 0 \\ 0 & \beta & 0 & \beta \\ 1-\alpha & 0 & 1-\alpha & 0 \\ 0 & 1-\beta & 0 & 1-\beta \end{pmatrix}$$

$$P^{\mathrm{R}} = \begin{pmatrix} \gamma & \gamma & 0 & 0 \\ 1-\gamma & 1-\gamma & 0 & 0 \\ 0 & 0 & \delta & \delta \\ 0 & 0 & 1-\delta & 1-\delta \end{pmatrix}$$
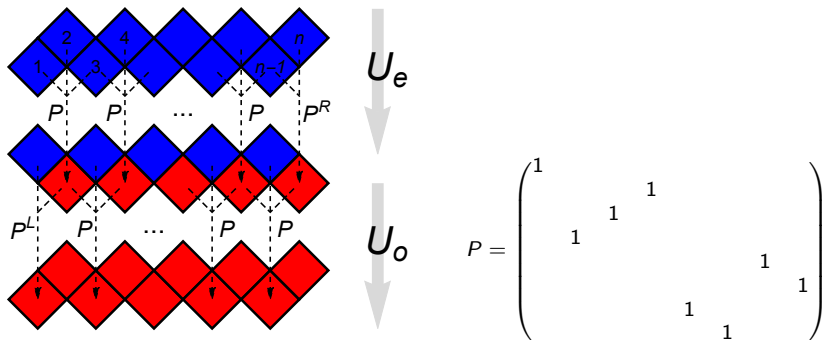
$$U = U_o U_e,$$
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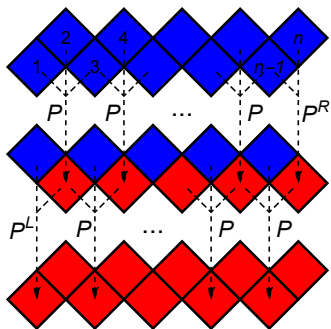
t

$x \longrightarrow$

Theorem

The $2^n \times 2^n$ matrix U is irreducible and aperiodic for generic values of driving parameters, more precisely, for an open set $0 < \alpha, \beta, \gamma, \delta < 1$.

Holographic ergodicity

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> *The $2^n \times 2^n$ matrix U is irreducible and aperiodic for generic values of driving parameters, more precisely, for an open set $0 < \alpha, \beta, \gamma, \delta < 1$.*

Consequence (via *Perron-Frobenius* theorem): Nonequilibrium steady state (NESS), i.e. fixed point of U

$$U\mathbf{p} = \mathbf{p}$$

is *unique*, and any initial probability state vector is asymptotically (in t) relaxing to $\mathbf{p}$.

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Idea of the proof:
Show that for any pair of configurations $\mathbf{s}, \mathbf{s}'$, such t_0 exists that

$$(U^t)_{\mathbf{s},\mathbf{s}'} > 0, \quad \forall t \geq t_0.$$
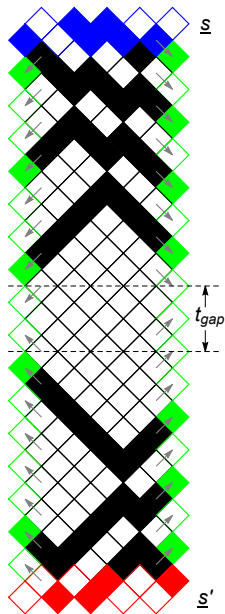
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Consider a pair of matrices:

$$
W_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi & \xi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \xi & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \omega \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} W_0 \\ W_1 \end{pmatrix}
$$

and $W_s'(\xi, \omega) := W_s(\omega, \xi)$.

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and $W_s'(\xi, \omega) := W_s(\omega, \xi)$. These satisfy a remarkable **bulk relation**:

$$P_{123}\mathbf{W}_1 S \mathbf{W}_2 \mathbf{W}_3' = \mathbf{W}_1 \mathbf{W}_2' \mathbf{W}_3 S$$

or component-wise

$$W_s S W_{\chi(ss's'')} W_{s''}' = W_s W_{s'}' W_{s''} S.$$

where S is a "delimiter" matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Suppose there exists pairs and quadruples of vectors $\langle l_s|$, $\langle l'_{ss'}|$, $|r_{ss'}\rangle$, $|r'_s\rangle$, and a scalar parameter λ, satisfying the following *boundary equations*

$$P_{123}\langle \mathbf{l}_1|\mathbf{W}_2\mathbf{W}'_3 = \langle \mathbf{l}'_{12}|\mathbf{W}_3 S,$$
$$P^{\mathrm{R}}_{12}|\mathbf{r}_{12}\rangle = \mathbf{W}'_1 S|\mathbf{r}'_2\rangle,$$
$$P_{123}\mathbf{W}'_1\mathbf{W}_2|\mathbf{r}'_3\rangle = \lambda \mathbf{W}'_1 S|\mathbf{r}_{23}\rangle,$$
$$P^{\mathrm{L}}_{12}\langle \mathbf{l}'_{12}| = \lambda^{-1}\langle \mathbf{l}_1|\mathbf{W}_2 S.$$

