Mapping TASEP back in time

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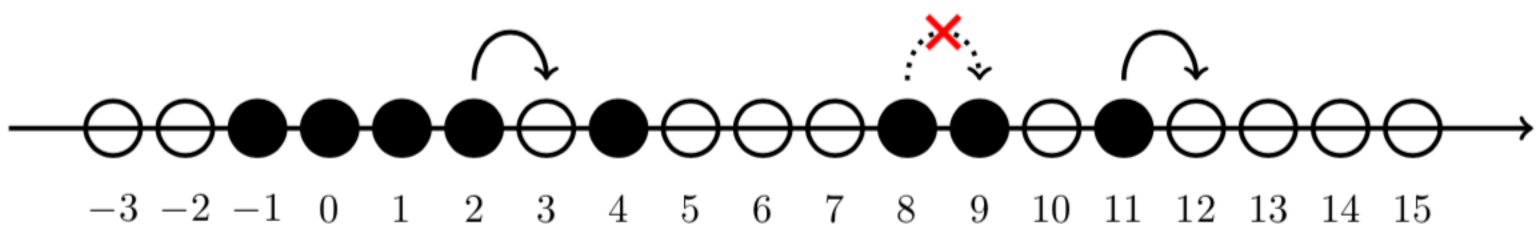
joint work with Axel Saenz

arXiv:1907.09155 [math.PR]

TASEP (totally asymmetric simple exclusion process)



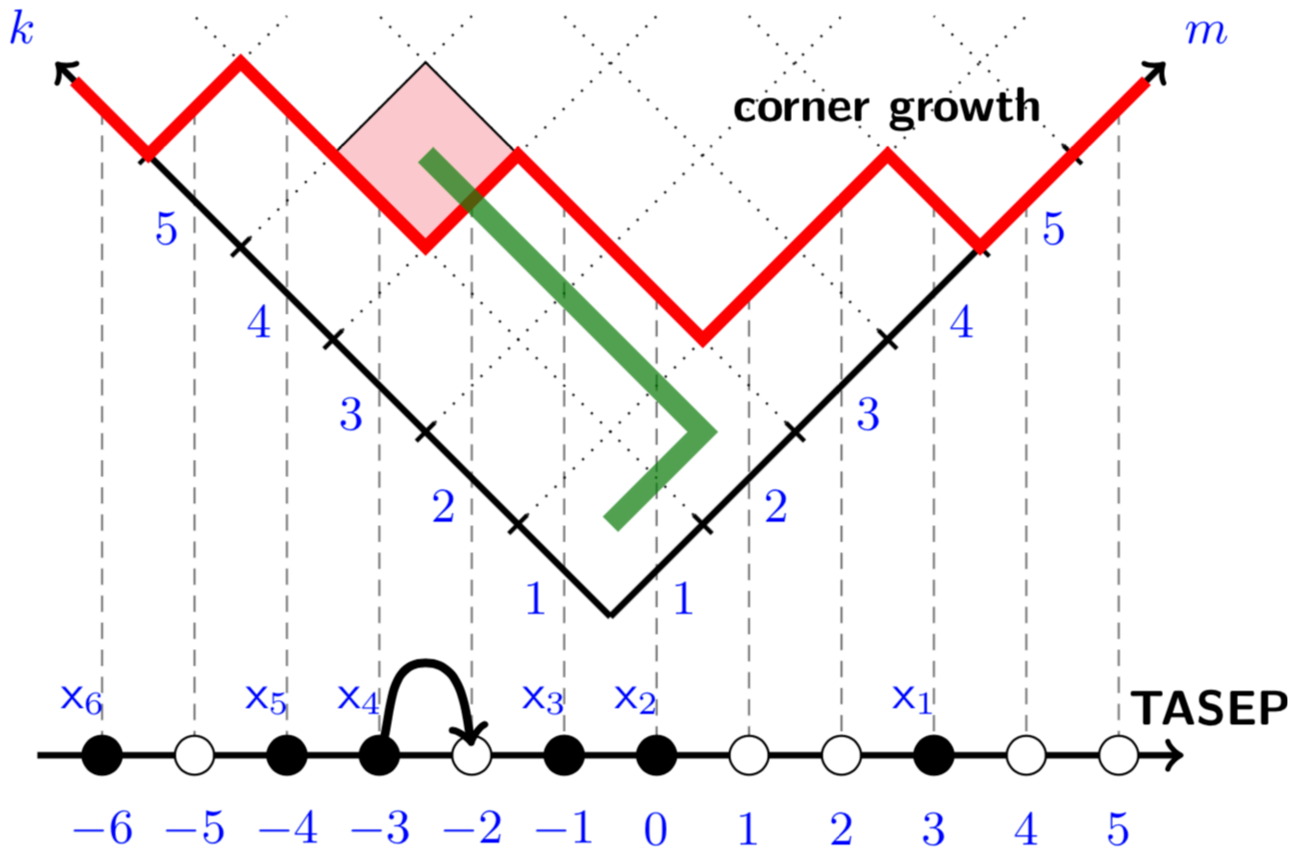
Each particle has an exponential clock with rate 1: $\mathbb{P}(\text{wait} > s) = e^{-s}$, $s > 0$, clocks are independent for each particle.

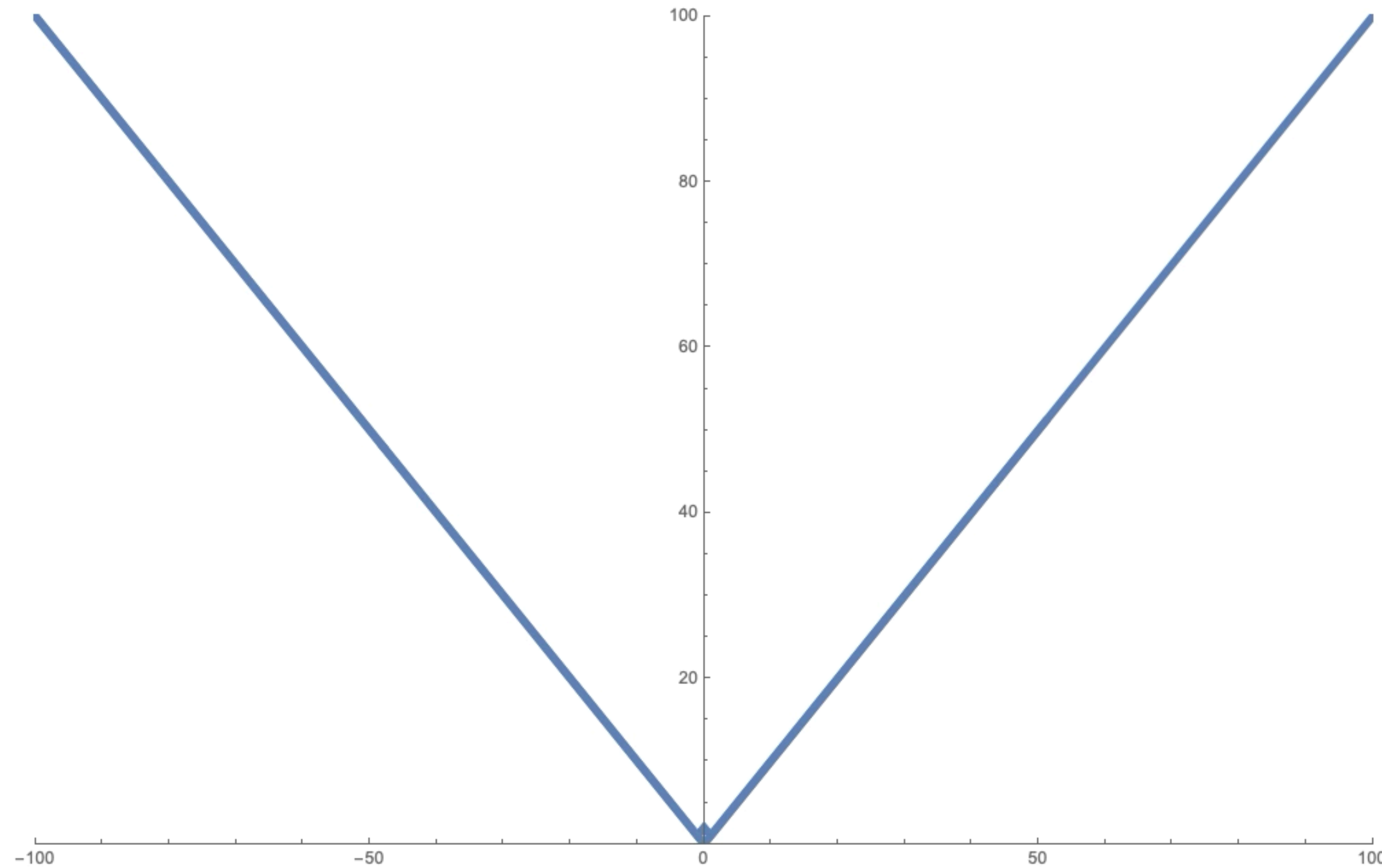
When the clock rings, the particle jumps to the right by one if the destination is not occupied.

- TASEP and ASEP (with particles moving in two directions) were introduced in 1969-1970, independently in biology **[C. MacDonald, J. Gibbs, and A. Pipkin '69]** and probability **[Spitzer '70]**

- In these 50 years, we understood a lot about TASEP and related systems (in the Kardar-Parisi-Zhang universality class), including limit shapes and fluctuations with general initial data

- Yet new asymptotic results are added every year (KPZ fixed point, Airy sheet, directed landscape, …). Let us give one basic example of asymptotics…





Parabola limit shape **[Rost 1981]**

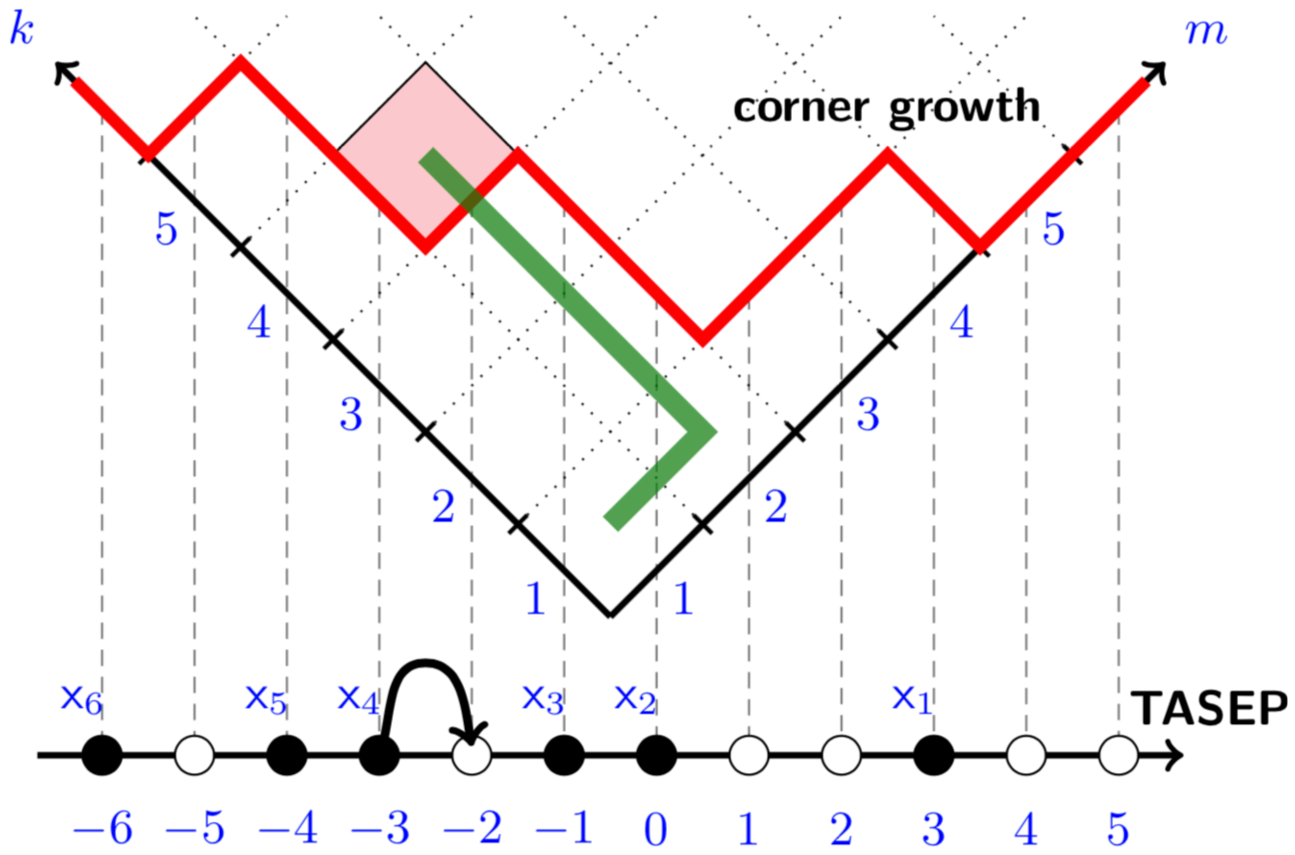
Fluctuations **[Johansson 2000]**

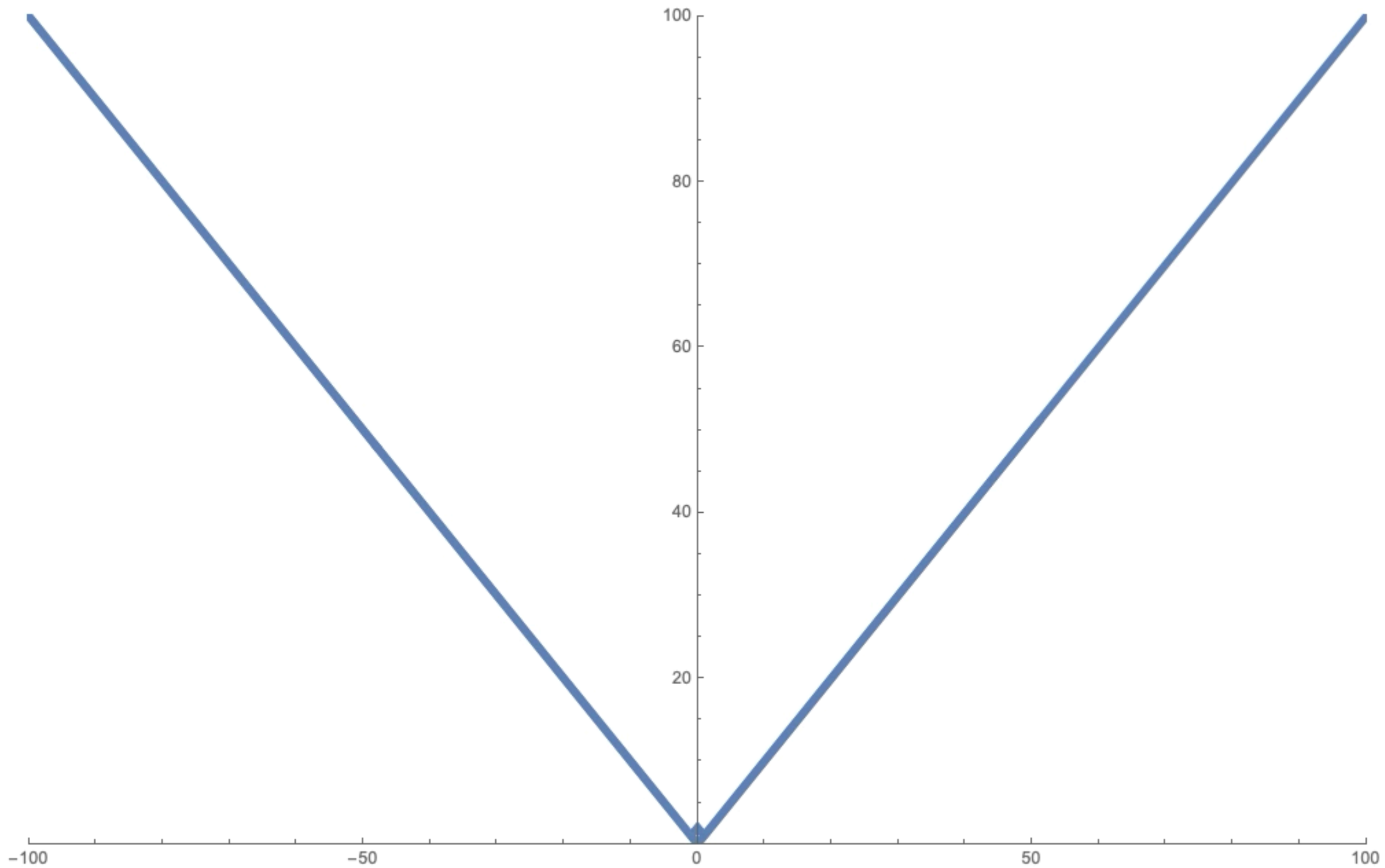
Start TASEP from the **step initial configuration** $x_i(0) = -i$, $i = 1, 2, \ldots$.

