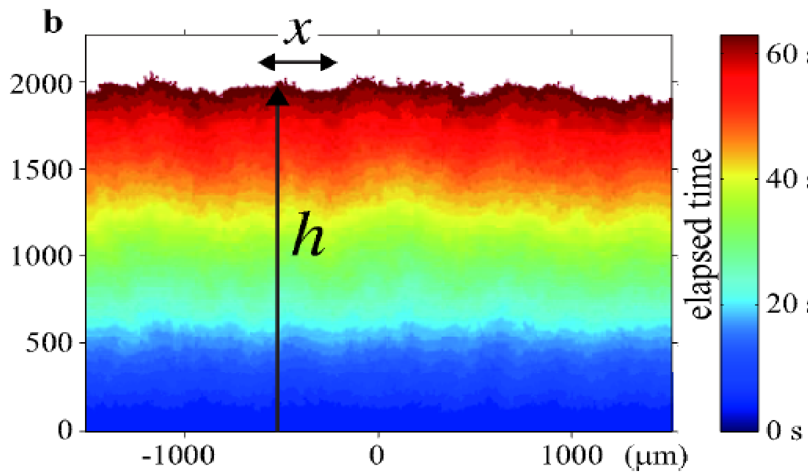
Large deviations for the Kardar-Parisi-Zhang equation

and the edge of random matrices

$h(x,t)$ height field $x \in \mathbb{R}$

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \sqrt{2}\,\xi(x,t)$$

space-time
white noise



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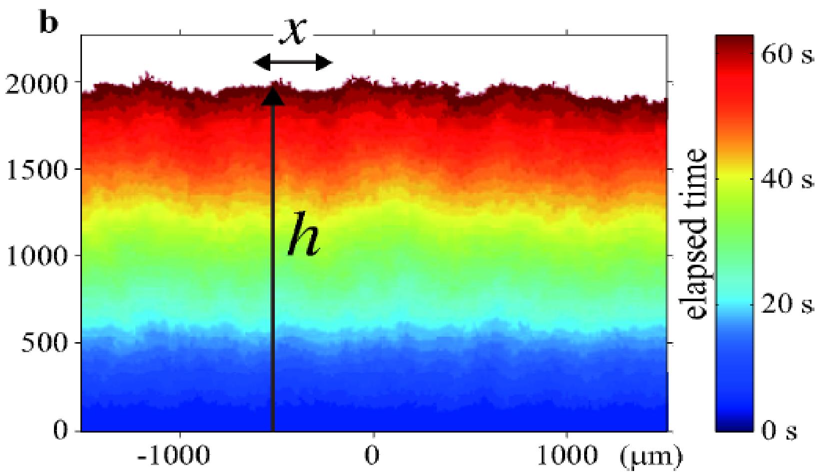

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SHE $\partial_t Z = \partial_x^2 Z + \sqrt{2}\,\xi(x,t) Z$

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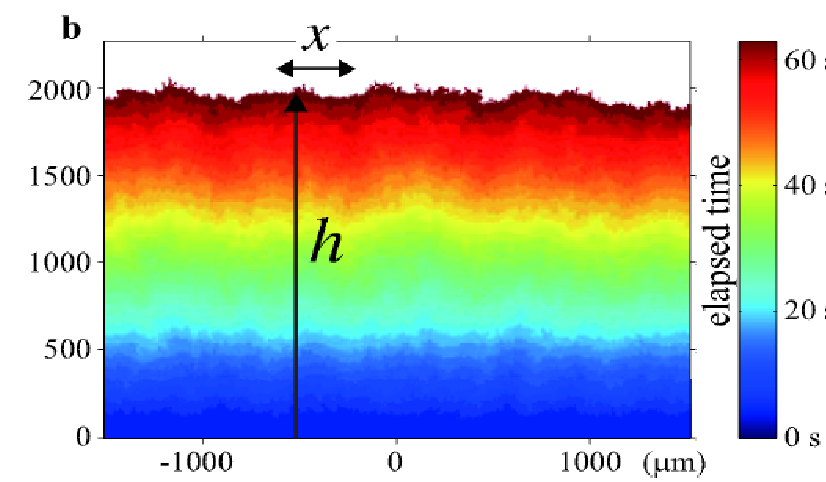

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P(h,t) ?

- typical fluctuations
- **tails of the PDF : large deviations (LD)**

exact results for LD

1) $t \gg 1$

2) $t \ll 1$

thanks to exact solution for KPZ for all time t

$t \gg 1$

1601.05957, PLD, Majumdar,Schehr
1703.03310, Sasarov, Meerson, Prolhac } $\Phi_-(z)$ via Painleve

Alexandre Krajenbrink

1802.08618, AK,PLD 1808.07710 AK,PLD, Prolhac via Cumulants

1803.05887, AK,PLD,Corwin, Ghosal, Tsai via CG

1809.03410, Tsai via SAO, rigorous

1811.00509, AK,PLD 4 methods (CG,SAO,Painleve,cumulants)

$$L = t \sum_i \phi(u + t^{-\frac{2}{3}} a_i)$$

1802.03273, 1810.07129, Corwin, Ghosal $|H|^{5/2}$ rigorous

$t \ll 1$

1603.03302, PLD, Majumdar,Schehr, Rosso droplet

1705.04654, 1804.08800 AK,PLD Brownian, 1/2 space

1808.07710 AK,PLD, Prolhac expansion up to t^3 + large t guess

Optimal fluct th. WNT

Korshunov et al. (2007)

Meerson et al. 1512.04910+…

Numerics 1802.02106 Hartmann et al.