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Then, the following probability vectors

$$\mathbf{p} \quad \equiv \quad \mathbf{p}_{12\ldots n} = \langle \mathbf{l}_1|\mathbf{W}_2\mathbf{W}'_3\mathbf{W}_4\cdots \mathbf{W}'_{n-3}\mathbf{W}_{n-2}|\mathbf{r}_{n-1,n}\rangle,$$
$$\mathbf{p}' \quad \equiv \quad \mathbf{p}'_{12\ldots n} = \langle \mathbf{l}'_{12}|\mathbf{W}_3\mathbf{W}'_4\cdots \mathbf{W}_{n-3}\mathbf{W}'_{n-2}\mathbf{W}_{n-1}|\mathbf{r}'_n\rangle,$$

satisfy the NESS fixed point condition

$$U_{\mathrm{e}}\mathbf{p} = \mathbf{p}', \quad U_{\mathrm{o}}\mathbf{p}' = \mathbf{p}.$$

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$$P^R_{12}|r_{12}\rangle = W'_1 S|r'_2\rangle,$$
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$$P^L_{12}\langle l'_{12}| = \lambda^{-1}\langle l_1|W_2 S.$$

Then, the following probability vectors

$$\mathbf{p} \equiv \mathbf{p}_{12\ldots n} = \langle l_1|W_2W'_3W_4\cdots W'_{n-3}W_{n-2}|r_{n-1,n}\rangle,$$
$$\mathbf{p}' \equiv \mathbf{p}'_{12\ldots n} = \langle l'_{12}|W_3W'_4\cdots W_{n-3}W'_{n-2}W_{n-1}|r'_n\rangle,$$

satisfy the NESS fixed point condition

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Proof: Observe the bulk relations

$$P_{123}W_1 SW_2W'_3 = W_1W'_2W_3S,$$
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to move the delimiter S around, when it 'hits' the boundary observe one of the boundary equations. After the full cycle, you obtain $U_oU_e\mathbf{p} = \lambda\lambda^{-1}\mathbf{p}$.

This yields a consistent system of equations which uniquely determine the unknown parameters, namely for the left boundary:

$$\xi = \frac{(\alpha + \beta - 1) - \lambda^{-1}\beta}{\lambda^{-2}(\beta - 1)}, \qquad \omega = \frac{\lambda^{-1}(\alpha - \lambda^{-1})}{\beta - 1},$$

and for the right boundary:

$$\xi = \frac{\lambda(\gamma - \lambda)}{\delta - 1}, \qquad \omega = \frac{\gamma + \delta - 1 - \lambda\delta}{\lambda^2(\delta - 1)},$$

yielding

$$\xi = \frac{(\gamma(\alpha + \beta - 1) - \beta)(\beta(\gamma + \delta - 1) - \gamma)}{(\alpha - \delta(\alpha + \beta - 1))^2},$$

$$\omega = \frac{(\delta(\alpha + \beta - 1) - \alpha)(\alpha(\gamma + \delta - 1) - \delta)}{(\gamma - \beta(\gamma + \delta - 1))^2},$$

and explicit expressions for the boundary vectors..

Can we **diagonalize** U with a similar ansatz?

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Yes, a good deal of decay modes can be written as a compact MPA with explicitly positionally dependent matrices

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depending on $x \in \{2, 3, \ldots, n-1\}$ via multiplicative momentum variable z, containing linear combinations of

$$\{1, z^x, z^{-x}\}$$

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For example:

$$\mathbf{W}^{(x)} = (e_{11} \otimes \mathbf{W}(\xi z, \omega/z) + e_{22} \otimes \mathbf{W}(\xi/z, \omega z)) \left(\mathbb{1}_8 + e_{12} \otimes \frac{c_+ z^x F_+ + c_- z^{-x} F_-}{\xi \omega - 1} \right)$$

$$F_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & \frac{\xi\omega-1}{\omega z^2} & 0 \\ 0 & 0 & 0 & \xi z^2 \end{pmatrix}, \quad F_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\xi^2 z^3} \\ 0 & 0 & \frac{\xi\omega-1}{\xi z^2} & 0 \\ 0 & 0 & 0 & \omega + \frac{1}{\xi}\left(\frac{1}{z^2} - 1\right) \end{pmatrix}.$$