Let $h(t, x)$ be the height of the interface over x at time t. Then

$$\lim_{L \to +\infty} \mathbb{P}\left(\frac{h(\tau L, \chi L) - L\mathfrak{h}(\tau, \chi)}{c_{\tau,\chi} L^{\mathbf{1/3}}} \geq -s \right) = F_{GUE}(s),$$

where F_{GUE} is the **GUE** (Gaussian Unitary Ensemble) **Tracy–Widom distribution** originated in random matrix theory **[Tracy and Widom, 1993]**





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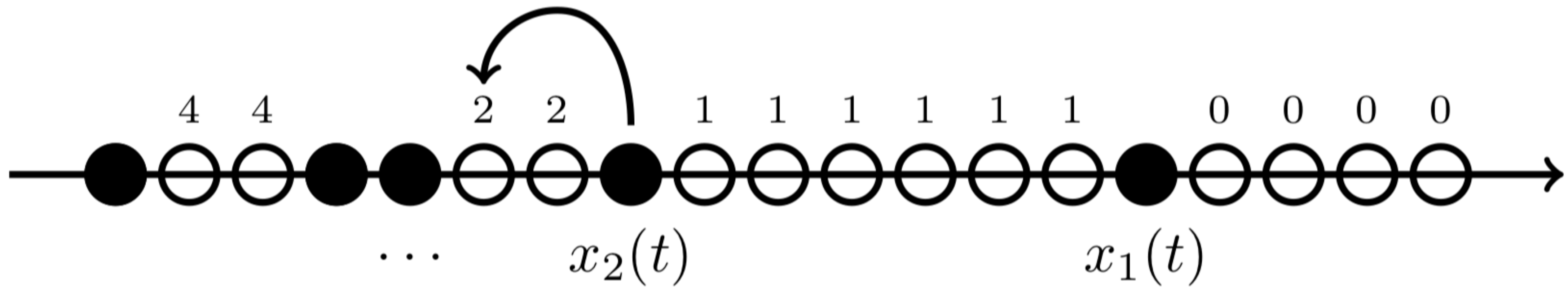
Definition. Backwards Hammersley process



- The Markov chain lives on left-packed configurations $x_1 > x_2 > x_3 > \dots$

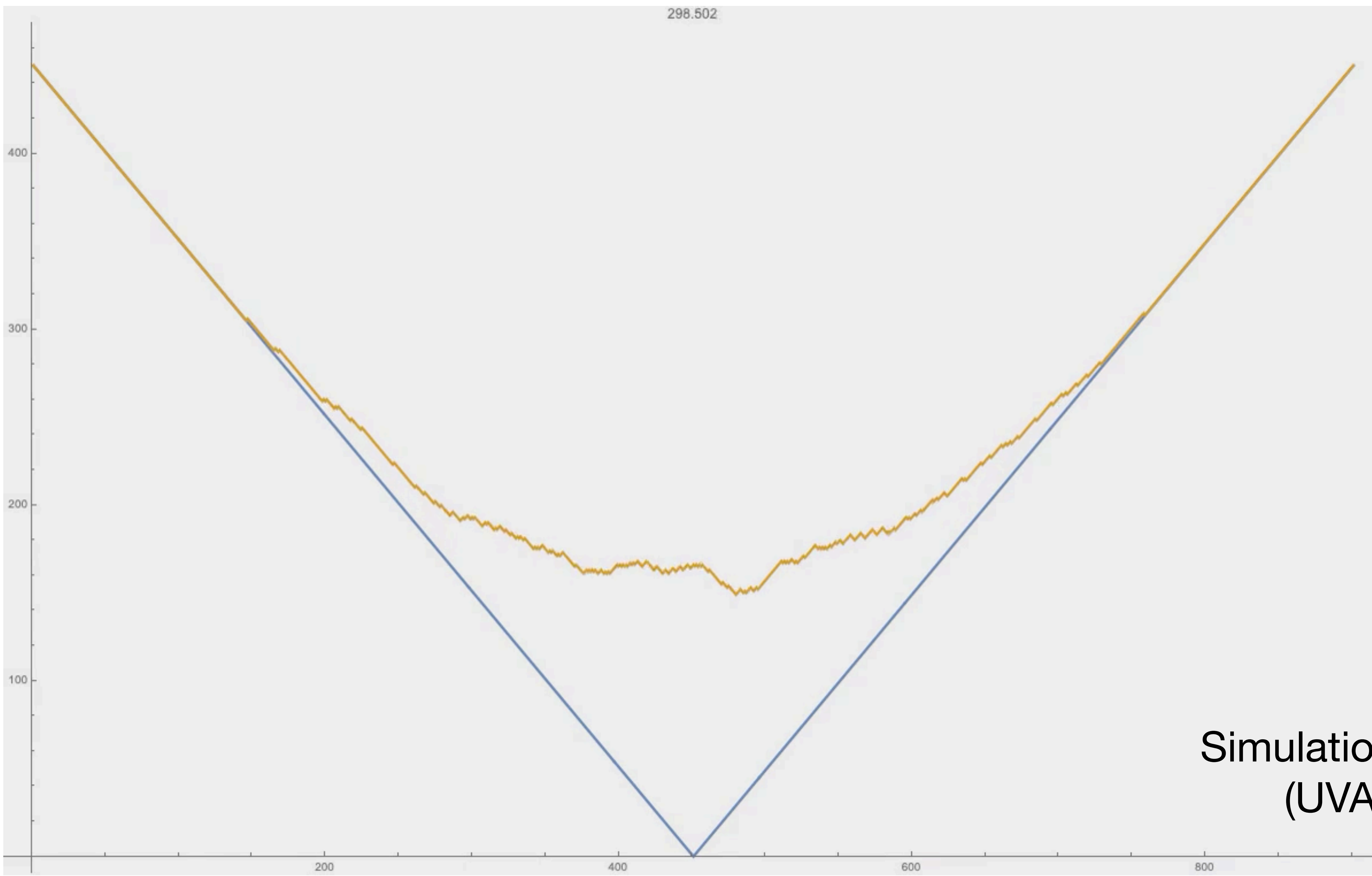
- Each hole has an independent exponential clock with rate equal to the number m of particles to its right, $\mathbb{P}(\text{wait} > s) = e^{-m\cdot s}$, $s > 0$.

- When the clock at a hole rings, the leftmost of the particles that are to the right of the hole instantaneously jumps into this hole

- Because total rate of jump is proportional to the size of the gap, this is a discrete space inhomogeneous version of the Hammersley process **[Hammersley '72], [Aldous-Diaconis '95]**

Running TASEP back in time

Theorem [P.-Saenz]. Let μ_t be the distribution of the TASEP (with step IC) at time t. Let L_τ be the backwards Hammersley Markov semigroup. Then $\mu_t L_\tau = \mu_{t \cdot e^{-\tau}}$, i.e., $\sum_{\overrightarrow{x}} \mu_t(\overrightarrow{x}) L_\tau(\overrightarrow{x}, \overrightarrow{y}) = \mu_{t \cdot e^{-\tau}}(\overrightarrow{y})$.



Note. Theorem does not apply to multitime distributions

Simulation joint with Haoyu Li (UVA undergraduate)

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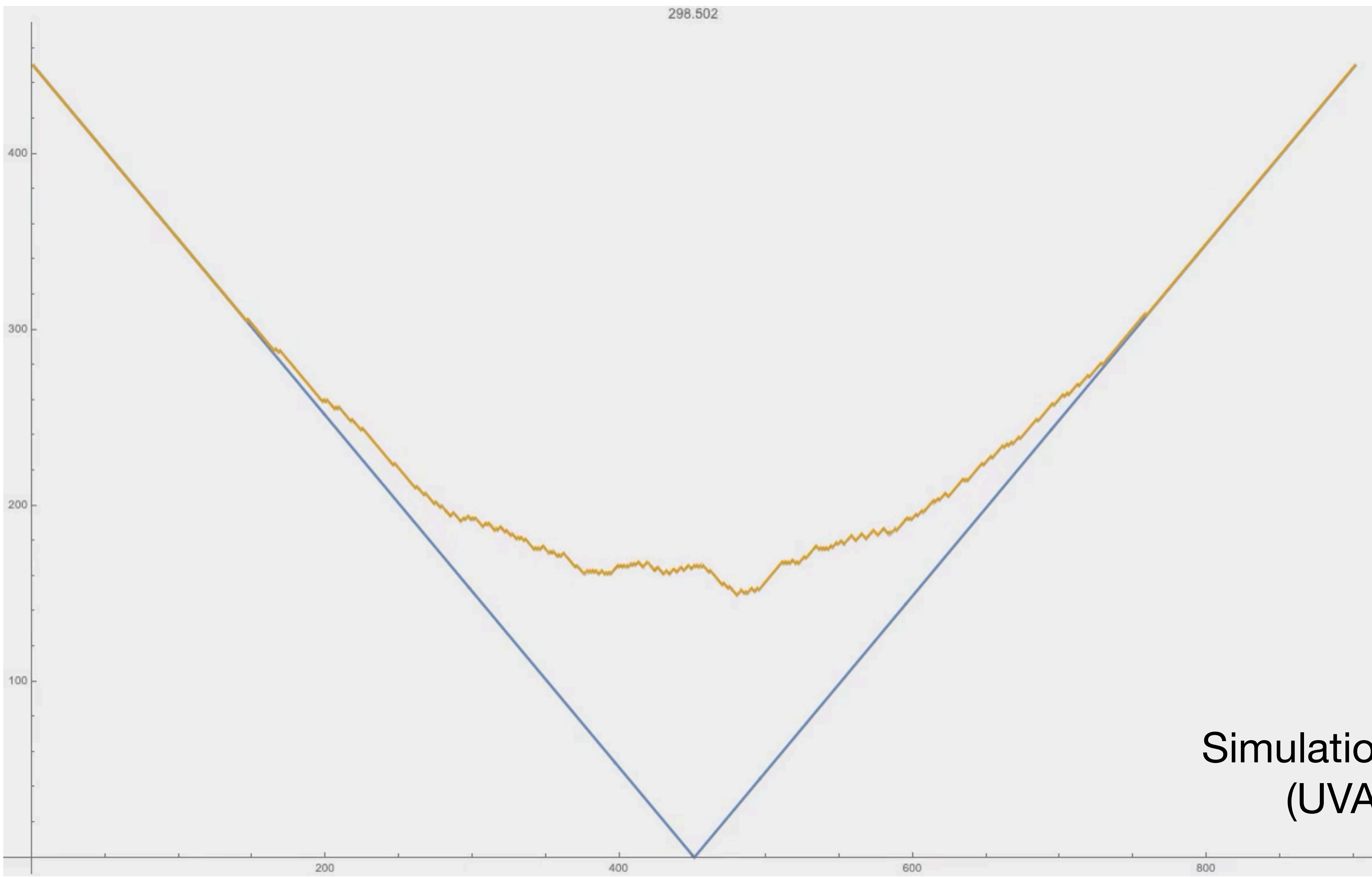

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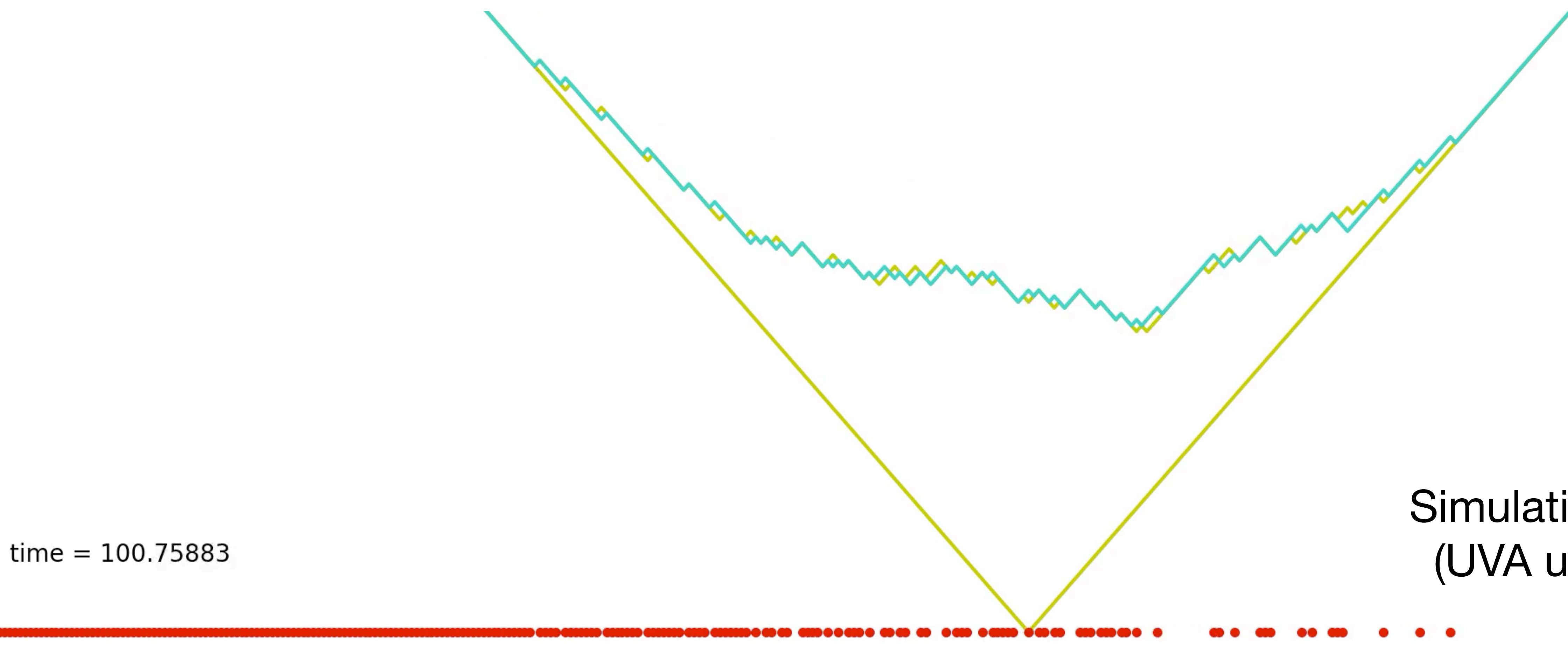
Equilibrium dynamics on TASEP

Corollary. Run in parallel:

- The usual TASEP;
- The backwards Hammersley process slowed down by a factor of t

The combined process preserves the TASEP distribution μ_t.