Hartmann, AK, PLD in preparation

KPZ equation: $$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \sqrt{2}\,\xi(x,t)$$

known results for typical fluctuations for t >> 1

$$h(0,t) \simeq v_\infty t + \chi\, t^{1/3}$$

χ random variable

with Tracy-Widom (TW) distribution

PDF of largest eigenvalue of a random matrix

depend on class of initial condition h(x,0)

KPZ equation: $\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \sqrt{2}\,\xi(x,t)$

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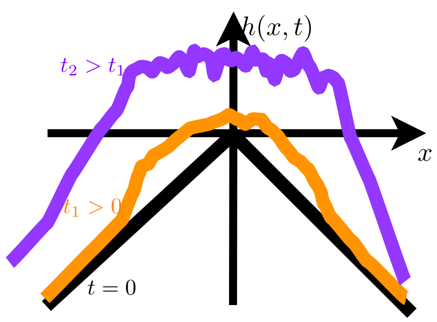
Initial condition

1) Flat $h(x,0) = 0$ GOE-TW $\chi = 2^{-2/3}\chi_1$

2) NWedge $h(x,0) = -w|x|$ GUE-TW $\chi = \chi_2$



curved/droplet

3) Brownian $h(x,0) = B(x)$ Baik-Rains $\chi = \chi_{\rm BR}$

stationary

CDF of TW are given by

- Fredholm determinants
- Solutions of Painleve equations

KPZ equation: $\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \sqrt{2}\,\xi(x,t)$

known results for typical fluctuations for t >> 1

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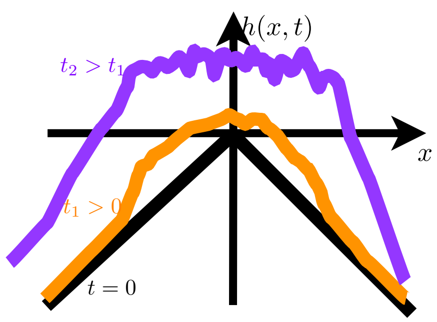

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Question: tails of P(H,t) ? $H := h(0,t) - v_\infty t$

Tails of P(H,t) ?

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \sqrt{2}\,\xi(x,t)$$

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short time $t \ll 1$

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Edwards-Wilkinson equation (EW)

1) typical fluctuations KPZeq => EWeq

$$H \sim t^{1/4}$$ Gaussian

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2) large deviations

$$|H| = O(1) \gg t^{1/4}$$

$$P(H,t) \sim \exp\left(-\frac{\Phi(H)}{\sqrt{t}}\right)$$

$\Phi(H)$ depends on IC

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$$P(H,t) \sim \exp\left(-\frac{\Phi(H)}{\sqrt{t}}\right)$$

$\sim H^2$

$\Phi(H)$ depends on IC

2 methods:

exact solutions -> exact $\Phi(H)$

weak noise theory/optimal fluctuation
-> exact tails, diff. eq. numerical

$t \gg 1$ large time "typical" tails $H \sim t^{1/3}$

= the tails of TW distributions universal

$t \gg 1$ large time

"typical" tails $H \sim t^{1/3}$

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right tail $\frac{H}{t^{1/3}} \gg 1$

$e^{-\frac{4}{3}\chi_2^{3/2}}$

$e^{-\frac{2}{3}\chi_1^{3/2}}$ $e^{-\frac{2}{3}\chi_{\rm BR}^{3/2}}$

droplet,flat

$$P(H,t) \sim \exp\left(-\frac{4}{3}\left(\frac{H}{t^{1/3}}\right)^{3/2}\right)$$

brownian $\frac{2}{3}$

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droplet, brownian

$e^{-\frac{1}{12}\chi_2^{3}}$

cubic tail
Painleve II

$$P(H,t) \sim \exp\left(-\frac{1}{12}(\frac{H}{t^{1/3}})^{3}\right)$$

$e^{-\frac{1}{24}\chi_1^{3}}$

flat $\frac{1}{6}$

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flat $\frac{1}{6}$

Q: what are far tails (large deviations) ?

does the cubic TW tail extend to all H,t ?

prediction large deviation regime

$t \gg 1$ $|H| \sim t \gg t^{1/3}$

1601.05957, PLD, Majumdar,Schehr

prediction   large deviation regime

$t \gg 1$ $|H| \sim t \gg t^{1/3}$

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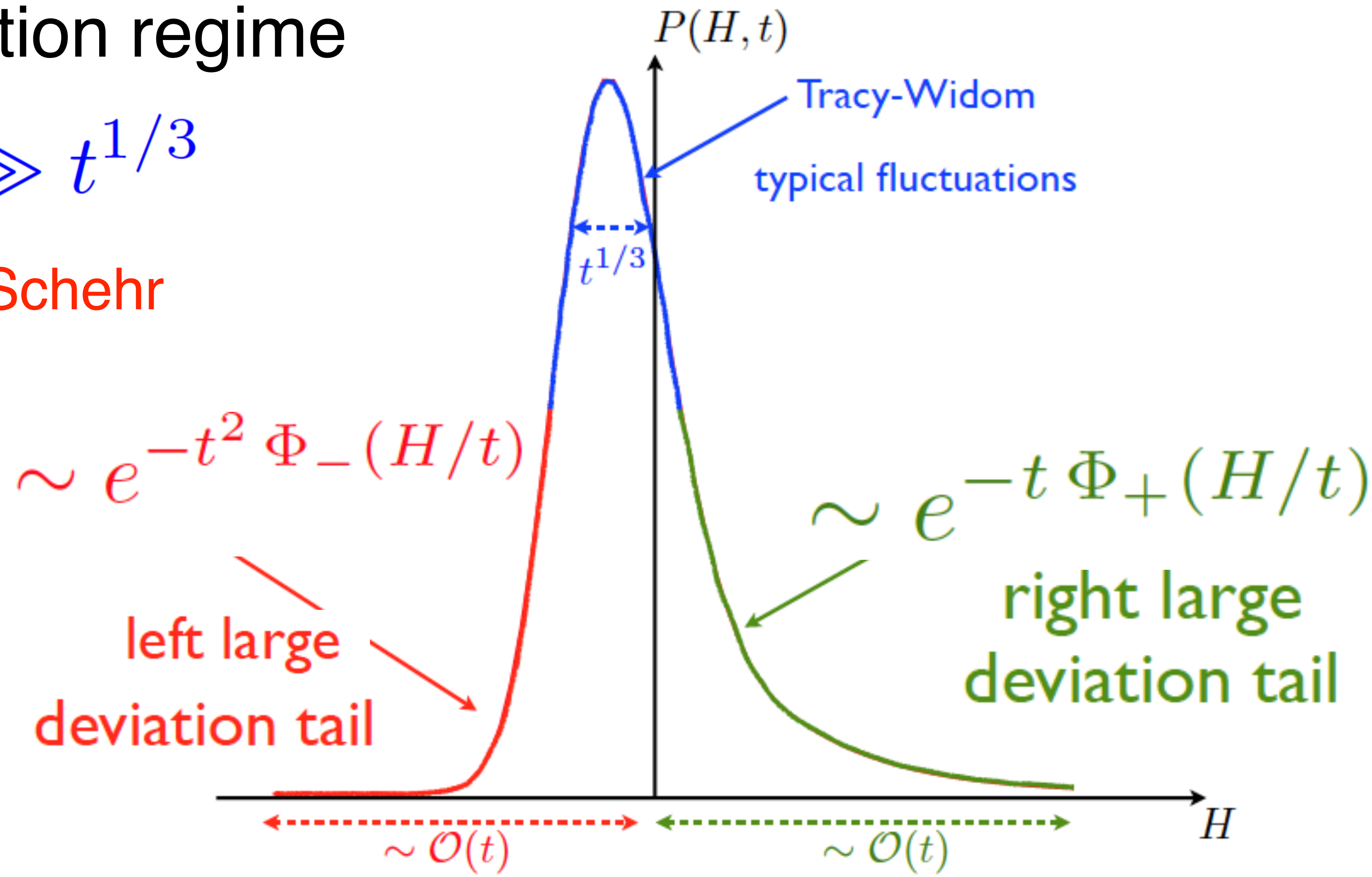
$\Phi_{\pm}$ non-universal