The Bethe-like equations for the Markov spectrum

$$U_{\mathrm{e}}\mathbf{p}(z) = \Lambda_{\mathrm{L}}\mathbf{p}'(z), \quad U_{\mathrm{o}}\mathbf{p}'(z) = \Lambda_{\mathrm{R}}\mathbf{p}(z).$$

$$\frac{z(\alpha + \beta - 1) - \beta\Lambda_{\mathrm{L}}}{(\beta - 1)\Lambda_{\mathrm{L}}^2} = \frac{\Lambda_{\mathrm{R}}(\gamma z - \Lambda_{\mathrm{R}})}{(\delta - 1)z},$$

$$\frac{z(\gamma + \delta - 1) - \delta\Lambda_{\mathrm{R}}}{(\delta - 1)\Lambda_{\mathrm{R}}^2} = \frac{\Lambda_{\mathrm{L}}(\alpha z - \Lambda_{\mathrm{L}})}{(\beta - 1)z},$$

$$z^{2n-6-4p} = \frac{(\alpha + \beta - 1)^p(\gamma + \delta - 1)^p}{\Lambda_{\mathrm{L}}^{4p}\Lambda_{\mathrm{R}}^{4p}}.$$
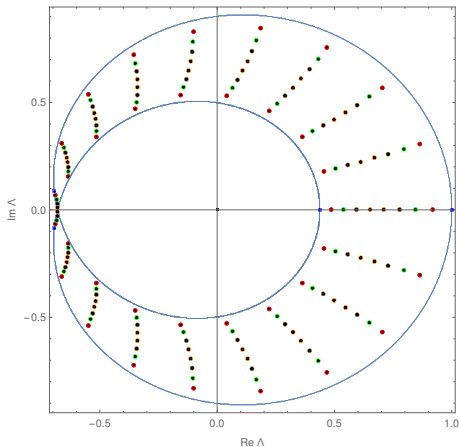
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Ultralocal basis $\{[0]_x, [1]_x\}$:

$$[\alpha]_x(\mathbf{s}) = \delta_{\alpha, s_x}, \qquad ([\alpha]_x [\beta]_y)(\mathbf{s}) = [\alpha]_x(\mathbf{s}) [\beta]_y(\mathbf{s}), \quad \alpha, \beta \in \{0, 1\}, \ \mathbf{s} \in \{0, 1\}^{\mathbb{Z}}$$

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r-local basis centred on site x:

$$[\alpha_1 \alpha_2 \ldots \alpha_r]_x \equiv [\alpha_1]_{x - \lfloor \frac{r}{2} \rfloor} [\alpha_2]_{x - \lfloor \frac{r}{2} \rfloor + 1} \cdots [\alpha_r]_{x + \lfloor \frac{r-1}{2} \rfloor}.$$

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Using unit element $\mathbb{1} = [0]_x + [1]_x$, we can extend the support of each r-local basis element as

$$[\alpha_1 \alpha_2 \ldots \alpha_r]_x \equiv \mathbb{1}_{x - \lfloor \frac{r+2}{2} \rfloor} \cdot [\alpha_1 \alpha_2 \ldots \alpha_r]_x \cdot \mathbb{1}_{x + \lfloor \frac{r+1}{2} \rfloor} \equiv$$
$$\equiv [0\alpha_1 \alpha_2 \ldots \alpha_r 0]_x + [0\alpha_1 \alpha_2 \ldots \alpha_r 1]_x + [1\alpha_1 \alpha_2 \ldots \alpha_r 0]_x + [1\alpha_1 \alpha_2 \ldots \alpha_r 1]_x.$$

Separable (strongly clustering) states p defined by expectation values $p(x)$ of ultralocal observables

$$\langle [\alpha_1 \alpha_2 \ldots \alpha_r]_x \rangle_p = p_{x - \lfloor \frac{r}{2} \rfloor}(\alpha_1) \cdot p_{x - \lfloor \frac{r}{2} \rfloor + 1}(\alpha_2) \cdots p_{x + \lfloor \frac{r-1}{2} \rfloor}(\alpha_r).$$

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Two examples of separable states that we consider:

1. A *maximum entropy state*

$$p_x(0) = p_x(1) = 1/2, \quad \forall x \in \mathbb{Z}.$$

2. An *inhomogeneous initial state*

$$\begin{cases} p_x(0) = p_x(1) = 1/2, & \text{for } x \leq 0 \\ p_x(0) = 1, \quad p_x(1) = 0. & \text{for } x > 0 \end{cases}$$

$$a^t(\mathbf{s}^0) = a(\mathbf{s}^t)$$

$$a^t(\mathbf{s}^0) = a(\mathbf{s}^t)$$

For 3-site observables, dynamical automorphism is defined as

$$U_x[\alpha \ \beta \ \gamma]_y = \begin{cases} [\alpha \ \chi(\alpha, \beta, \gamma) \ \gamma]_y; & x = y, \\ [\alpha \ \beta \ \gamma]_y; & |x - y| \geq 2, \end{cases}$$

while for any r-local observable it is defined as
a *t-staggered linear homomorphism*

$$a^{t+1} = U(t)a^t$$

$$U(t) = \begin{cases} \prod_{x \in 2\mathbb{Z}} U_x; & t \equiv 0 \pmod 2, \\ \prod_{x \in 2\mathbb{Z}+1} U_x; & t \equiv 1 \pmod 2. \end{cases}$$