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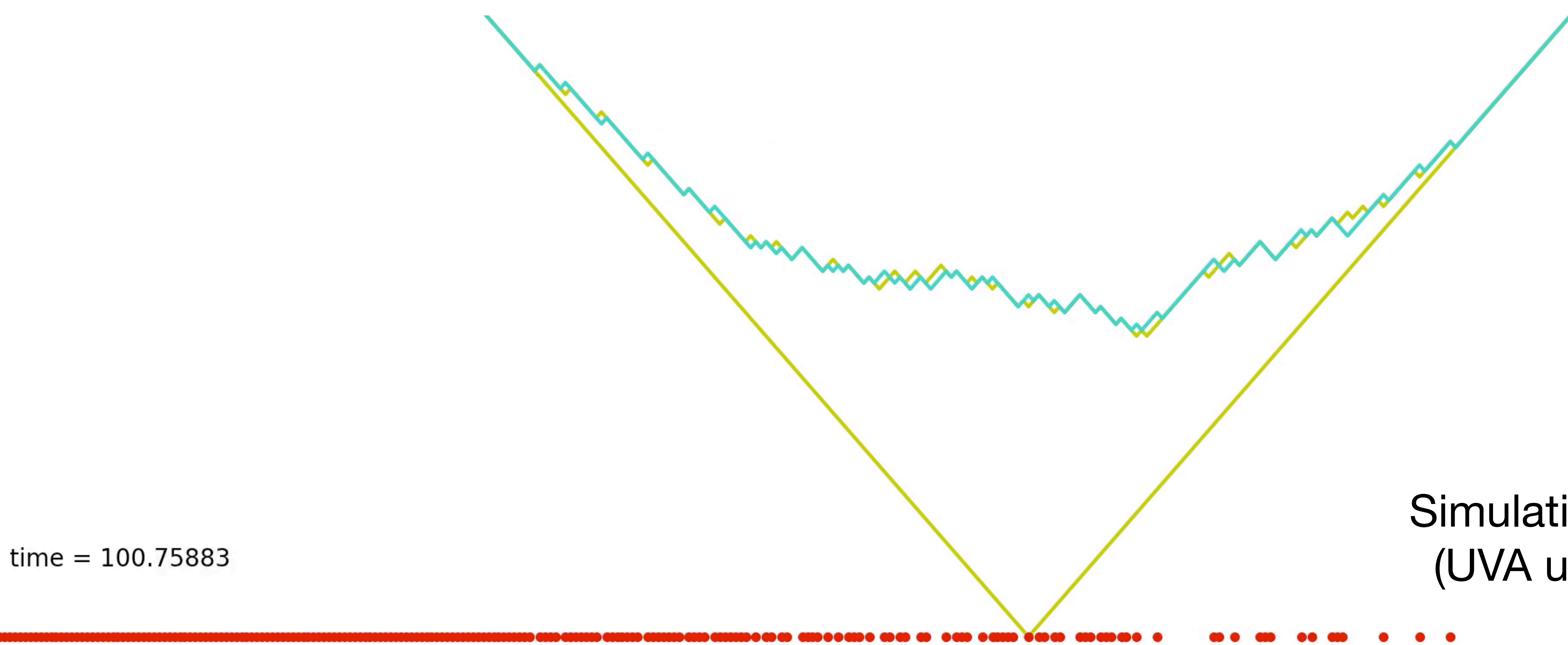

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Strategy of the proof

- TASEP is mapped to Schur measures (e.g. via Robinson-Schensted-Knuth correspondence, known since 1990s)

- Schur measures in a "bosonic" interpretation have a vertex model structure ($U_q(\widehat{\mathfrak{sl}_2})$ model with $q = 0$ and infinite vertical spin)

- Vertex weights satisfy the Yang-Baxter equation

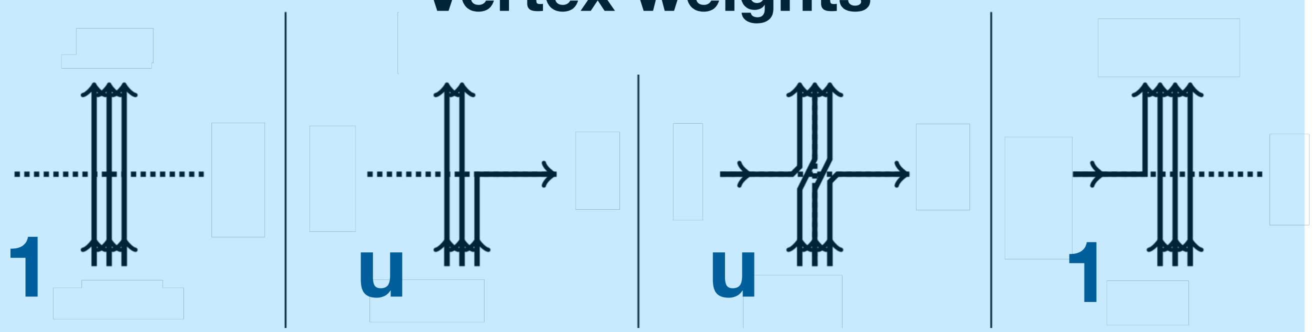
- The Yang-Baxter equation can be turned into a stochastic map

- This stochastic map leads to the backwards Hammersley process and the theorem

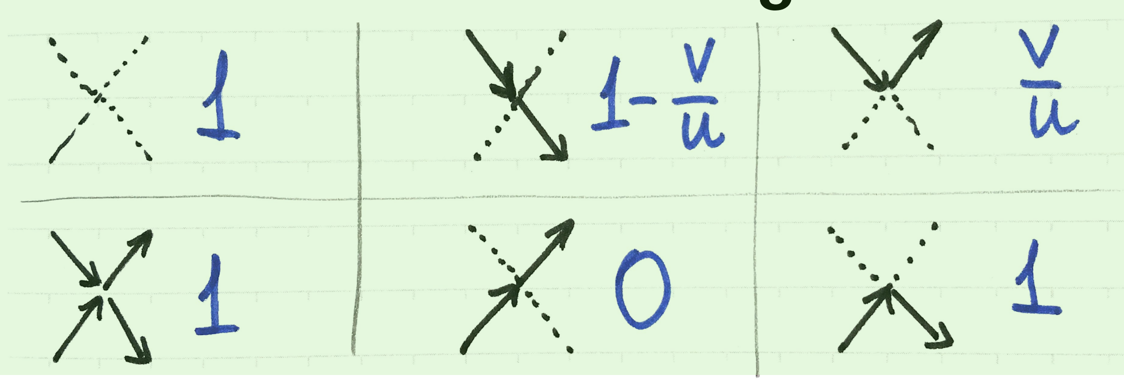
Schur vertex model and Yang-Baxter (RLL) identity

A higher spin six vertex model with $q = 0$ and infinite vertical spin

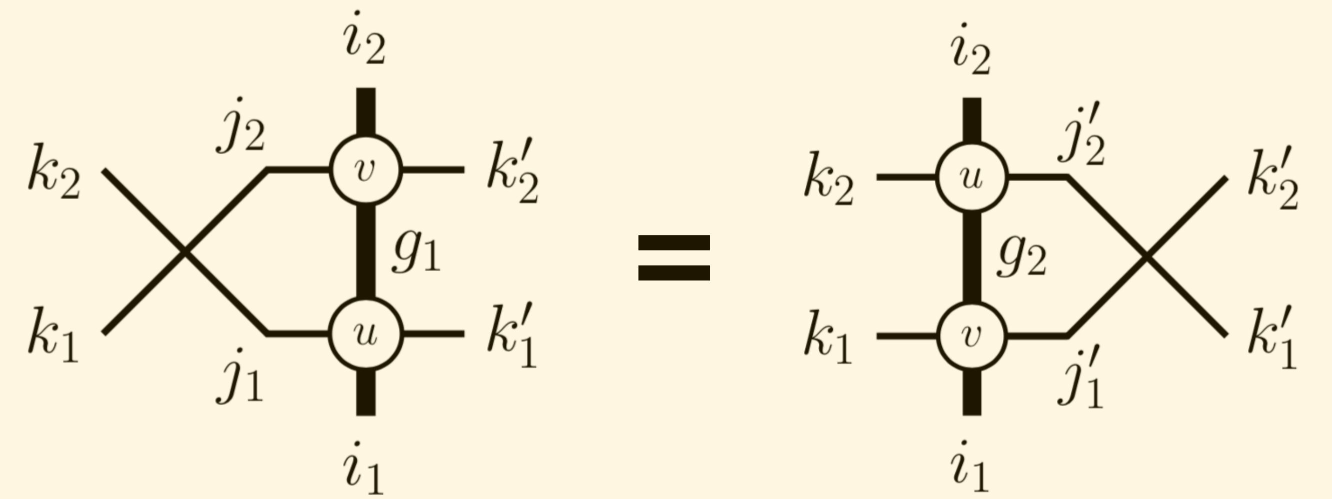
Vertex weights



Cross vertex weights



$U_q(\widehat{\mathfrak{sl}_2})$ **Yang-Baxter equation**



Sum over g_1, j_1, j_2 of the left-hand side is equal to the sum over g_2, j_1', j_2' of the right-hand side

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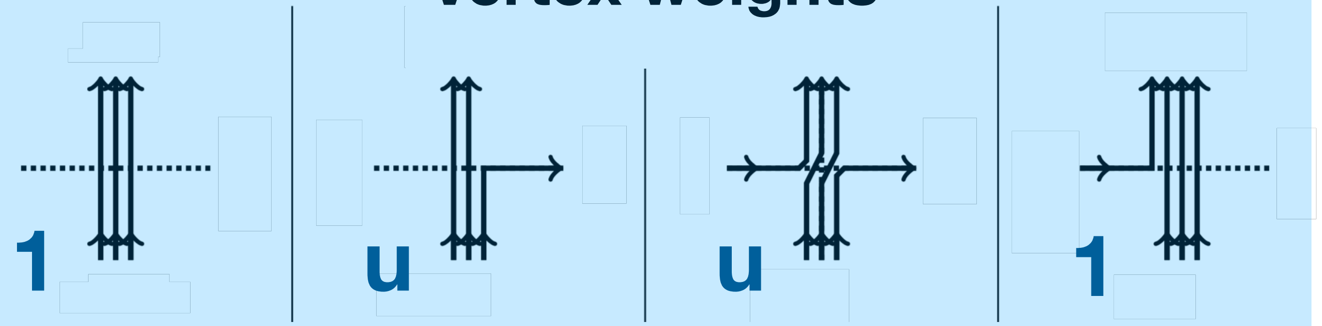

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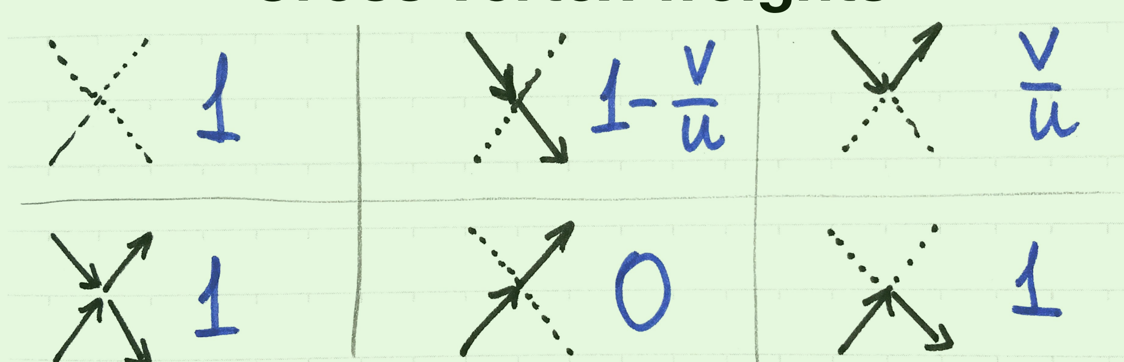

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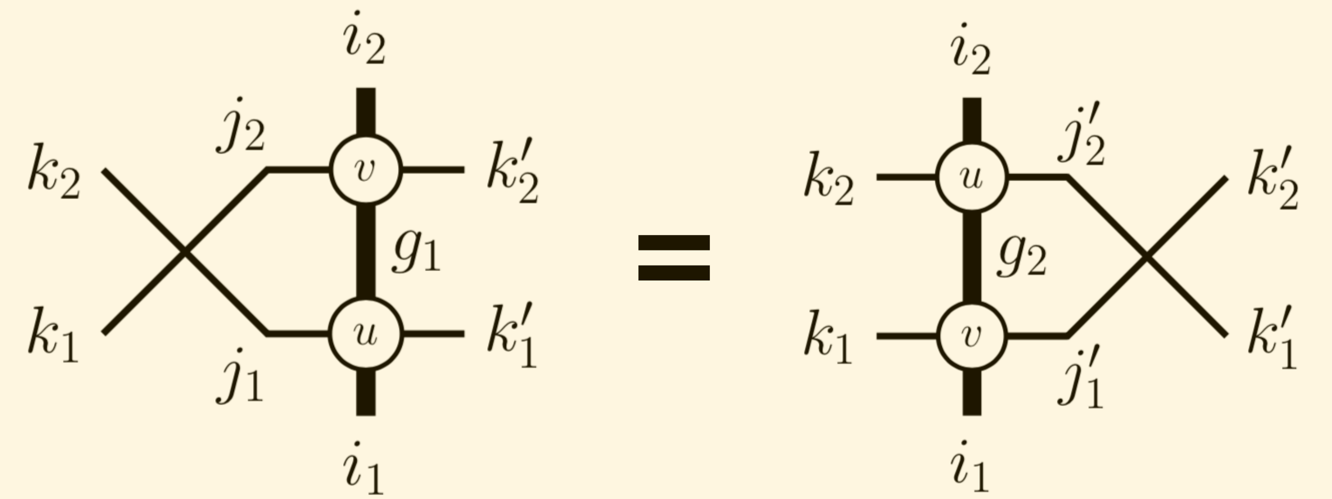
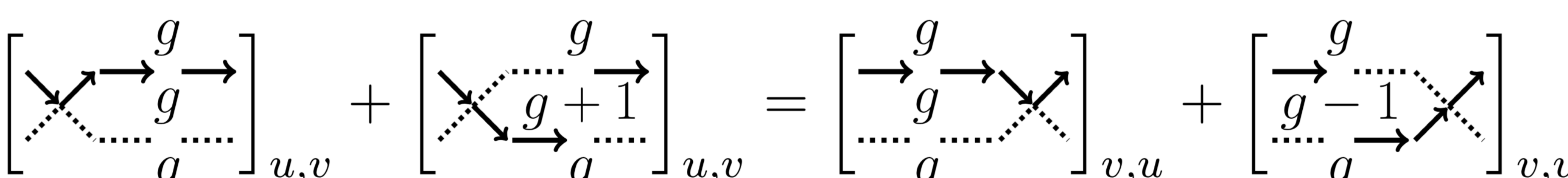