t, t^2 universal



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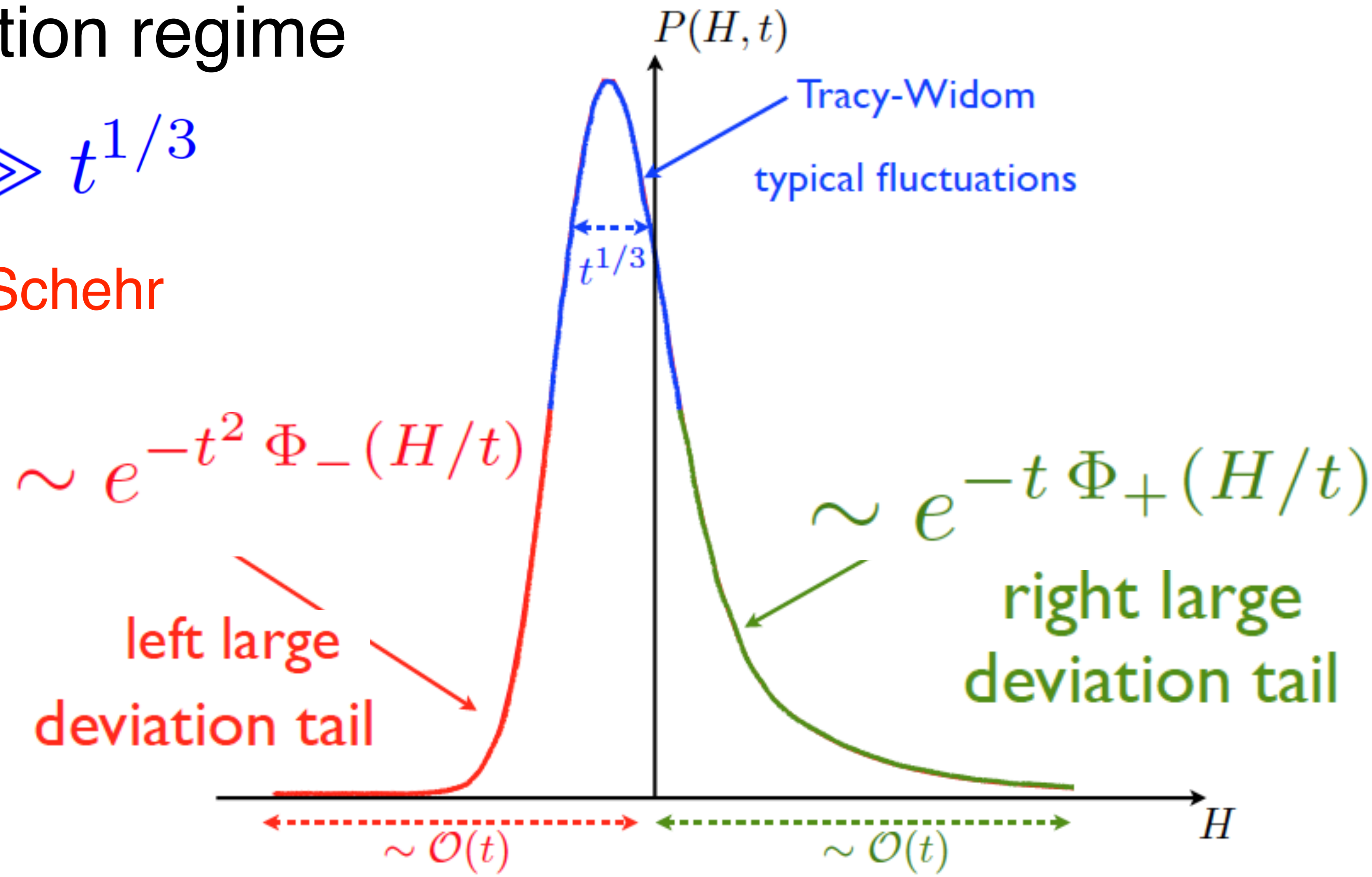

analogy with

large deviation for random matrices

$\tilde{\lambda}_i \in [-2N, 2N]$ $\tilde{\lambda}_1 = \max_i \tilde{\lambda}_i$

$\lambda_i = \frac{\tilde{\lambda}_i}{N}$

$$P(\tilde{\lambda}_1, N) \sim e^{-N \Phi_+\left(\frac{\tilde{\lambda}_1 - 2N}{N}\right)}$$

$$P(\tilde{\lambda}_1, N) \sim e^{-N^2 \Phi_-\left(\frac{\tilde{\lambda}_1 - 2N}{N}\right)}$$

prediction large deviation regime

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1601.05957, PLD, Majumdar,Schehr