Theorem (Klobas *et al.* 18): Time evolution of a local observable $[1]_x$ reads

$$[1]_x^t = \sum_{s_{-t},\ldots,s_t \in \{0,1\}} c_{s_{-t},\ldots,s_t}(t)[s_{-t}s_{-t+1}\cdots s_t]_x,$$

where the amplitudes $c_{s_{-t},\ldots,s_t}(t) \in \{0,1\}$ can be represented as MPA

$$c_{s_{-t},\ldots s_t}(t) = \langle l(t)|V_{s_{-t}}W_{s_{-t+1}}V_{s_{-t+2}}\cdots W_{s_{t-1}}V_{s_t}|r\rangle +$$
$$+ \langle l'|V'_{s_{-t}}W'_{s_{-t+1}}V'_{s_{-t+2}}\cdots W'_{s_{t-1}}V'_{s_t}|r'(t)\rangle.$$

V_s, W_s, V'_s, $W'_s \in \mathrm{End}(\mathcal{V})$, $s \in \{0,1\}$, are linear operators over auxiliary Hilbert space $\mathcal{V} = \mathrm{lsp}\{|c,w,n,a\rangle;\ c,w \in \mathbb{N}_0,\ n \in \{0,1,2\},\ a \in \{0,1\}\}$, and can be explicitly expressed in terms of ladder operators and projectors

$$\mathbf{c}^+ = \sum_{c,w,n,a} |c+1,w,n,a\rangle\langle c,w,n,a|, \qquad \mathbf{c}^- = \left(\mathbf{c}^+\right)^T,$$

$$\mathbf{w}^+ = \sum_{c,w,n,a} |c,w+1,n,a\rangle\langle c,w,n,a|, \qquad \mathbf{w}^- = \left(\mathbf{w}^+\right)^T,$$

$$\mathbf{e}_{c_2w_2n_2a_2,c_1w_1n_1a_1} = |c_2,w_2,n_2,a_2\rangle\langle c_1,w_1,n_1,a_1|,$$

$$\mathbf{e}_{n_2a_2,n_1a_1} = \sum_{c,w} |c,w,n_2,a_2\rangle\langle c,w,n_1,a_1|,$$

$$V_0 = \mathbf{e}_{00,00} + \mathbf{e}_{10,00} + \mathbf{e}_{20,00} + \mathbf{c}^+ \mathbf{e}_{10,01} + \mathbf{e}_{01,01} + \mathbf{c}^+ \mathbf{w}^+ \mathbf{e}_{11,01} + \mathbf{e}_{21,01} +$$
$$+ \mathbf{e}_{0001,0001} + \mathbf{e}_{0011,0001} + \mathbf{e}_{0021,0001},$$

$$V_1 = \mathbf{e}_{00,10} + \mathbf{e}_{10,20} + \mathbf{e}_{20,20} + \mathbf{e}_{00,11} + \mathbf{e}_{10,21} + \mathbf{e}_{20,21} + \mathbf{e}_{01,11} +$$
$$+ \mathbf{w}^+ \mathbf{e}_{11,21} + \mathbf{w}^+ \mathbf{e}_{21,21} + \mathbf{e}_{0001,0011} + \mathbf{e}_{0011,0021} + \mathbf{e}_{0021,0021},$$

$$W_0 = \mathbf{c}^- \mathbf{w}^+ \left(\mathbf{e}_{00,00} + \mathbf{e}_{10,00} + \mathbf{e}_{20,00} \right) + \mathbf{w}^+ \mathbf{e}_{10,01} + \mathbf{w}^+ \mathbf{e}_{01,01} +$$
$$+ \mathbf{c}^+ \left(\mathbf{w}^+ \right)^2 \mathbf{e}_{11,01} + \mathbf{w}^+ \mathbf{e}_{21,01} + \mathbf{e}_{1111,0001} + \mathbf{e}_{0001,0001} + \mathbf{e}_{0011,0001} + \mathbf{e}_{0021,0001},$$

$$W_1 = \mathbf{c}^- \mathbf{w}^+ \left(\mathbf{e}_{00,10} + \mathbf{e}_{10,20} + \mathbf{e}_{20,20} \right) + \mathbf{w}^+ \mathbf{e}_{01,11} + \mathbf{c}^+ \mathbf{w}^+ \mathbf{e}_{11,21} +$$
$$+ \mathbf{c}^+ \mathbf{w}^+ \mathbf{e}_{21,21} + \mathbf{e}_{0001,0011} + \mathbf{e}_{0011,0021} + \mathbf{e}_{0021,0021},$$

$$V_0' = V_0^T - \left(\mathbf{e}_{0001,1111} + \mathbf{e}_{0101,1211} + \mathbf{e}_{0101,1110} \right),$$

$$V_1' = V_1^T,$$

$$W_0' = W_0^T - \left(\mathbf{e}_{0001,1111} + \mathbf{e}_{0000,1211} \right),$$