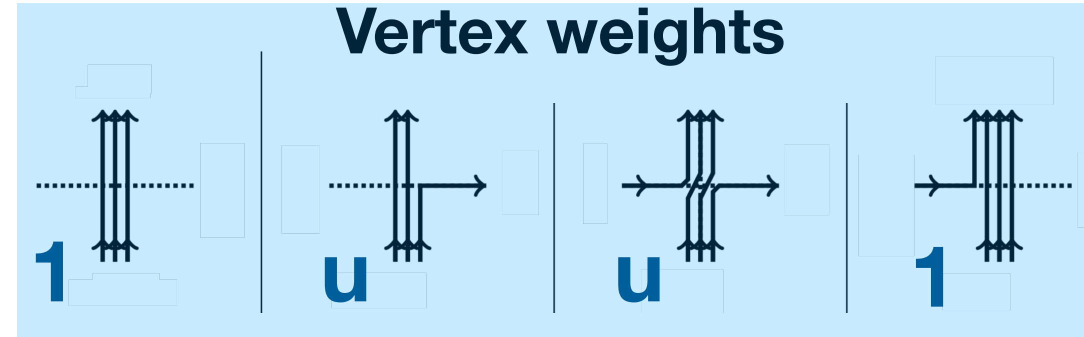
Sum over g_1, j_1, j_2 of the left-hand side is equal to the sum over g_2, j'_1, j'_2 of the right-hand side

Example:



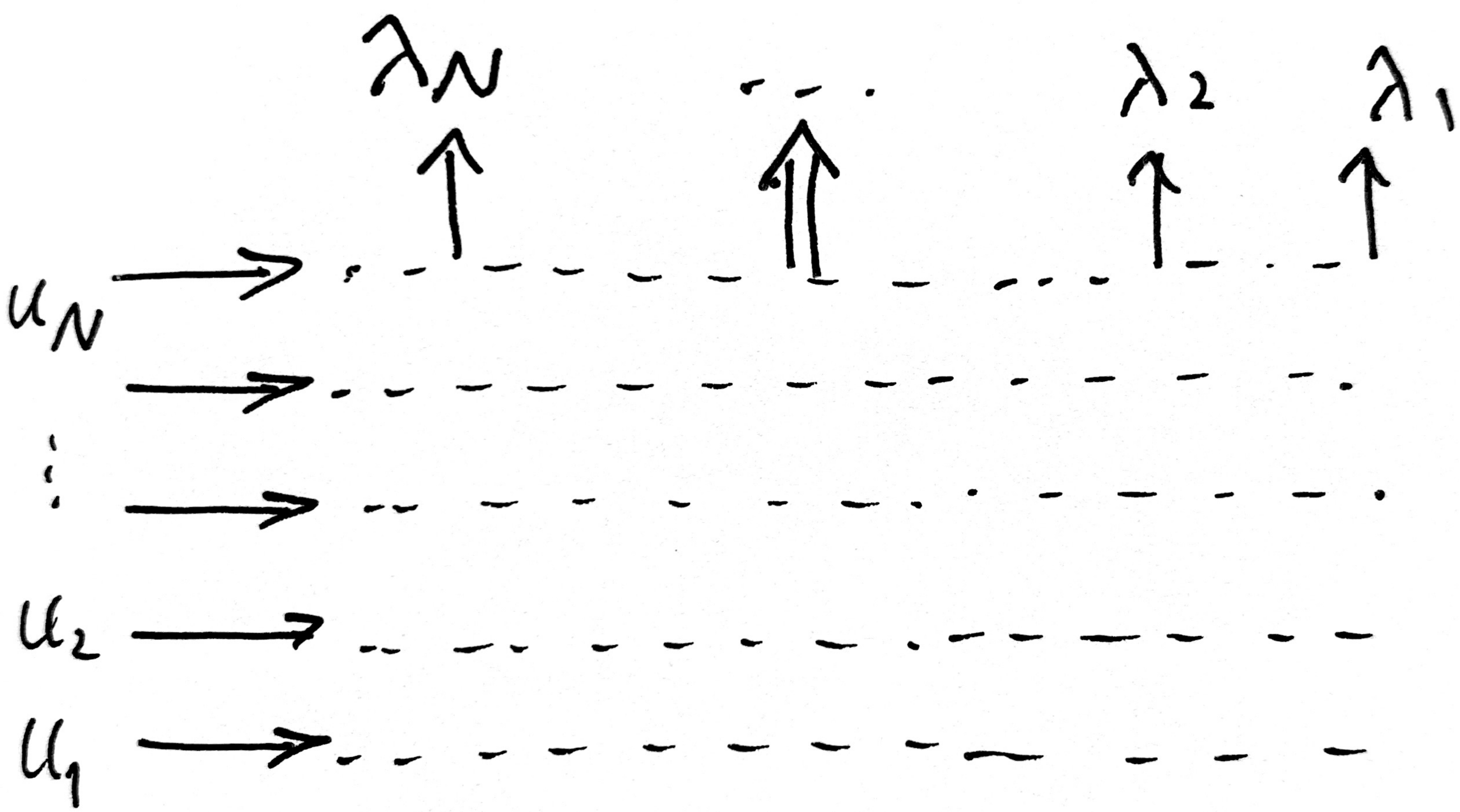
$$\frac{v}{u} \cdot v + \left(1 - \frac{v}{u}\right) \cdot v = \frac{v}{u} \cdot u + 0$$

Schur symmetric polynomials



Let λ be a partition with N parts,
$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0), \lambda_i \in \mathbb{Z}$.

The **Schur polynomial** $s_\lambda(u_1, \ldots, u_N)$ is defined as the partition function of the vertex model:



Another formula:

$$s_\lambda(u_1, \ldots, u_N) = \frac{\det[x_i^{\lambda_j+N-j}]_{i,j=1}^N}{\det[x_i^{N-j}]_{i,j=1}^N}$$

The partition function $s_\lambda(u_1, \ldots, u_N)$ is symmetric thanks to the Yang-Baxter equation

Bijectivisation of the Yang-Baxter equation

Let A, B be finite sets and $\sum_{a\in A} w(a) = \sum_{b\in B} w(b)$ (with positive terms)

A **bijectivisation** (**coupling**) of this identity is a family of transition probabilities $p(a \to b)$ and $p'(b \to a)$, satisfying

$$w(a)p(a \to b) = w(b)p'(b \to a)$$

for all $a \in A,\ b \in B$.

If all probabilities are equal to 0 or 1 and $|A| = |B|$, then this is a usual bijection.

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Example: **1 + 3 = 2 + 2**

$$\begin{array}{c|cc} & 2 & 2 \\ \hline 1 & 1 & 0 \\ 3 & 1/3 & 2/3 \end{array}$$

(maximally dependent)

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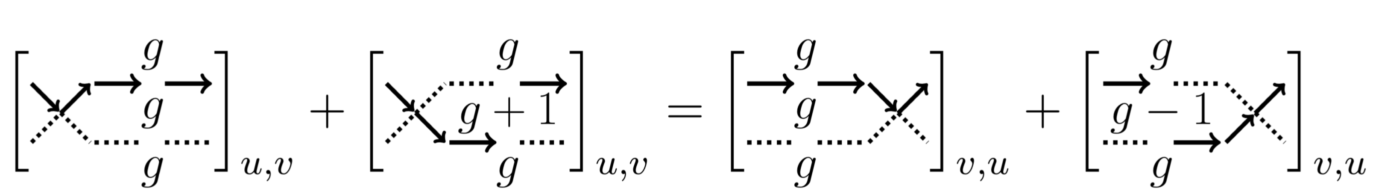
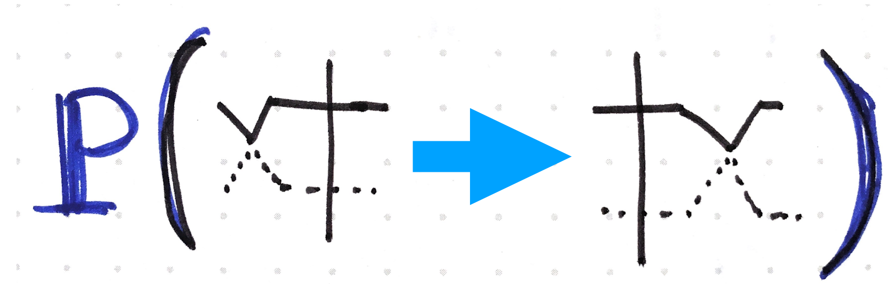
For the Yang-Baxter equation:



leads to

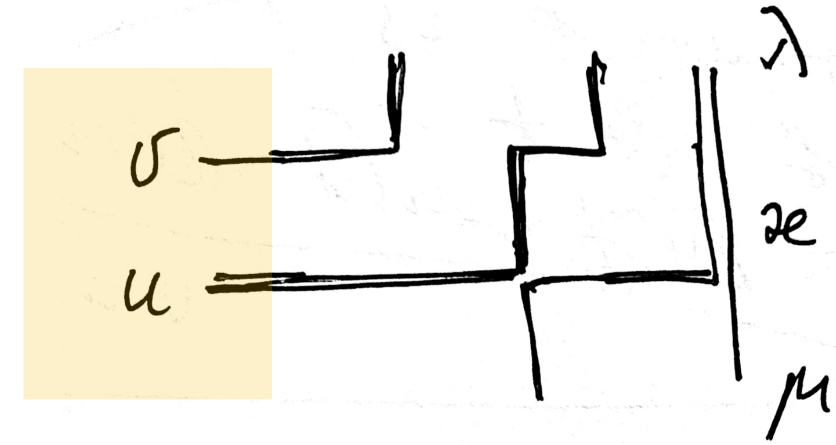


Bijectivisation in the vertex model

With $u > v > 0$, denote the distribution of $\varkappa$ by

$$P_{u,v}(\varkappa \mid \lambda, \mu) \propto u^{|\varkappa|-|\mu|} v^{|\lambda|-|\varkappa|} \propto \left(\frac{u}{v}\right)^{|\varkappa|}$$

(where $|\varkappa| = \varkappa_1 + \ldots + \varkappa_M$).



add $\times$ on the left (weight 1) & use bijectivised YBE to move it right.
Remove $\times$ (weight 1)

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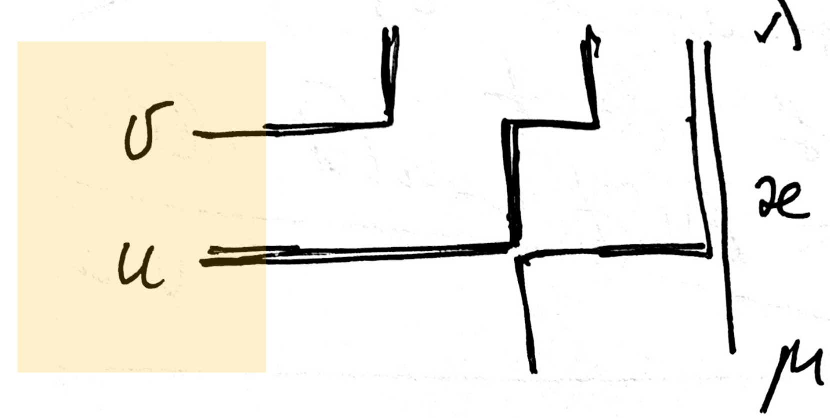
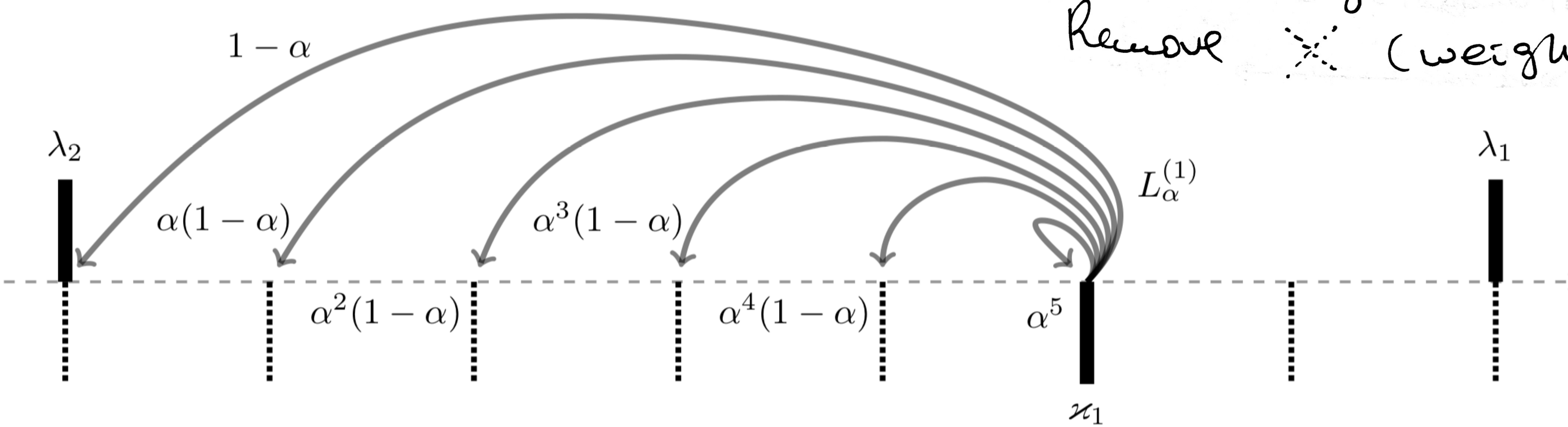


add on the left (weight 1) & use bijectivised YBE to move it right. Remove (weight 1)

Lemma. For $u > v$, the bijectivised YBE maps the measure $P_{u,v}$ to $P_{v,u}$ and acts as follows, where $\alpha = v/u$:



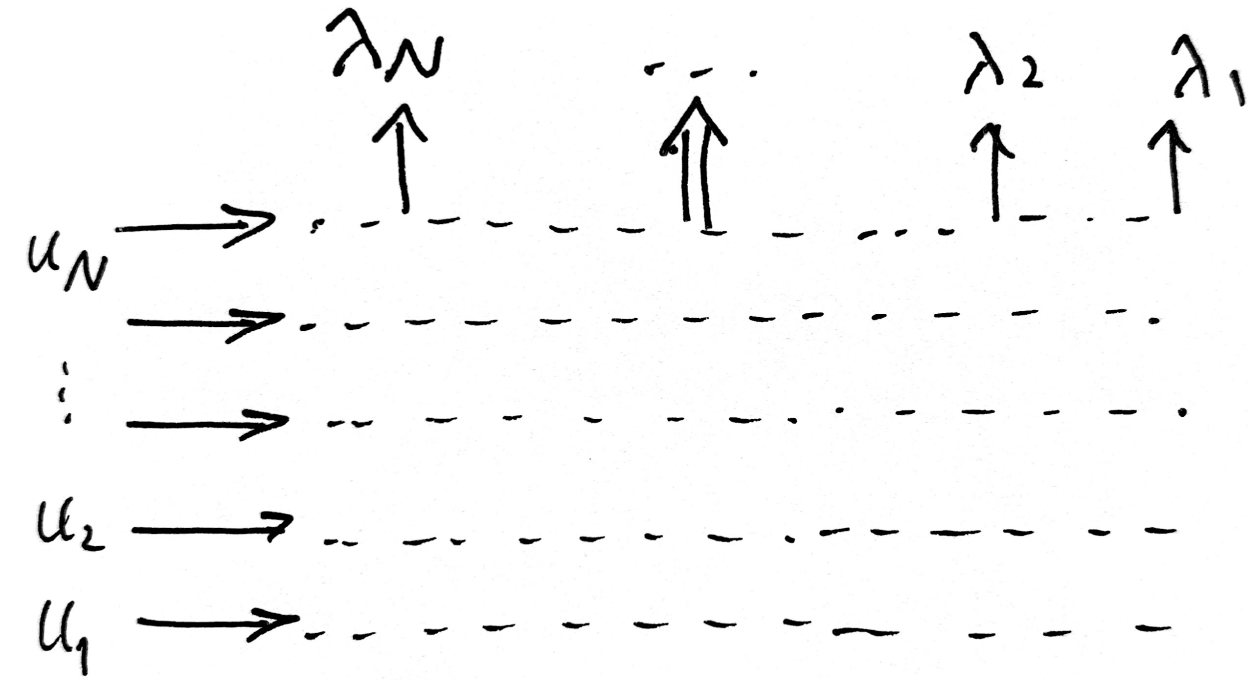
Same map with **right** jumps maps $P_{v,u}$ back to $P_{u,v}$

(so, where you jump depends on the order of u, v).