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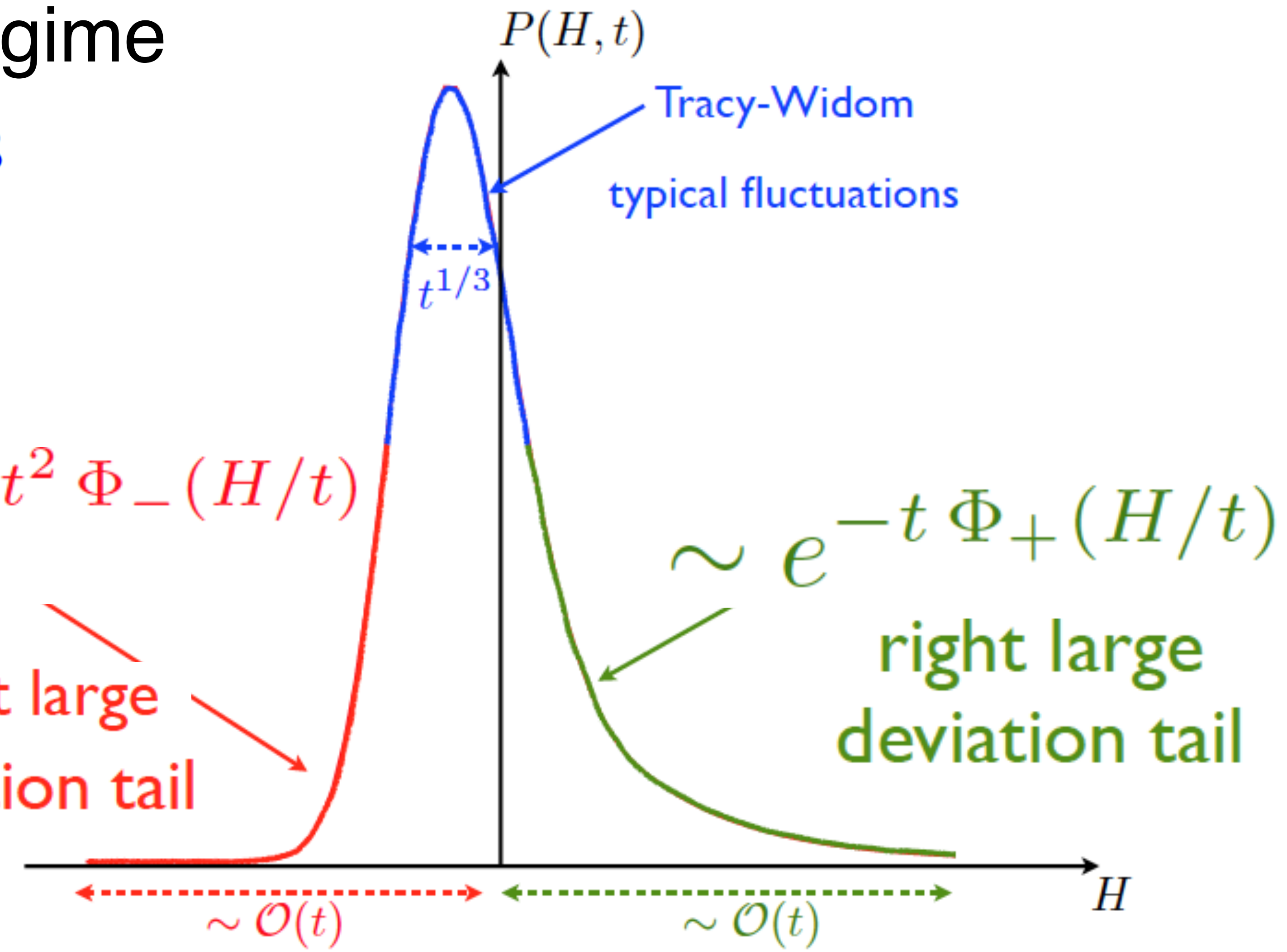


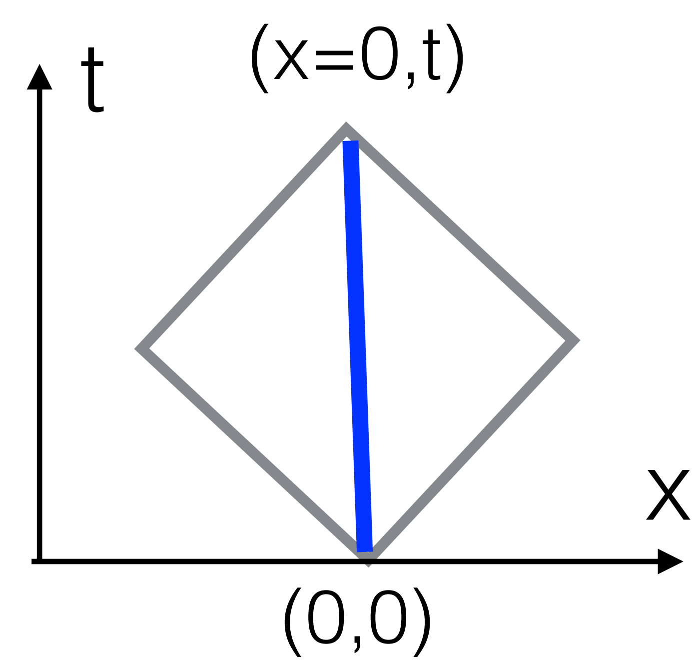
$$h(x,t) = \max[h(x-1,t-1), h(x+1,t-1)] + \eta(x,t)$$

$$p(\eta) = e^{-\eta}$$

largest eigenvalue
complex whishart matrix
Johansson (2000)

H <<typical
proba all sites in square
have small energy

$\sim e^{-t^2}$



H >> typical
proba all sites along blue line
have large energy $\sim e^{-t}$

Matching TW tails with large deviations

TW tail $$P(H,t) \sim \exp\left(-\frac{4}{3}(\frac{H}{t^{1/3}})^{3/2}\right)$$

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TW tail $$P(H,t) \sim \exp\left(-\frac{4}{3}(\frac{H}{t^{1/3}})^{3/2}\right) = \exp\left(-t\,\frac{4}{3}(\frac{H}{t})^{3/2}\right)$$

Large deviations $$\exp\left(-t\,\Phi_+(\frac{H}{t})\right)$$ $$\Phi_+(z) \simeq_{z\to 0^+} \frac{4}{3}z^{3/2}$$

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$$\exp\left(-t^2\,\Phi_-(\frac{H}{t})\right)$$ $$\Phi_-(z) \simeq_{z\to 0^-} \frac{1}{12}|z|^3$$

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Matching TW tails with large deviations

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rate functions for KPZ equation?