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$$W_1 = \mathbf{c}^- \mathbf{w}^+ \left(\mathbf{e}_{00,10} + \mathbf{e}_{10,20} + \mathbf{e}_{20,20} \right) + \mathbf{w}^+ \mathbf{e}_{01,11} + \mathbf{c}^+ \mathbf{w}^+ \mathbf{e}_{11,21} +$$
$$+ \mathbf{c}^+ \mathbf{w}^+ \mathbf{e}_{21,21} + \mathbf{e}_{0001,0011} + \mathbf{e}_{0011,0021} + \mathbf{e}_{0021,0021},$$

$$V_0' = V_0^T - \left(\mathbf{e}_{0001,1111} + \mathbf{e}_{0101,1211} + \mathbf{e}_{0101,1110} \right),$$

$$V_1' = V_1^T,$$

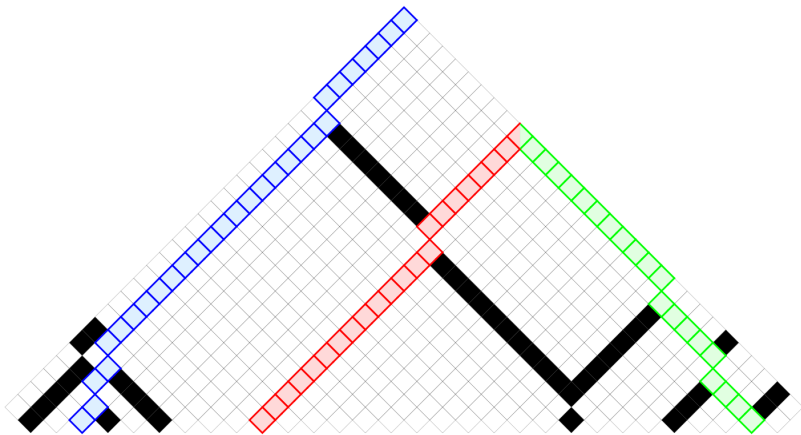$$W_0' = W_0^T - \left(\mathbf{e}_{0001,1111} + \mathbf{e}_{0000,1211} \right),$$

$$W_1' = W_1^T - \left(\mathbf{e}_{0021,1111} + \mathbf{e}_{0021,1121} + \mathbf{e}_{0121,1211} + \mathbf{e}_{0121,1221} \right).$$

The time-dependent auxiliary space boundary vectors take the following form:

$$\langle l(t)| = \langle 0, t, 0, 0|,$$
$$|r\rangle = |0,0,0,0\rangle + |0,0,0,1\rangle + |0,0,0,2\rangle,$$
$$\langle l'| = \langle 0,0,0,1| + \langle 0,0,1,1| + \langle 0,0,2,1| + \langle 0,1,0,1| + \langle 0,1,2,1|, \tag{1}$$
$$|r'(t)\rangle = |0, t+1, 0, 0\rangle.$$

Proof: *'Real space, real time inverse scattering transform'*

The weight of left MPA $\langle l(t)|V_{s_{-t}} W_{s_{-t+1}} V_{s_{-t+2}} \cdots W_{s_{t-1}} V_{s_t}|r\rangle$ is 1 (or 0) if the configuration $(s_{-t}, s_{-t+1}, \ldots, s_t)$ can (cannot) be obtained in a light-cone with the *left-mover at the origin*!

$$C(x,t) = \langle [1]_x [1]_0^t \rangle_P - \langle [1]_x \rangle_P \langle [1]_0^t \rangle_P = \langle [1]_x [1]_0^t \rangle_P - \frac{1}{4}$$

Exact dynamical structure factor

$$C(x,t) = \langle [1]_x [1]_0^t \rangle_P - \langle [1]_x \rangle_P \langle [1]_0^t \rangle_P = \langle [1]_x [1]_0^t \rangle_P - \frac{1}{4}$$

Using time-dependent MPA:

$$C(x,t) = \frac{1}{2^{2t+1}} \left(\langle l(t) | T^{\frac{x+t}{2}} V_1 \overline{T}^{t-\frac{x+t}{2}} | r \rangle + \langle l' | \overline{T}'^{\frac{x+t}{2}} V_1' T'^{t-\frac{x+t}{2}} | r'(t) \rangle \right) - \frac{1}{4}$$
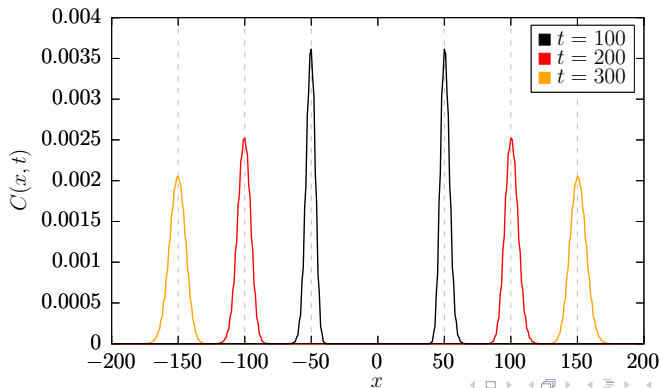
with