Remark. Action of the symmetric group S_N

Let $u_1, \ldots, u_N > 0$ be distinct spectral parameters. With each $\sigma \in S_N$ we associate a measure $\mathbb{M}_\sigma$ on configurations with top condition λ and spectral parameters $u_{\sigma(1)}, \ldots, u_{\sigma(N)}$.

The Markov operators or $L_\alpha^{(j)}$ and $R_\alpha^{(j)}$ with appropriate α act by transpositions on σ.



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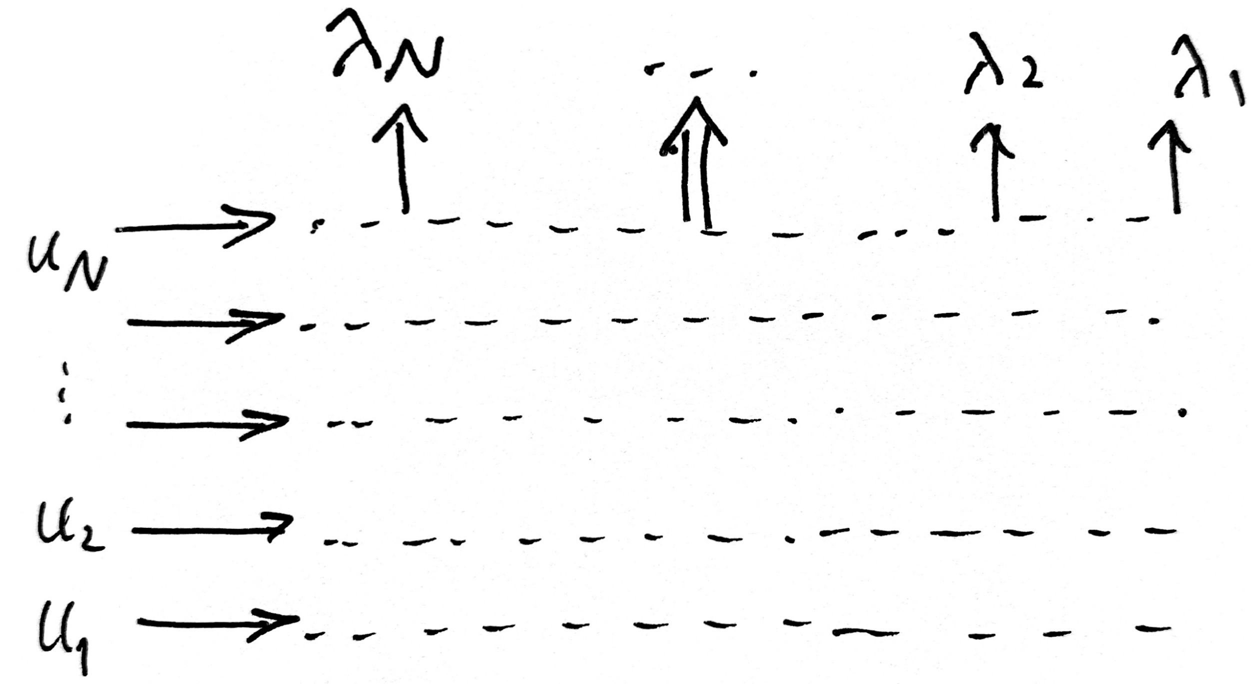

The Markov operators or $L_\alpha^{(j)}$ and $R_\alpha^{(j)}$ with appropriate α act by transpositions on σ.

In particular, the sequence $(1, q, q^2, \ldots, q^{N-1})$, $0 < q < 1$, can be mapped to its reversal $(q^{N-1}, \ldots, q^2, q, 1)$ by only applying the left jump operators $L^{(j)}$, totally $\binom{N}{2}$ of them

$q = 0.7$

(lozenge tilings are in bijection with vertex configurations)

Simulation joint with Edith Zhang (UVA undergraduate)

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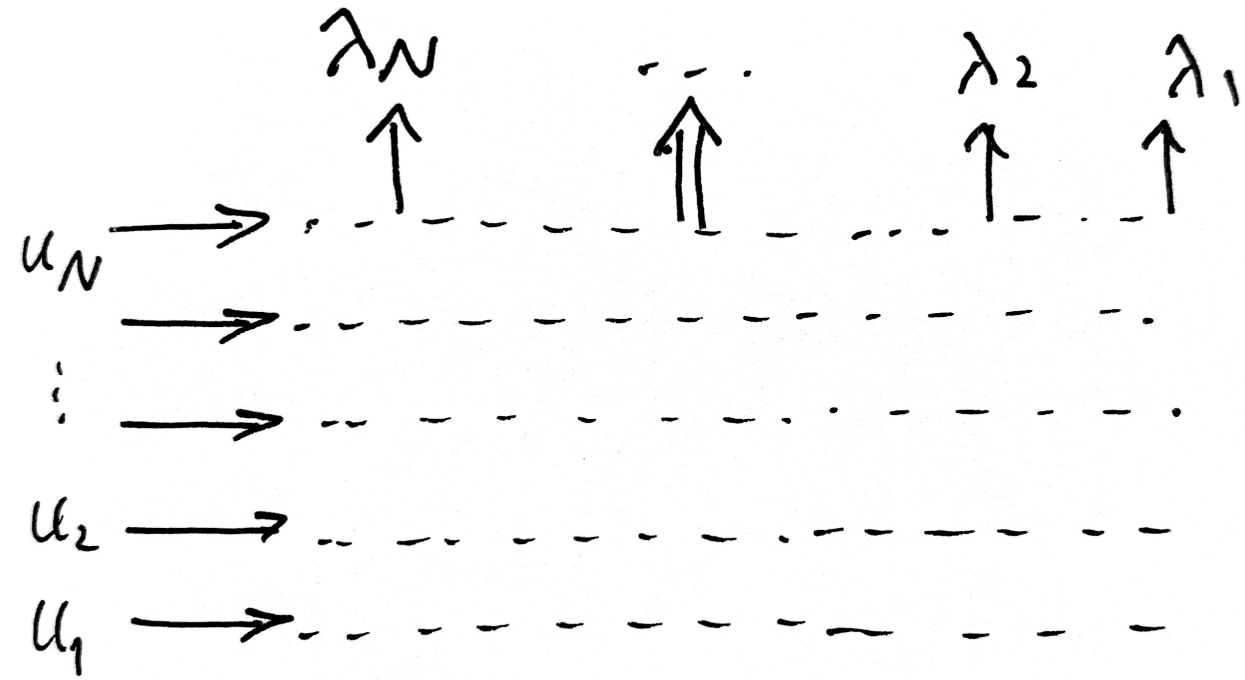

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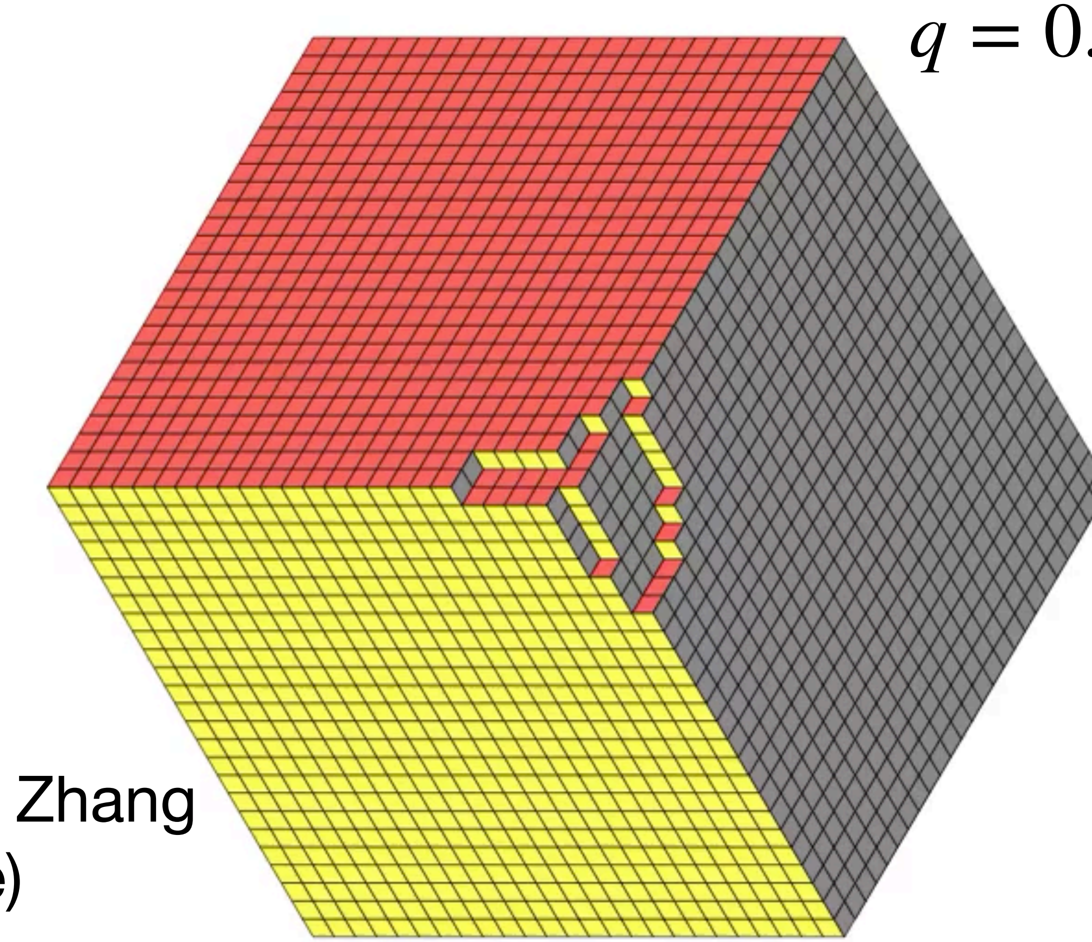

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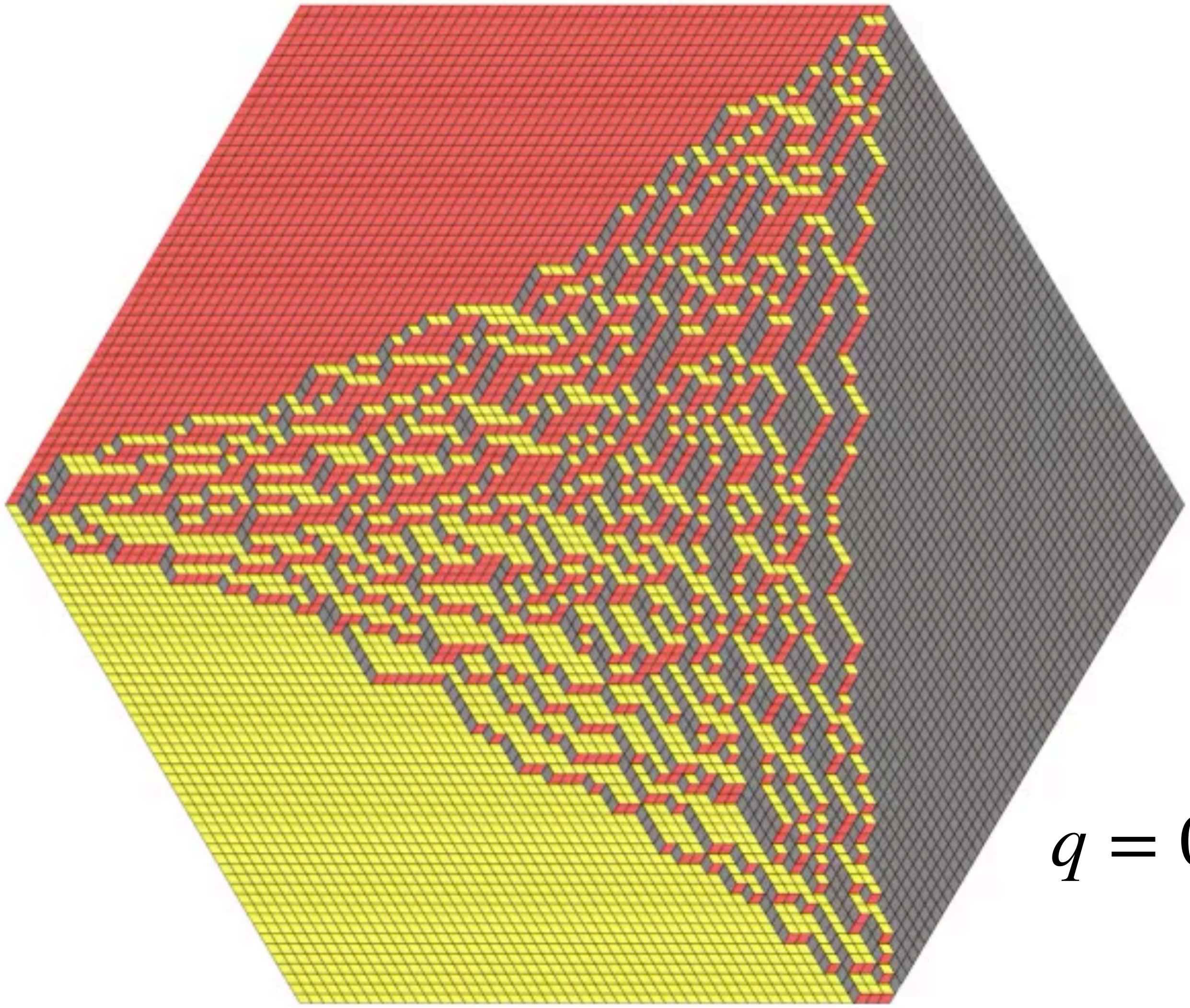
Limit shapes of q^{vol} lozenge tilings: **[Cohn-Kenyon-Propp '00], [Kenyon-Okounkov '05]**