universal

prediction large deviation regime

$t \gg 1$ $|H| \sim t \gg t^{1/3}$

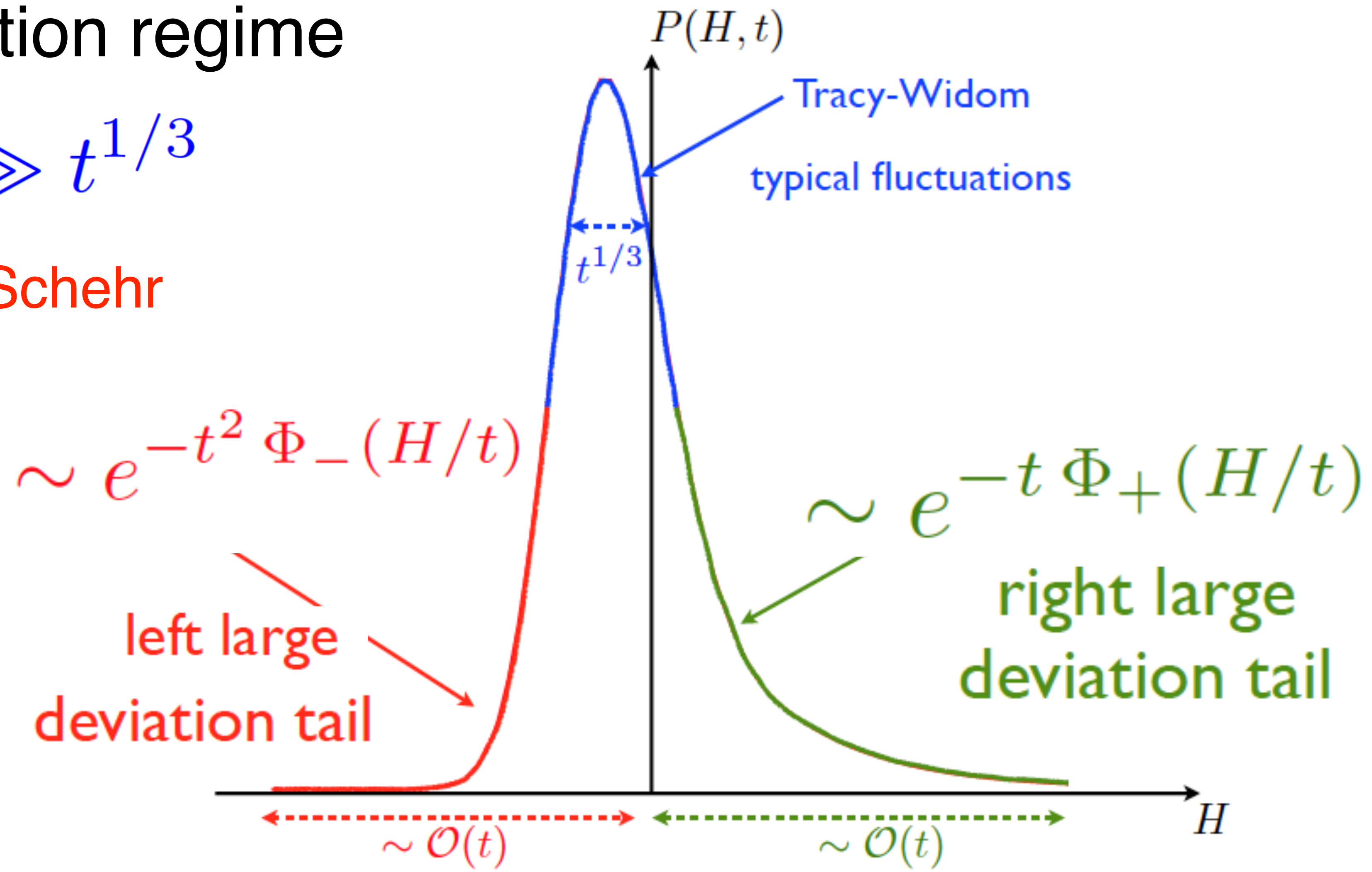
1601.05957, PLD, Majumdar,Schehr

for the KPZ equation

droplet IC

$$\Phi_+(z) = \frac{4}{3} z^{3/2} \qquad z \geq 0$$

$$\langle e^{nH} \rangle \sim e^{\frac{1}{12} n^3 t}$$



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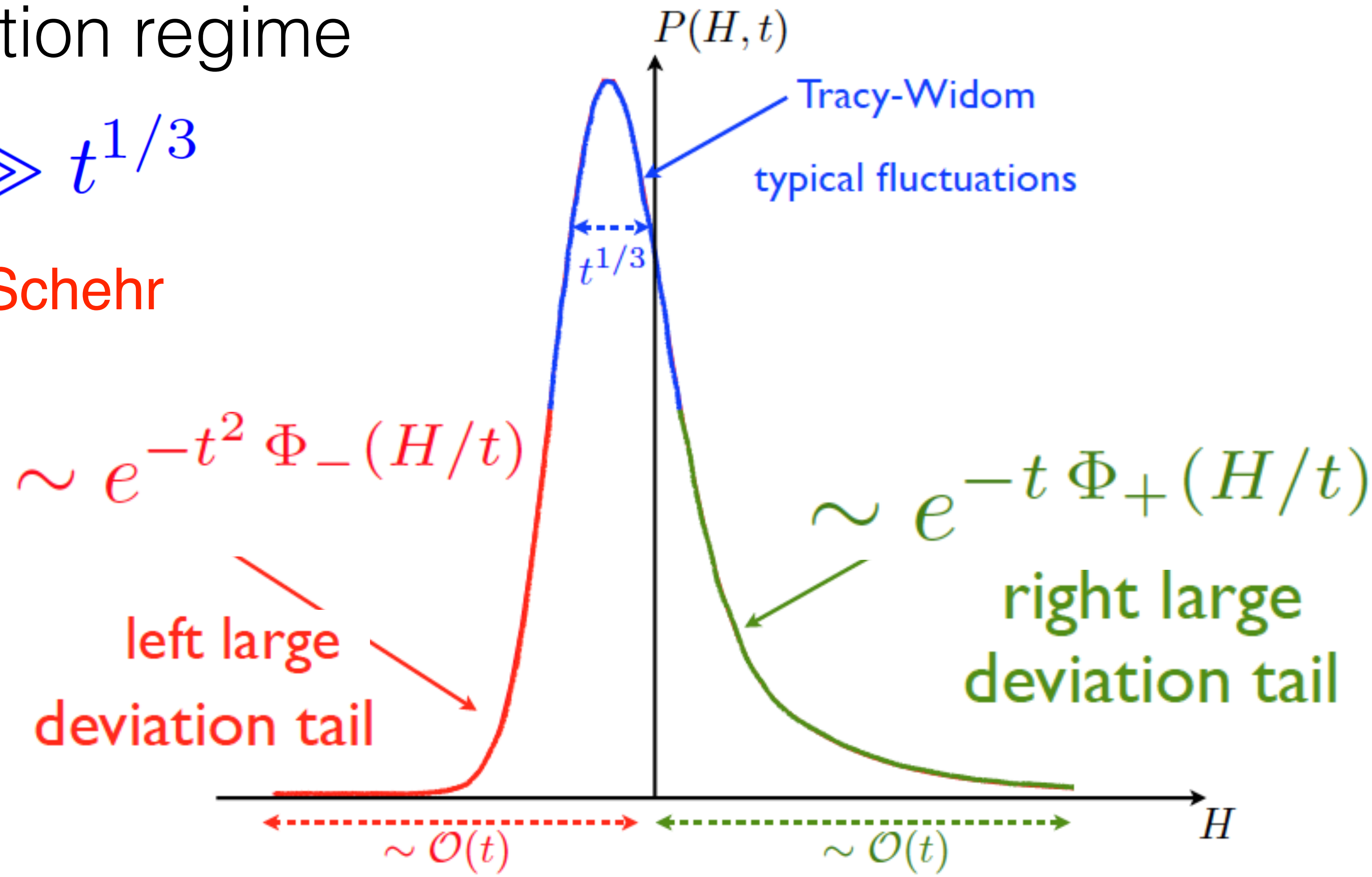
finite time solution of droplet IC