$$T = (V_0 + V_1)(W_0 + W_1), \qquad \overline{T} = (W_0 + W_1)(V_0 + V_1),$$
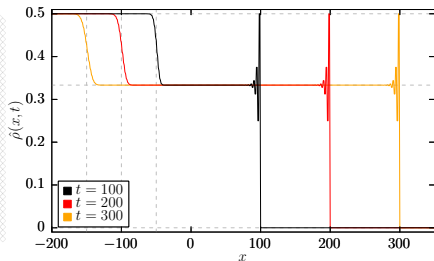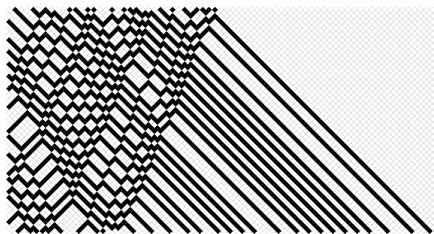$$T' = (W_0' + W_1')(V_0' + V_1'), \qquad \overline{T}' = (V_0' + V_1')(W_0' + W_1').$$

We find normal hydrodynamic scaling!

For maximum entropy ('infinite temperature') state, tMPA yields

$$C(x,t) = 2^{-t-1} \sum_{m=0}^{\frac{t-|x|-2}{2}} 4^m \left(2 \binom{t-2m-3}{m} - \binom{t-2m-2}{m} \right)$$

$$\simeq \frac{1}{16\sqrt{t\pi}} \exp\left(-\frac{4}{t}\left(|x| - \frac{t}{2}\right)^2 \right).$$

$$\hat{\rho}(x, t) = \langle [1]_x \rangle_{p_{\text{inhom}}^t} = \langle [1]_x^{-t} \rangle_{p_{\text{inhom}}}$$

Exact solution exhibits the following simple asymptotic behavior:

- Quasi-free regime

$$\hat{\rho}\left(t \geq x \geq -\frac{t}{3} + 1, t\right) = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^{\lfloor \frac{t-x+\mathbf{1}}{\mathbf{2}} \rfloor}\right)$$

- Thermalizing (diffusive) regime

$$\lim_{t \to \infty} \hat{\rho}\left(-\frac{t}{2} + \zeta\sqrt{t}, t\right) = \frac{1}{12}\left(5 - \text{erf}(2\zeta)\right)$$

[B.Buča, J.P.Garrahan, T.Prosen, M.Vanicat, arXiv:1901.00845]
Large deviation theory for arbitrary observable of the form:

$$\mathcal{O}_T = \sum_{t=0}^{T-1} \sum_{x=1}^{N-1} \left[f_x(s_x^t, s_{x+1}^t) + g_x(s_x^{t+1/2}, s_{x+1}^{t+1/2}) \right]$$

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Tilted Markov generator:

$$\tilde{U}(s) = U_\mathrm{o}\, G(s)\, U_\mathrm{e}\, F(s).$$

$$F(s) = F_{12}^{(1)} F_{23}^{(2)} F_{34}^{(3)} \ldots F_{N-1,N}^{(N-1)} \quad \text{and} \quad G(s) = G_{12}^{(1)} G_{23}^{(2)} G_{34}^{(3)} \ldots G_{N-1,N}^{(N-1)}$$

where

$$F^{(x)} = \begin{pmatrix} f_{0,0}^{(x)} & 0 & 0 & 0 \\ 0 & f_{0,1}^{(x)} & 0 & 0 \\ 0 & 0 & f_{1,0}^{(x)} & 0 \\ 0 & 0 & 0 & f_{1,1}^{(x)} \end{pmatrix}, \qquad f_{s,s'}^{(x)} \equiv e^{s f_x(s,s')}$$

and similar for $G^{(x)}$.

There exist 3×3 matrices satisying bulk algebraic conditions:

$$f_{ss'}^{(j-1)} f_{s's''}^{(j)} W_s^{(j-1)} W_{s'}^{(j)} X_{s''}^{(j+1)} = X_s^{(j-1)} V_{\chi(ss's'')}^{(j)} V_{s''}^{(j+1)},$$

$$g_{ss'}^{(j-2)} g_{s's''}^{(j-1)} X_s^{(j-2)} V_{s'}^{(j-1)} V_{s''}^{(j)} = W_s^{(j-2)} W_{\chi(ss's'')}^{(j-1)} X_{s''}^{(j)},$$

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and boundary equations

$$f_{ss'}^{(1)} f_{s's''}^{(2)} \langle l_s | W_{s'}^{(2)} X_{s''}^{(3)} = \langle l'_{s\chi(s's'')} | V_{s''}^{(3)},$$

$$\sum_{m,m'=0,1} R_{ss'}^{mm'} f_{mm'}^{(N-1)} |r_{mm'}\rangle = \lambda_{\mathrm{R}} X_s^{(N-1)} |r_{s'}'\rangle,$$

$$\sum_{m,m'=0,1} L_{ss'}^{mm'} g_{mm'}^{(1)} \langle l'_{mm'} | = \lambda_{\mathrm{L}} \langle l_s | X_{s'}^{(2)},$$

$$g_{ss'}^{(N-2)} g_{s's''}^{(N-1)} X_s^{(N-2)} V_{s'}^{(N-1)} |r_{s''}'\rangle = W_s^{(N-2)} |r_{\chi(ss's'')s''}'\rangle,$$