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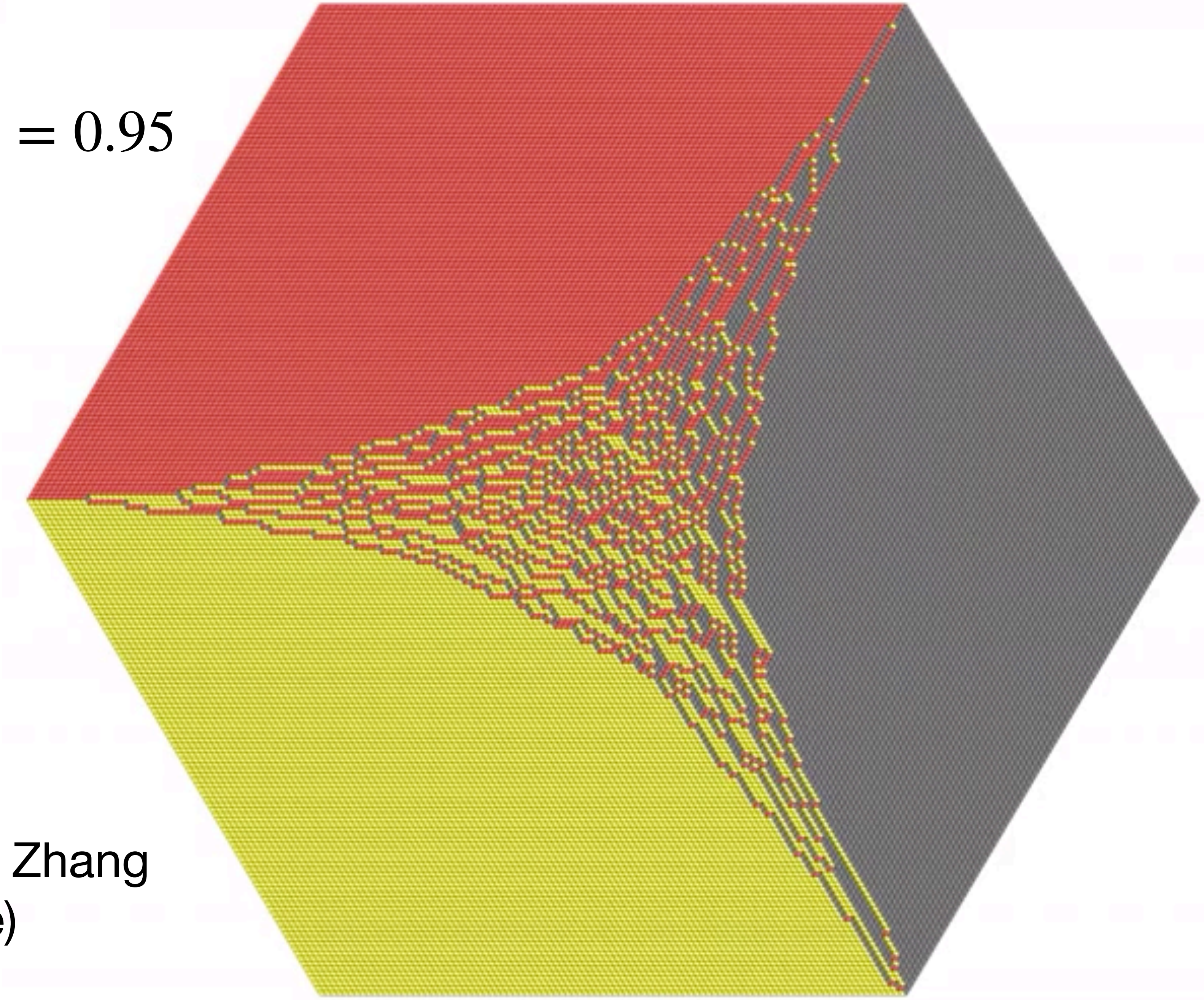

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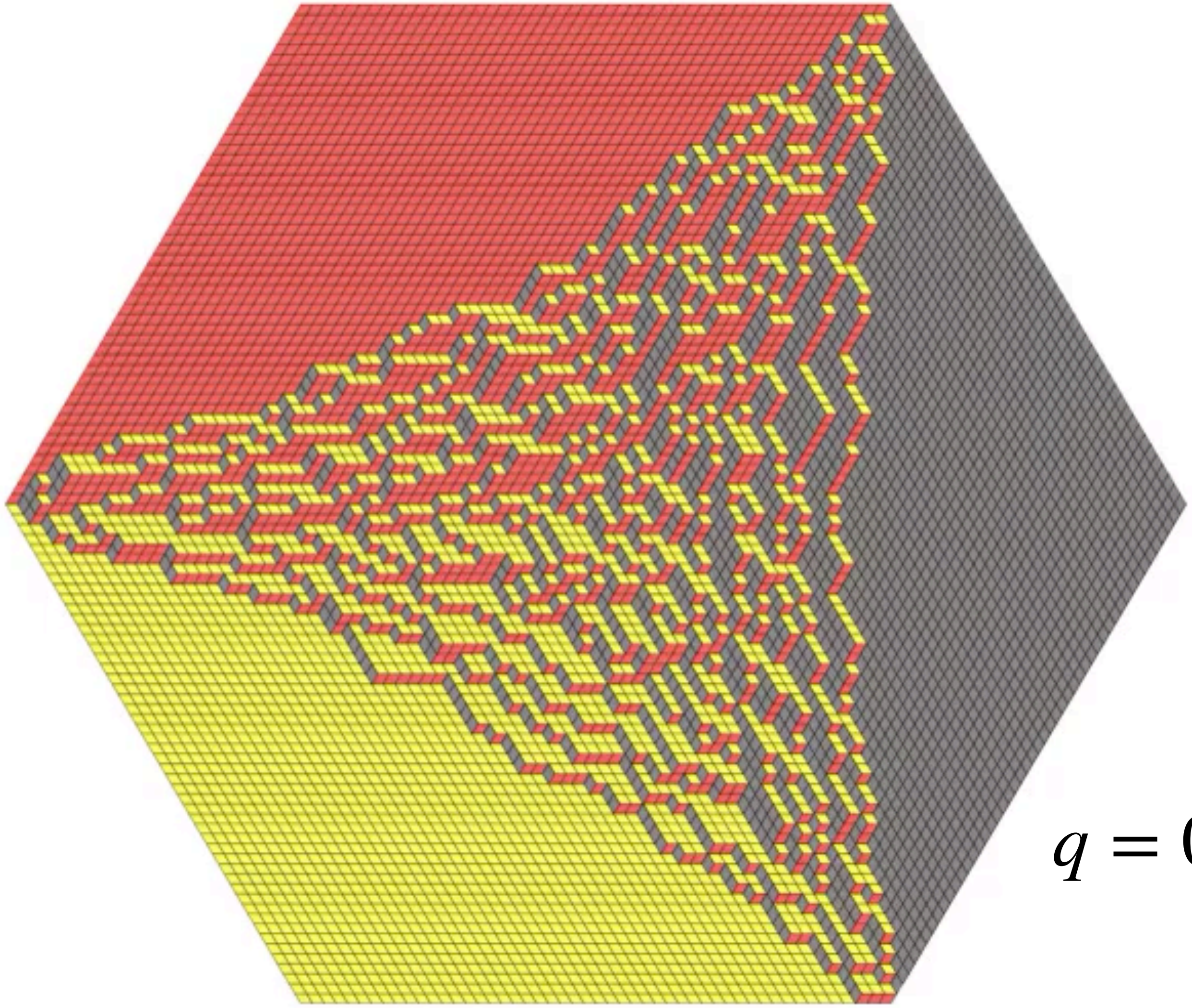

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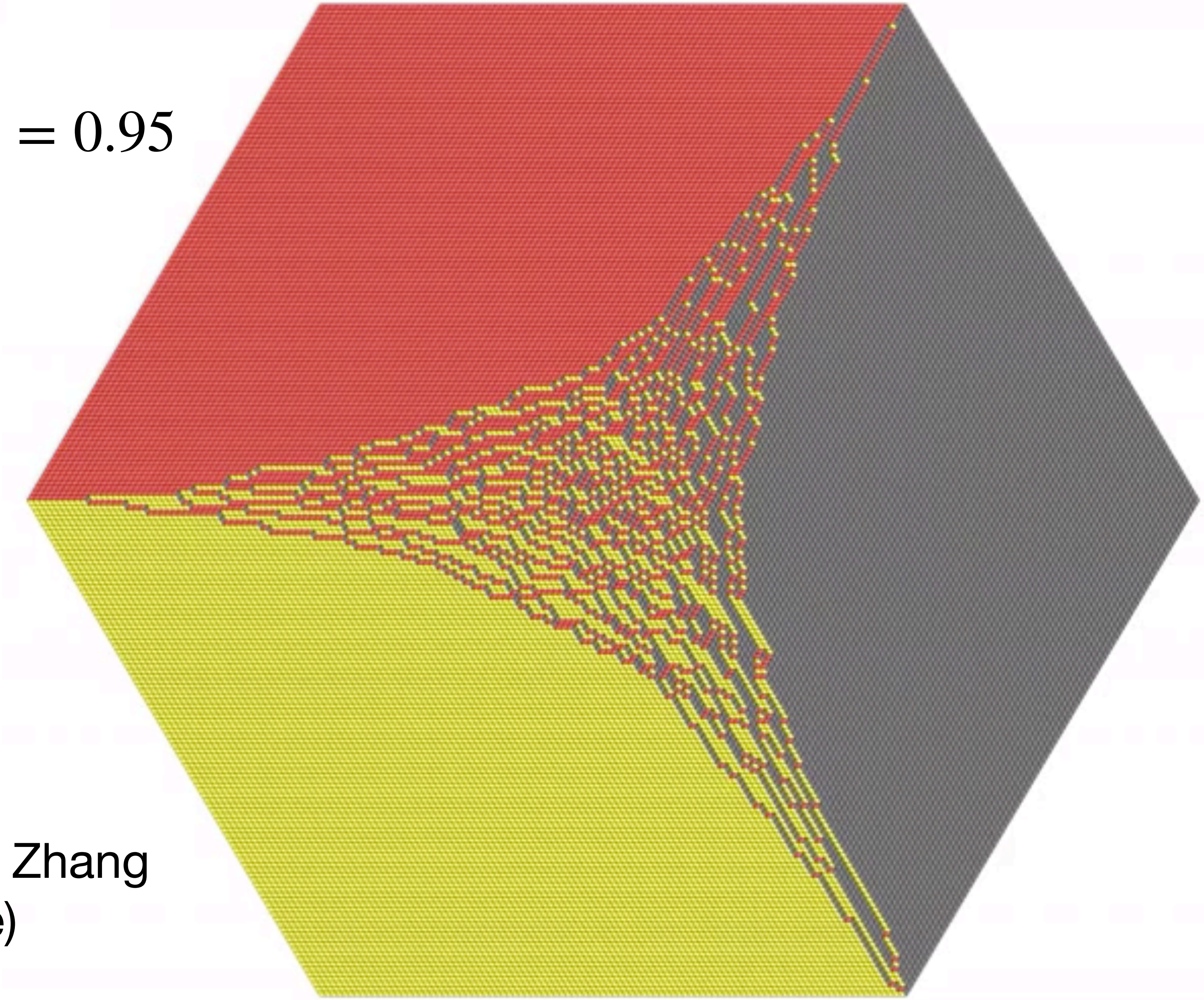

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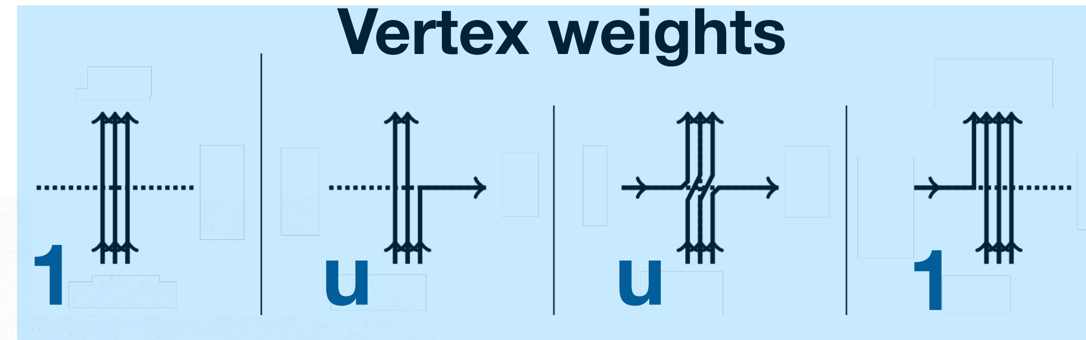
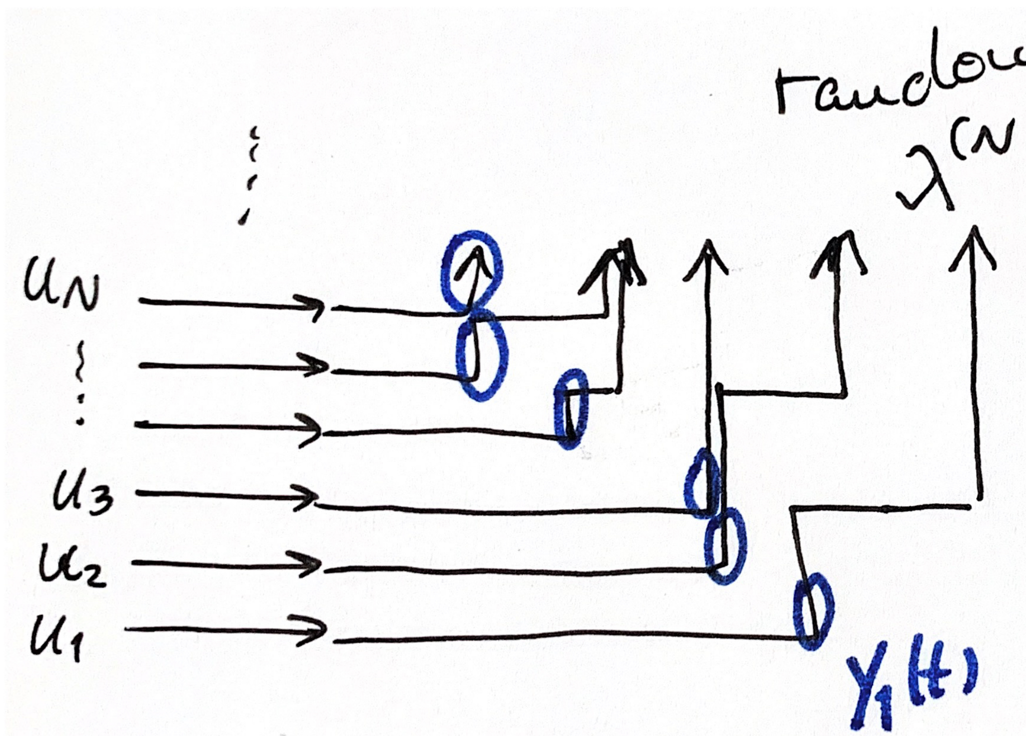
TASEP at the edge of a vertex model