non-local extension of Painleve II equation

Amir, Corwin, Quastel (2010)



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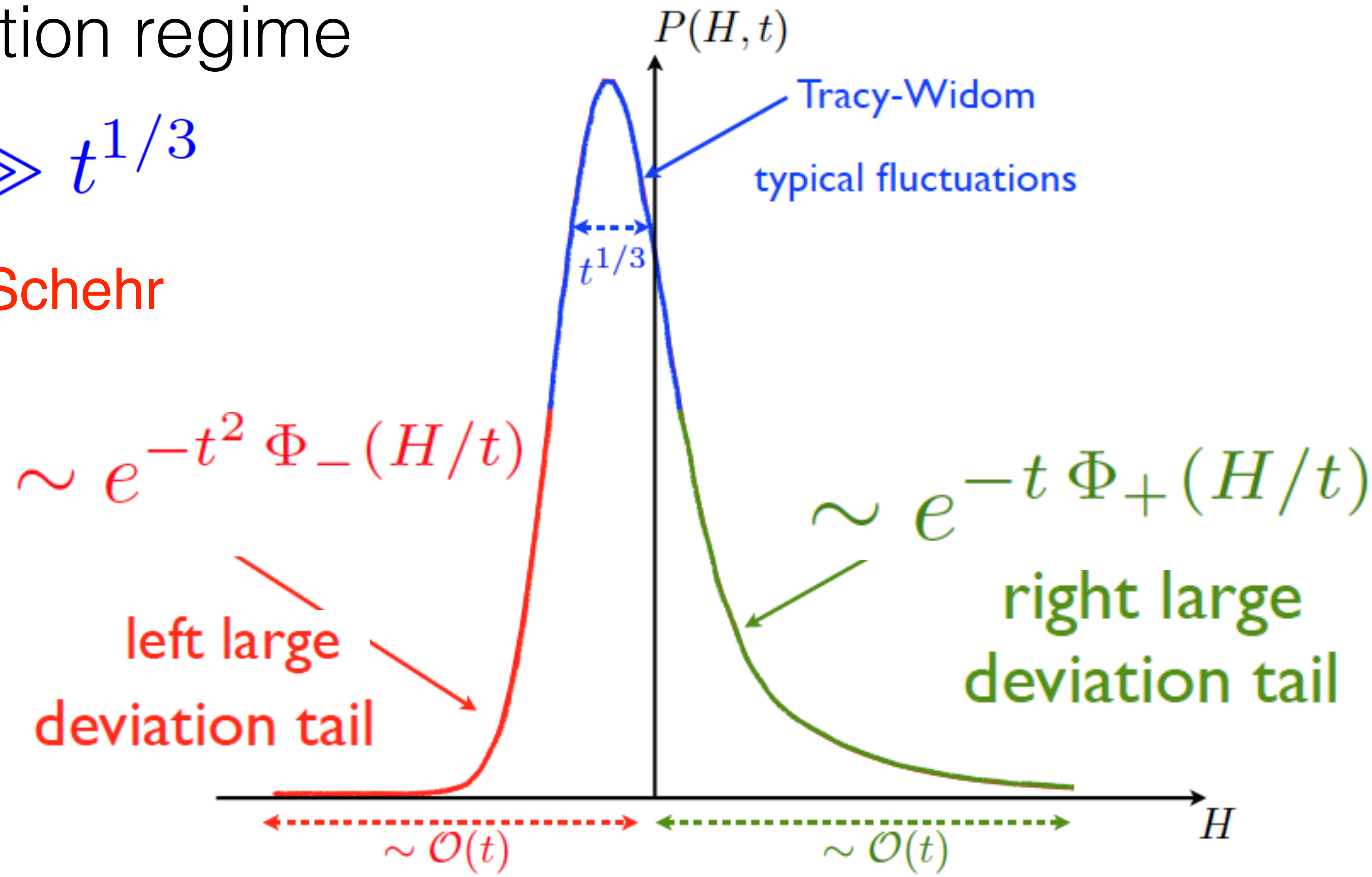

wrong !