Inhomogeneous matrix ansatz cancellation mechanism

There exist 3×3 matrices satisfying bulk algebraic conditions:

$$f_{ss'}^{(j-1)} f_{s's''}^{(j)} W_s^{(j-1)} W_{s'}^{(j)} X_{s''}^{(j+1)} = X_s^{(j-1)} V_{\chi(ss's'')}^{(j)} V_{s''}^{(j+1)},$$

$$g_{ss'}^{(j-2)} g_{s's''}^{(j-1)} X_s^{(j-2)} V_{s'}^{(j-1)} V_{s''}^{(j)} = W_s^{(j-2)} W_{\chi(ss's'')}^{(j-1)} X_{s''}^{(j)},$$

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$$\sum_{m,m'=0,1} L_{ss'}^{mm'} g_{mm'}^{(1)} \langle l'_{mm'} | = \lambda_{\mathrm{L}} \langle l_s | X_{s'}^{(2)},$$

$$g_{ss'}^{(N-2)} g_{s's''}^{(N-1)} X_s^{(N-2)} V_{s'}^{(N-1)} |r_{s''}'\rangle = W_s^{(N-2)} |r_{\chi(ss's'')s''}\rangle,$$

such that MPA:

$$p_{s_1,\ldots,s_N} = \langle l_{s_1} | W_{s_2}^{(2)} W_{s_3}^{(3)} \cdots W_{s_{N-3}}^{(N-3)} W_{s_{N-2}}^{(N-2)} |r_{s_{N-1}s_N}\rangle$$
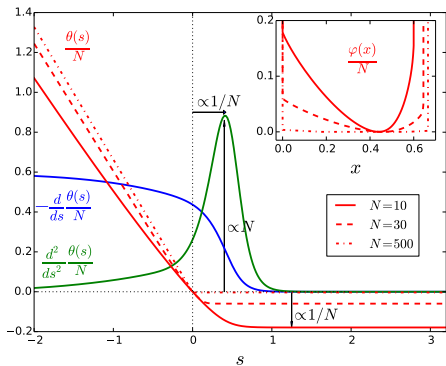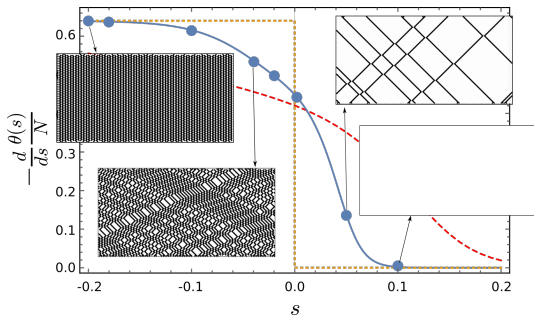
$$p'_{s_1,\ldots,s_N} = \langle l'_{s_1s_2} | V_{s_3}^{(3)} V_{s_4}^{(4)} \cdots V_{s_{N-2}}^{(N-2)} V_{s_{N-1}}^{(N-1)} |r'_{s_N}\rangle,$$

solves the eigenvalue equation

$$\tilde{U}(s)\mathbf{p} = \Lambda(s)\mathbf{p}$$

and $\Lambda(s) = e^{\theta(s)}$ is a root of third order polynomial.

- Interacting integrable model about which we can compute everything: *quenches, non-equilibrium steady states with baths, relaxation rates, dynamical structure factor, large deviations etc.*

- Generalizations (stochastic/unitary branching)? Link to Yang-Baxter integrability missing?

- Testbed for computing diffusive corrections to generalized hydroduynamics. See e.g.: S. Gopalakrishnan, D. Huse, V. Khemani, R. Vasseur, PRB **98**, 220303 (2018)

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The work supported by Slovenian Research Agency (ARRS) and

European Research Council
Established by the European Commission

Conclusions and Outlook

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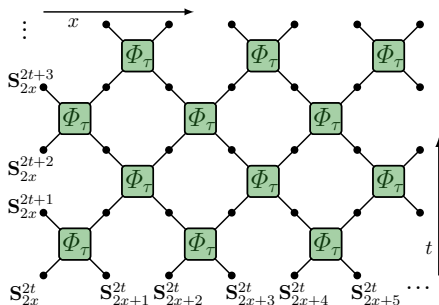**European Research Council**
Established by the European Commission