Via Robinson-Schensted-Knuth (since 1990s), **[O'Connell '03]**, also **[Borodin-Ferrari '08]**





$$x_i(t) \overset{d}{=} y_i(t) - i$$

TASEP

vertex model

$$\mathrm{Prob}[\lambda^{(N)}(t) = \lambda] = e^{-(u_1+\ldots+u_N)t} s_\lambda(u_1, \ldots, u_N) s_\lambda(\rho_t),$$

where $s_\lambda(\rho_t) := \lim_{K\to+\infty} s_\lambda(\frac{t}{K}, \ldots, \frac{t}{K})$ (K times)

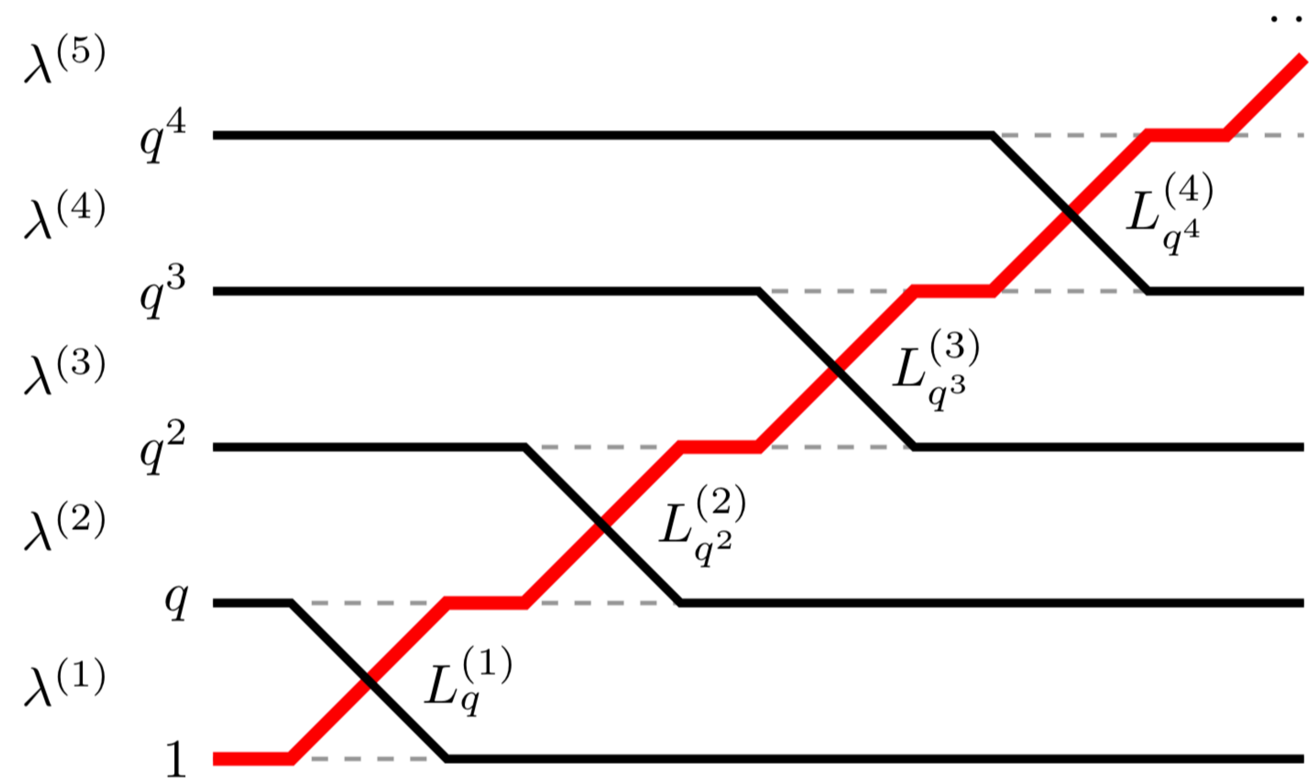
Then $y_i(t)$ coming from this vertex model are identified with particle coordinates of TASEP with speeds u_i at time t.

Edge of a vertex model and $L^{(j)}$. Finishing the proof

Take the spectral parameters (speeds in TASEP) to be $1, q, q^2, \ldots$, where $0 < q < 1$.

Apply the left jumps in the vertex model in this order.

Then the measure $s_\lambda(1, q, q^2, \ldots) s_\lambda(\rho_t)$ turns into $s_\lambda(q, q^2, q^3, \ldots) s_\lambda(\rho_t)$, and



$$s_\lambda(q, q^2, q^3, \ldots) s_\lambda(\rho_t) = q^{|\lambda|} s_\lambda(1, q, q^2, \ldots) s_\lambda(\rho_t) = s_\lambda(1, q, q^2, \ldots) s_\lambda(\rho_{q \cdot t})$$

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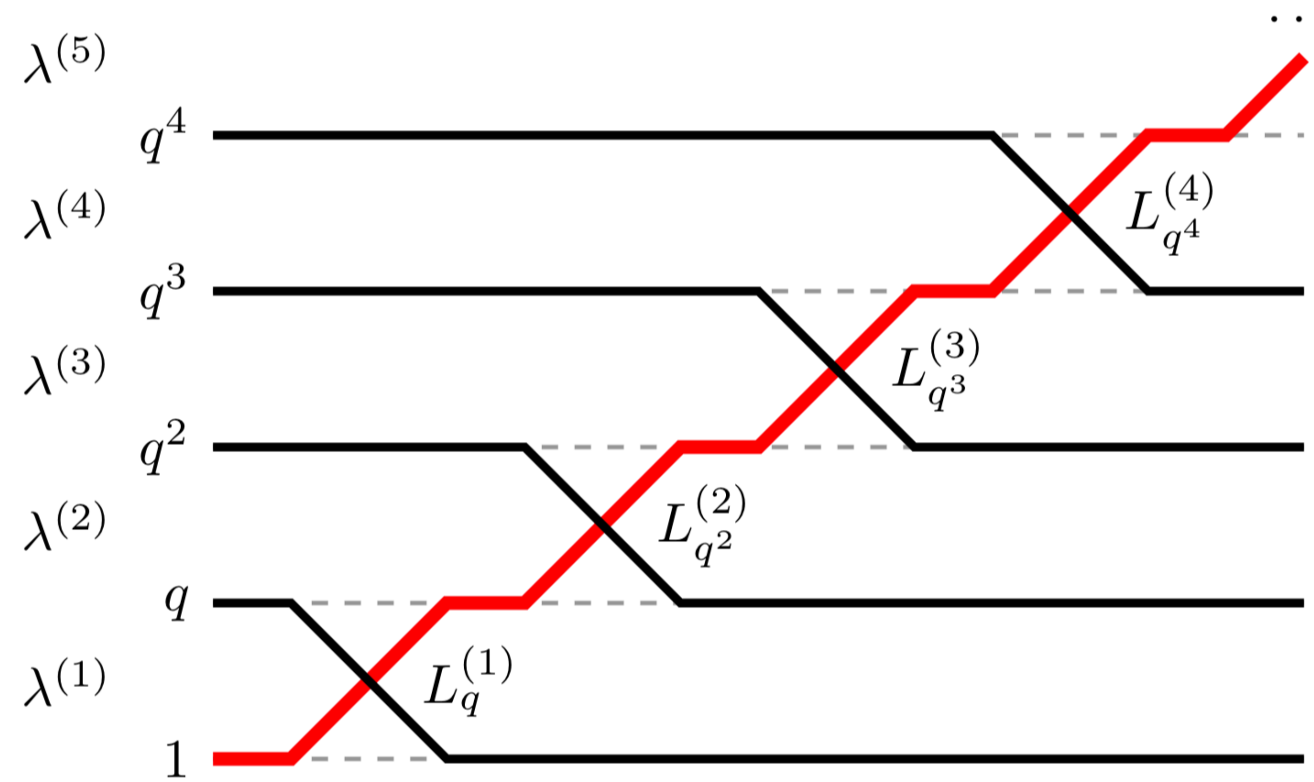

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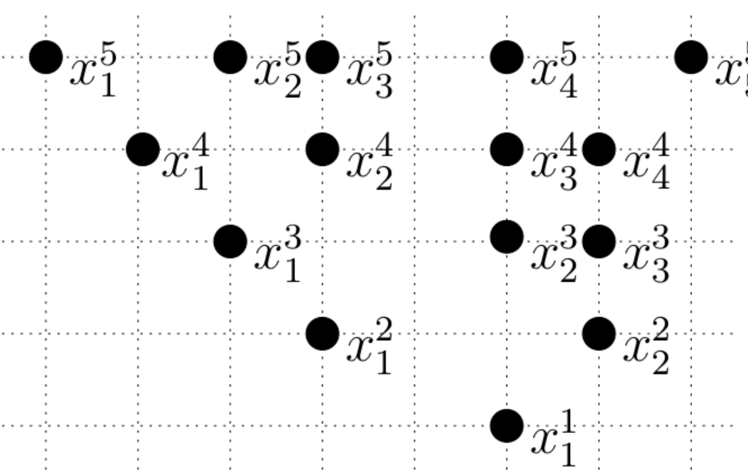
We thus proved:

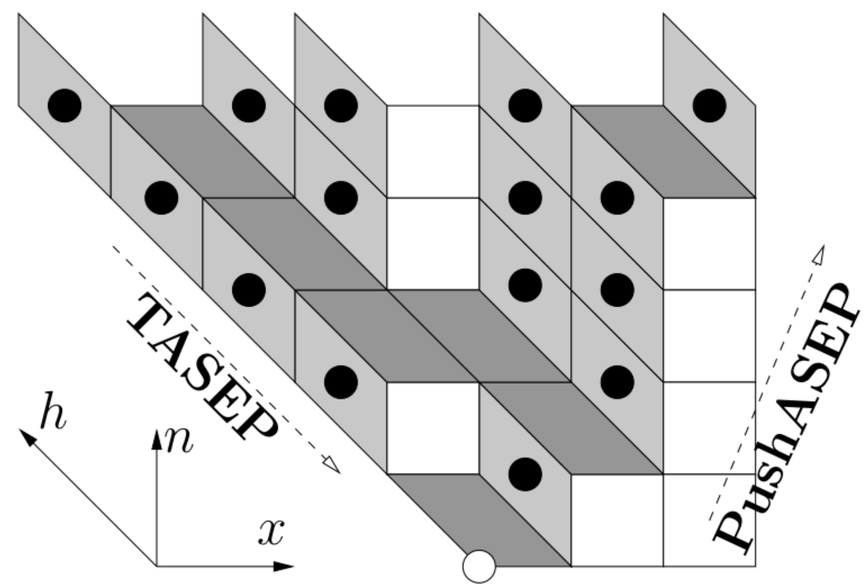
Theorem. Denote by $\mu_t^{(q)}$ the measure of the TASEP with q-speeds, and $\mathbb{L}_q$ the combination of the left jump operators as above. Then $\mu_t^{(q)} \mathbb{L}_q = \mu_{q \cdot t}^{(q)}$.

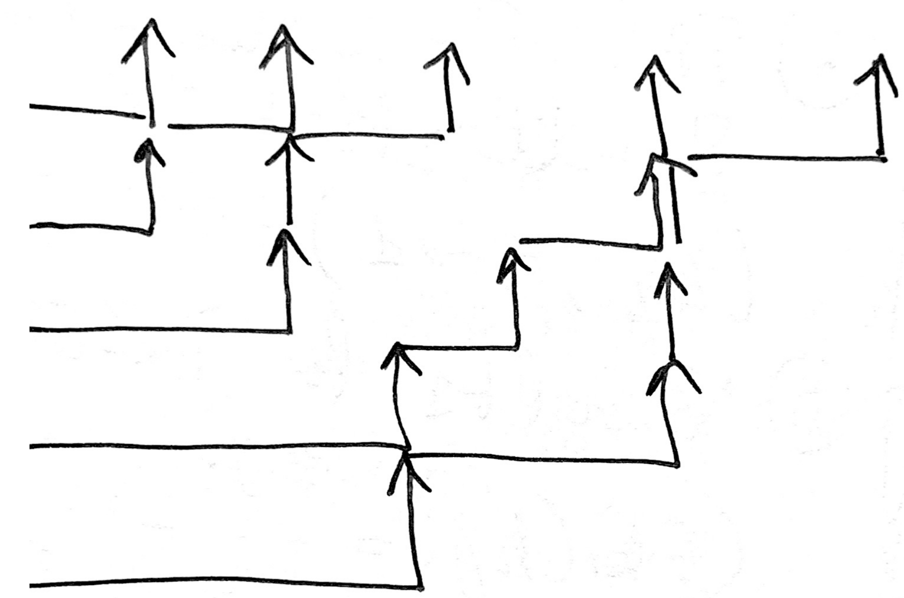
Applying $\mathbb{L}_q$ many times and taking Poisson-like limit $q \to 1$ leads to the main theorem.

Remark. Back in time in a two-dimensional model

The Borodin-Ferrari **Anisotropic KPZ** dynamics (arXiv:0804.3035 [math-ph]) on two-dimensional interlacing arrays also has a reversal.