1703.03310, Sasarov, Meerson, Prolhac

semi-classical ansatz

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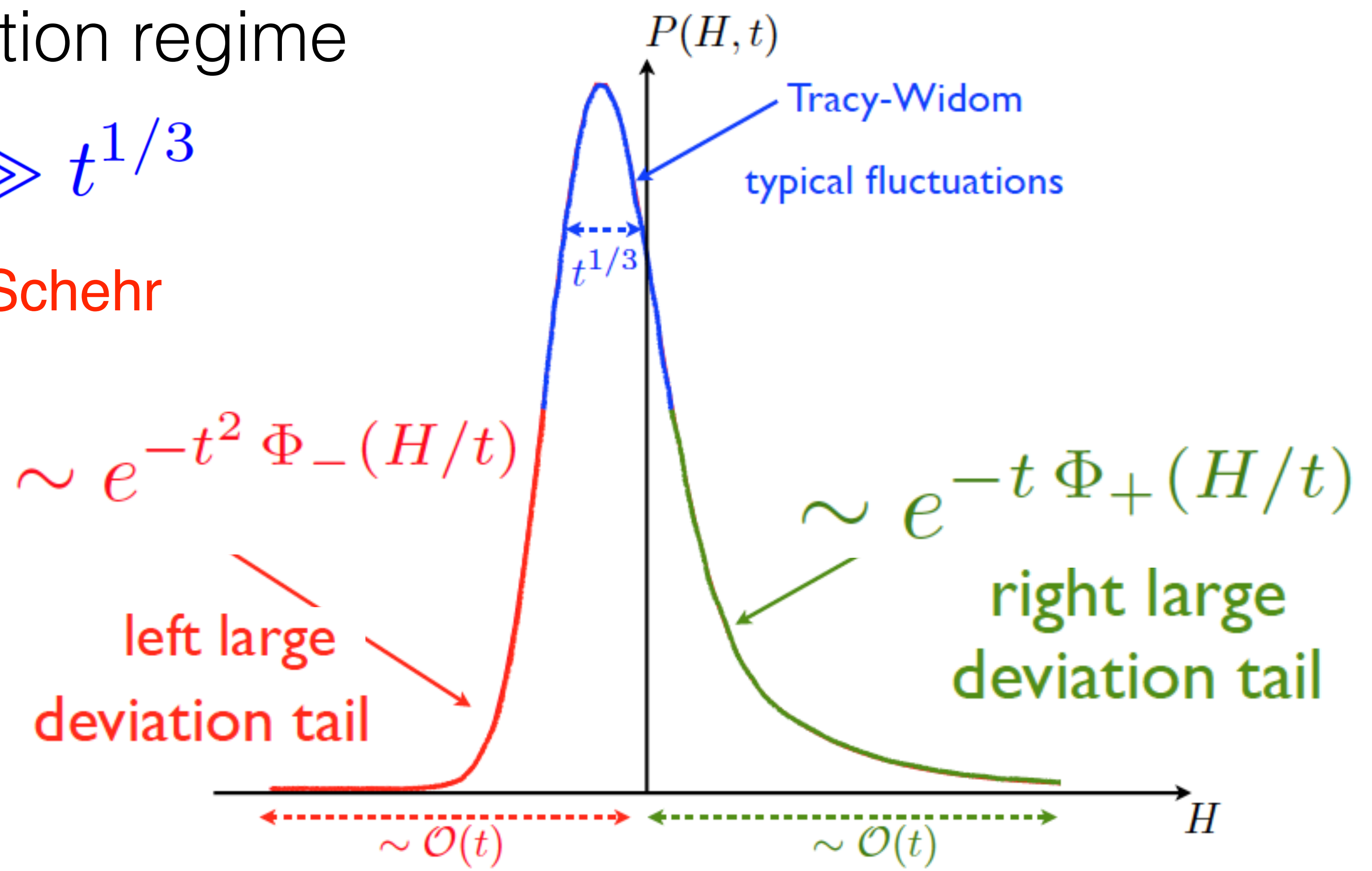

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now: 4 methods now to solve the problem

calculation of $\Phi_-(z)$ from exact solution for KPZ droplet IC

$$\left\langle \exp\left(-e^{H-st^{1/3}}\right) \right\rangle_{\mathrm{KPZ}}$$

calculation of $\Phi_-(z)$ from exact solution for KPZ droplet IC

$$\left\langle \exp\left(-e^{H - s t^{1/3}}\right) \right\rangle_{\mathrm{KPZ}} = \mathrm{Det}(I - P_0 K_{ts} P_0)$$

Calabrese,PLD,Rosso;Dotsenko

Spohn,Sasamoto;Amir,Corwin,Quastel (2010)

calculation of $\Phi_-(z)$ from exact solution for KPZ droplet IC

$$\left\langle \exp\left(-e^{H-st^{1/3}}\right) \right\rangle_{\mathrm{KPZ}} = \left\langle \prod_{i=1}^{+\infty} \frac{1}{1+e^{t^{1/3}(a_i-s)}} \right\rangle_{\mathrm{Airy}}$$

Borodin,Gorin (2016)

calculation of $\Phi_-(z)$ from exact solution for KPZ droplet IC

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Borodin,Gorin (2016)

a_i Airy point process

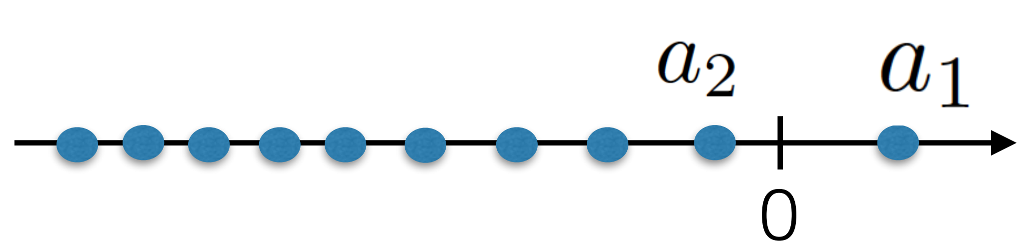
determinantal point process

$$K_{\mathrm{Ai}}(a,a') = \int_0^{+\infty} dr \,\mathrm{Ai}(a+r)\mathrm{Ai}(a'+r)$$

n-point correlations= $\det_{n\times n} K_{\mathrm{Ai}}$

mean density $\rho(a) = K_{\mathrm{Ai}}(a,a) \simeq_{a\to-\infty} \frac{1}{\pi}\sqrt{|a|}$

GUE [-2,2] eigenvalues $\lambda_i = 2 + \frac{a_i}{N^{2/3}}$



APP is scaled edge limit of GUE

calculation of $\Phi_-(z)$ from exact solution for KPZ droplet IC

$$\left\langle \exp\left(-e^{H-st^{1/3}}\right) \right\rangle_{\mathrm{KPZ}} = \left\langle \prod_{i=1}^{+\infty} \frac{1}{1+e^{t^{1/3}(a_i-s)}} \right\rangle_{\mathrm{Airy}}$$

(0) (1)

a_i Airy point process mean density $\rho(a) = K_{\mathrm{Ai}}(a,a) \simeq_{a\to-\infty} \frac{1}{\pi}\sqrt{|a|}$

$$\lambda_i = 2 + \frac{a_i}{N^{2/3}}$$

$t \to +\infty$

s fixed (0)$=\mathrm{Prob}(\frac{H}{t^{1/3}} < s)$

typical fluctuations

calculation of $\Phi_-(z)$ from exact solution for KPZ droplet IC

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GUE-TW

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$t \to +\infty$

s fixed

(0)= $\mathrm{Prob}(\frac{H}{t^{1/3}} < s)$ =(1)= $\mathrm{Prob}(a_{\max} < s) = F_2(s)$