Family of rational-symplectic maps over $\mathcal{S}^2 \times \mathcal{S}^2$ ($\mathbf{S_1} \cdot \mathbf{S_1} = \mathbf{S_2} \cdot \mathbf{S_2} = 1$):

$$\Phi_\tau(\mathbf{S_1}, \mathbf{S_2}) = \frac{1}{\sigma^2 + \tau^2}\Big(\sigma^2 \mathbf{S_1} + \tau^2 \mathbf{S_2} + \tau \mathbf{S_1} \times \mathbf{S_2}, \sigma^2 \mathbf{S_2} + \tau^2 \mathbf{S_1} + \tau \mathbf{S_2} \times \mathbf{S_1}\Big),$$

$$\sigma^2 := \frac{1}{2}\Big(1 + \mathbf{S_1} \cdot \mathbf{S_2}\Big),$$

defining discrete space-time many-body symplectic dynamics
(classical local Floquet circuit)

$$(S_{2x}^{2t+1}, \mathbf{S}_{2x+1}^{2t+1}) = \Phi_\tau(\mathbf{S}_{2x}^{2t}, \mathbf{S}_{2x+1}^{2t}), \qquad (\mathbf{S}_{2x-1}^{2t+2}, \mathbf{S}_{2x}^{2t+2}) = \Phi_\tau(\mathbf{S}_{2x-1}^{2t+1}, \mathbf{S}_{2x}^{2t+1}),$$

Defining $\mathbb{I}$ a identity map over $\mathcal{S}^2$ we find (note that $\Phi_\infty(\mathbf{S}_1, \mathbf{S}_2) = (\mathbf{S}_2, \mathbf{S}_1)$):

$$\left(\Phi_\lambda \otimes \mathbb{I}\right) \circ \left(\mathbb{I} \otimes \Phi_{\lambda+\mu}\right) \circ \left(\Phi_\mu \otimes \mathbb{I}\right) = \left(\mathbb{I} \otimes \Phi_\mu\right) \circ \left(\Phi_{\lambda+\mu} \otimes \mathbb{I}\right) \circ \left(\mathbb{I} \otimes \Phi_\lambda\right).$$

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Furthermore, defining a matrix valued function $L(\lambda) : \mathcal{S}^2 \to \mathrm{End}(\mathbb{C}^2)$

$$L(\mathbf{S}; \lambda) = \mathbb{1} + \frac{1}{2i\lambda}\mathbf{S} \cdot \boldsymbol{\sigma},$$

we find that RLL relation is obeyed:

$$L(\mathbf{S}_2; \lambda)L(\mathbf{S}_1; \mu) = L(\mathbf{S}_2'; \mu)L(\mathbf{S}_1'; \lambda), \quad (\mathbf{S}_1', \mathbf{S}_2') := \Phi_{\lambda-\mu}(\mathbf{S}_1, \mathbf{S}_2).$$

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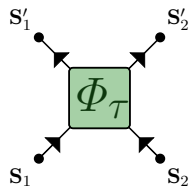
Defining monodromy-matrix

$$T(\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_{N-1}; \lambda, \mu) = \mathrm{tr}\left(\overset{\overset{\leftarrow}{N/2-1}}{\prod_{x=0}} L(\mathbf{S}_{2x+1}; \lambda) L(\mathbf{S}_{2x}; \mu) \right),$$

subsequent application of RLL relations shows manifest conservation of its trace (for any λ while $\mu = \lambda - \tau$)
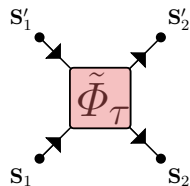
$$T(\mathbf{S}_0^t \ldots \mathbf{S}_{N-1}^t; \lambda, \lambda - \tau) = T(\mathbf{S}_0^{t+1} \ldots \mathbf{S}_{N-1}^{t+1}; \lambda - \tau, \lambda) = T(\mathbf{S}_0^{t+2} \ldots \mathbf{S}_{N-1}^{t+2}; \lambda, \lambda - \tau).$$

$$Q_k^{\text{even}} = \partial_\lambda^k \log |T(\lambda, \lambda - \tau)|^2|_{\lambda = -\frac{i}{2}},$$

$$Q_k^{\text{odd}} = \partial_\lambda^k \log |T(\lambda, \lambda - \tau)|^2|_{\lambda = \tau - \frac{i}{2}}, \quad k = 0, 1, 2 \ldots$$
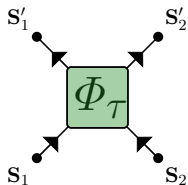
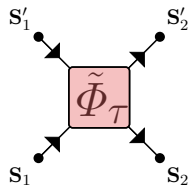$$(\mathbf{S}_1', \mathbf{S}_2') = \Phi_\tau(\mathbf{S}_1, \mathbf{S}_2), \qquad (\mathbf{S}_2, \mathbf{S}_2') = \tilde{\Phi}_\tau(\mathbf{S}_1, \mathbf{S}_1').$$

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$$\Xi \circ \tilde{\Phi}_\tau = \Phi_\tau \circ (-\Xi),$$

$$\Xi(\mathbf{S}, \mathbf{S}') := (\mathbf{S}, -\mathbf{S}'), \quad (-\Xi)(\mathbf{S}, \mathbf{S}') := (-\mathbf{S}, \mathbf{S}').$$

$$C(x, t) = \langle S_x^t S_0^0 \rangle - \langle S_x^t \rangle \langle S_0^0 \rangle$$