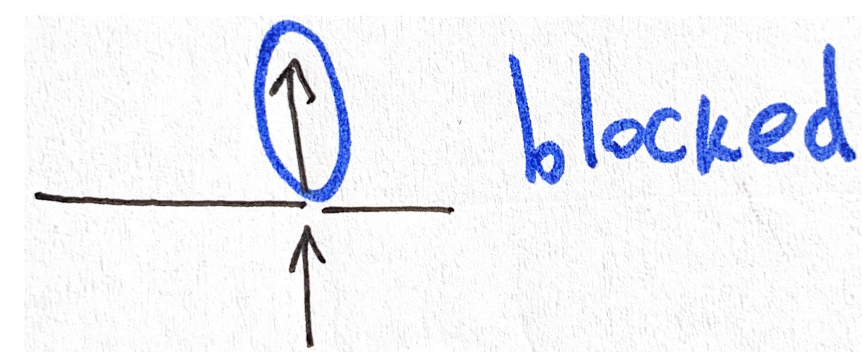






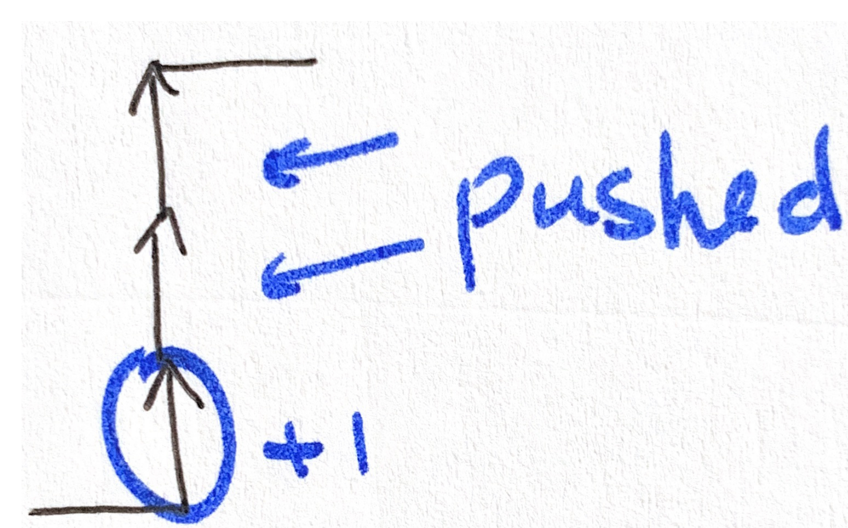
Forward dynamics is defined via push-block rules:

- Each vertical arrow has an independent exponential clock with rate 1. When the clock rings, the arrow attempts to move to the right by 1 (in its horizontal line)



- If the jumping arrow is blocked from below, there is no jump

- If the arrow's jump violates interlacing with above, pushing is forced



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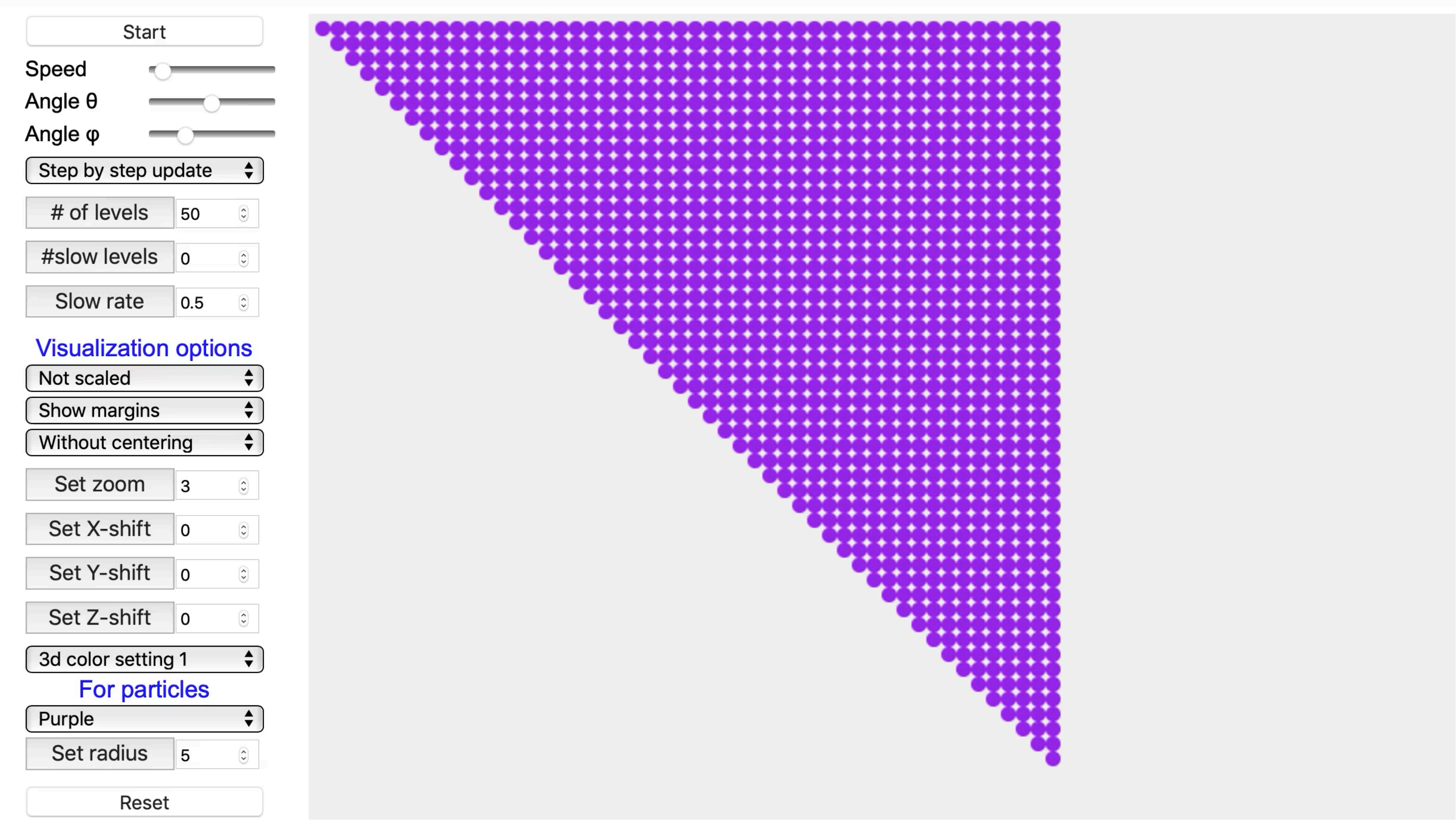

Simulation by Patrik Ferrari https://wt.iam.uni-bonn.de/ferrari/research/jsanimationakpz/

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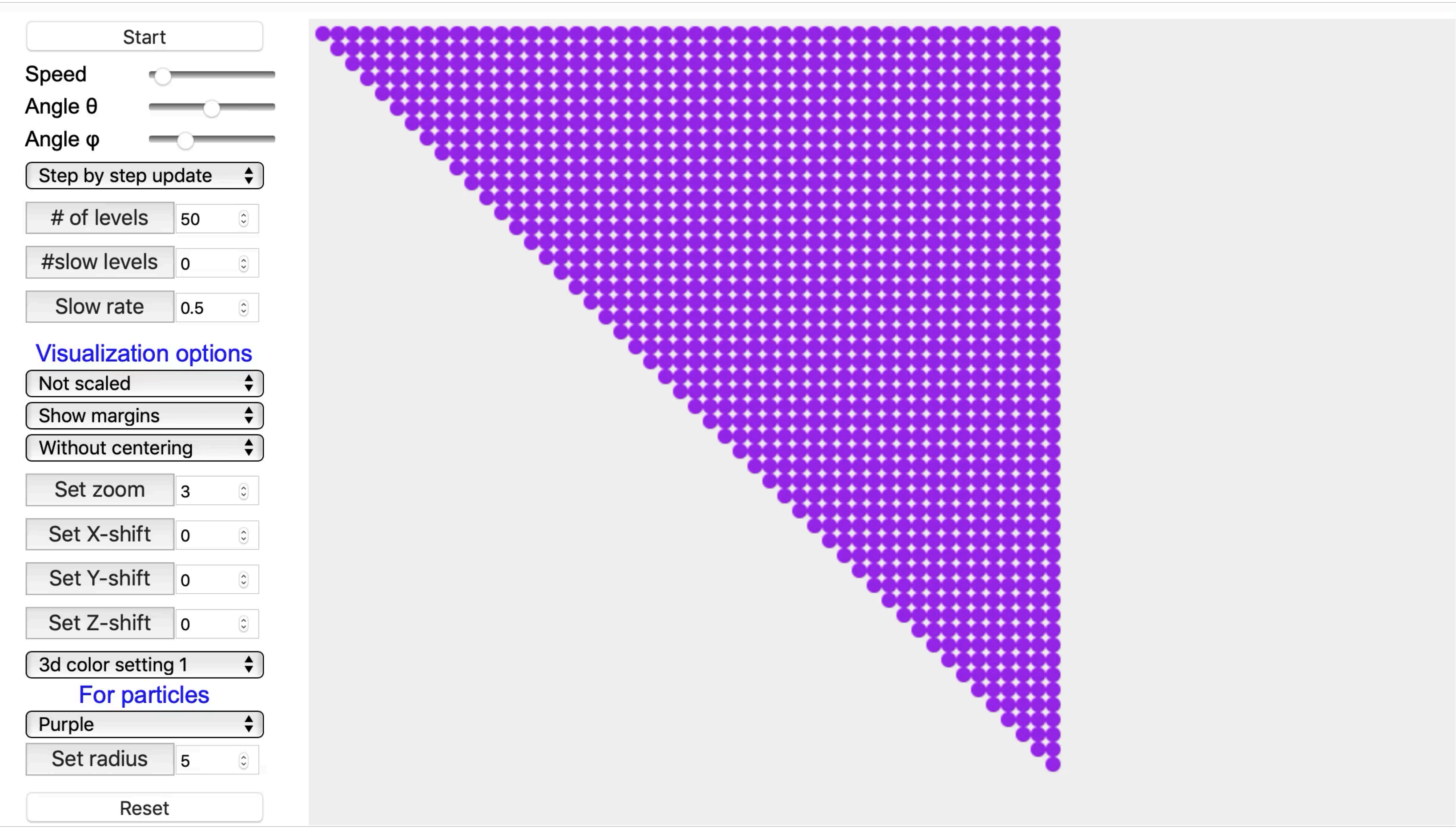
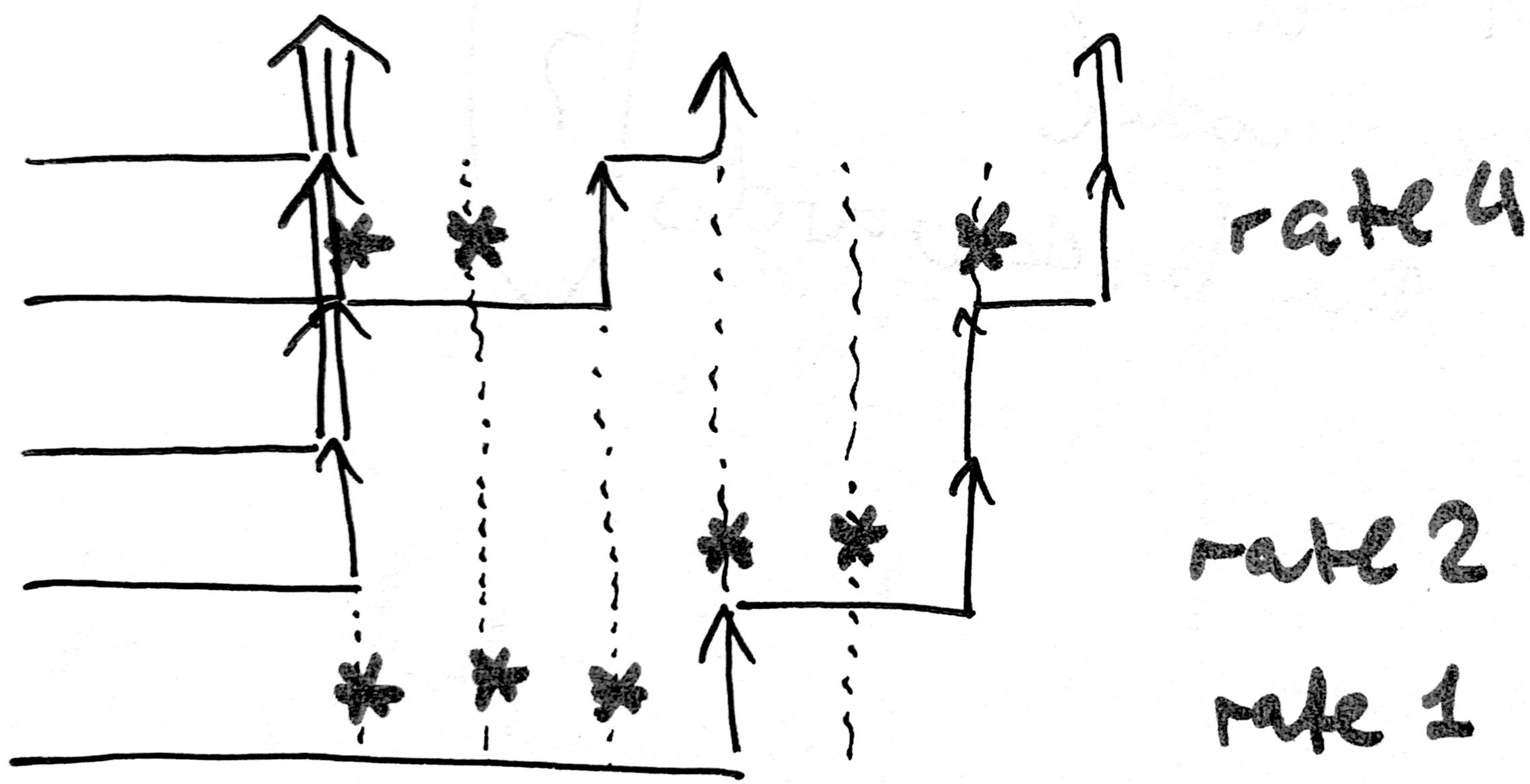


Simulation by Patrik Ferrari https://wt.iam.uni-bonn.de/ferrari/research/jsanimationakpz/

Remark. Back in time in a two-dimensional model

The reversal dynamics is defined as follows. Each possible(*) hole at each level k has an independent exponential clock of rate k. When the clock rings, the hole attracts the closest of its right neighbors.



Like for the TASEP reversal (backwards Hammersley), some care is needed to define the dynamics as dependence propagates “from infinity” (“possible holes” depend on the lower and the upper levels); and there are infinitely many jumps in finite time.

For push-block initial data this is possible, and we get the time reversal $M_t \mathbb{L}_\tau = M_{t \cdot e^{-\tau}}$

Consequences for TASEP

Theorem. $\mu_t L_\tau = \mu_{t \cdot e^{-\tau}}$

- This is a new structural result for TASEP with step initial data
- Likely characterizes the distribution μ_t by a stationary dynamics (one has to prove convergence to the distribution μ_t)
- Leads to new identities for expectations under μ_t, for example:

$$\frac{\partial}{\partial t}\mathbb{E}G(h_t) = -\frac{1}{t}\,\mathbb{E}\Big(h_t \cdot (G(h_t - 1) - G(h_t)) \cdot \#\{\text{distance from 0 to the leftmost particle in } \mathbb{Z}_{<0}\}\Big)$$

Here h_t is the number of particles to the right of zero, and $G(\,\cdot\,)$ is an arbitrary function.

- The presence of "two times" in TASEP raises questions about fluctuation exponents. For example, if we run TASEP for time t and stationary dynamics for time s, what are the combined fluctuation exponents?
- Limit of stationary dynamics to the top Airy line?
- Other initial data?

Summary. More questions than answers

- Found a new interesting property of TASEP

- Characterization of nonequilibrium distributions by means of stationary dynamics

- How general is this effect?
  - Seems that it applies (in one way or another) to most integrable particle systems in the KPZ universality class: stochastic six-vertex model, random polymers, edge of random matrices…

- New asymptotic questions - do some known good tools work?

- Time reversal applies to some two-dimensional models, too

- Any other stochastic applications of YBE? Yes, at least **[Grimmett-Manolescu 2014]** in critical percolation; and domino shuffling for Aztec diamond **[Elkies-Kuperberg-Larsen-Propp 1992]** are essentially based on bijectivisations of YBE