GUE-TW

typical fluctuations

large deviations:

$$s \to -\infty \qquad s = z t^{2/3} \qquad z < 0$$

$$\left\langle \exp\left(-e^{H-st^{1/3}}\right) \right\rangle_{\mathrm{KPZ}} = \left\langle \prod_{i=1}^{+\infty} \frac{1}{1+e^{t^{1/3}(a_i-s)}} \right\rangle_{\mathrm{Airy}}$$

(0) (1)

$$= \mathrm{Prob}\left(\frac{H}{t} < z\right)$$

$s \to -\infty$

$s = zt^{2/3}$

$z < 0$

mean density

$$\rho(a) \simeq_{a\to-\infty} \frac{1}{\pi}\sqrt{|a|}$$

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$$\phi_{ts}(a) = \log(1 + e^{t^{1/3}(a-s)})$$
$$\simeq t^{1/3}(a-s)_+$$

$x_+ = \max(x, 0)$

$$\left\langle \exp\left(-e^{H-st^{1/3}}\right) \right\rangle_{\mathrm{KPZ}} = \left\langle \prod_{i=1}^{+\infty} \frac{1}{1+e^{t^{1/3}(a_i-s)}} \right\rangle_{\mathrm{Airy}}$$

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mean density

$$s \to -\infty$$

$$s = zt^{2/3}$$

$$z < 0$$

$$= \mathrm{Prob}\left(\frac{H}{t} < z\right)$$

$$\rho(a) \simeq_{a \to -\infty} \frac{1}{\pi}\sqrt{|a|}$$

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$$\simeq t^{1/3}(a-s)_+$$

$$x_+ = \max(x, 0)$$

$$\geq \exp\left(- \left\langle \sum_i \phi_{ts}(a_i) \right\rangle_{\mathrm{Airy}} \right)$$

$$\simeq \exp\left(- \int da\, \rho(a) t^{1/3} (a-s)_+ \right)$$

$$\left\langle \exp\left(-e^{H-st^{1/3}}\right) \right\rangle_{\mathrm{KPZ}} = \left\langle \prod_{i=1}^{+\infty} \frac{1}{1+e^{t^{1/3}(a_i-s)}} \right\rangle_{\mathrm{Airy}}$$

(0) (1)

$s \to -\infty$

$s = zt^{2/3}$

$z < 0$

$$= \mathrm{Prob}\left(\frac{H}{t} < z\right)$$

mean density

$$\rho(a) \simeq_{a\to-\infty} \frac{1}{\pi}\sqrt{|a|}$$

$$\lambda_i = 2 + \frac{a_i}{N^{2/3}}$$

$$= (1) = \left\langle \exp\Big(-\sum_i \phi_{ts}(a_i)\Big) \right\rangle_{\mathrm{Airy}}$$

$$\phi_{ts}(a) = \log\left(1+e^{t^{1/3}(a-s)}\right)$$

$$\simeq t^{1/3}(a-s)_+$$

$x_+ = \max(x,0)$

$$\geq \exp\Big(-\Big\langle \sum_i \phi_{ts}(a_i)\Big\rangle_{\mathrm{Airy}}\Big)$$

$a_i = b_i t^{2/3}$

$a = bt^{2/3}$

$$\simeq \exp\Big(-\int da\,\rho(a)t^{1/3}(a-s)_+\Big)$$

$$= \exp\Big(-t^2\int db\,(b-z)_+\frac{\rho(t^{2/3}b)}{t^{1/3}}\Big)$$

$$\downarrow$$

$$\frac{1}{\pi}\sqrt{-b}\,\theta(-b)$$

$$\left\langle \exp\left(-e^{H-st^{1/3}}\right) \right\rangle_{\mathrm{KPZ}} = \left\langle \prod_{i=1}^{+\infty} \frac{1}{1+e^{t^{1/3}(a_i-s)}} \right\rangle_{\mathrm{Airy}}$$

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$$\simeq \exp\left(-\int da\,\rho(a) t^{1/3}(a-s)_+\right)$$

=> $\Phi_-(z) \leq \frac{4}{15\pi}(-z)^{5/2}$

$$= \exp\left(-t^2 \int db(b-z)_+ \frac{\rho(t^{2/3}b)}{t^{1/3}}\right)$$

$$-t^2 \frac{4}{15\pi}(-z)^{5/2}$$

$$\frac{1}{\pi}\sqrt{-b}\,\theta(-b)$$

$$\left\langle \exp\left(-e^{H-st^{1/3}}\right) \right\rangle_{\mathrm{KPZ}} = \left\langle \prod_{i=1}^{+\infty} \frac{1}{1+e^{t^{1/3}(a_i-s)}} \right\rangle_{\mathrm{Airy}}$$

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$a_i = b_i t^{2/3}$

$a = bt^{2/3}$

$$\simeq \exp\left(-\int da\, \rho(a) t^{1/3}(a-s)_+\right)$$

$$=> \quad \Phi_-(z) \leq \frac{4}{15\pi}(-z)^{5/2}$$

rules out lzl^3

$$= \exp\left(-t^2 \int db\,(b-z)_+ \frac{\rho(t^{2/3}b)}{t^{1/3}}\right)$$

$$P(H,t) \sim e^{-\frac{4}{15\pi}t^2\left(\frac{|H|}{t}\right)^{5/2}}$$

$$-t^2\frac{4}{15\pi}(-z)^{5/2}$$

$$\frac{1}{\pi}\sqrt{-b}\,\theta(-b)$$

$$\left\langle \exp\left(-e^{H-st^{1/3}}\right)\right\rangle_{\text{KPZ}} = \left\langle \prod_{i=1}^{+\infty} \frac{1}{1+e^{t^{1/3}(a_i-s)}} \right\rangle_{\text{Airy}}$$

(0) (1)

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$a_i = b_i t^{2/3}$

$a = bt^{2/3}$

$$\simeq \exp\left(-\int da\,\rho(a) t^{1/3}(a-s)_+\right)$$

$$=> \quad \Phi_-(z) \leq \frac{4}{15\pi}(-z)^{5/2}$$

rules out |z|^3

$$= \exp\left(-t^2 \int db\,(b-z)_+ \frac{\rho(t^{2/3}b)}{t^{1/3}}\right)$$

$$P(H,t) \sim e^{-\frac{4}{15\pi}t^2\left(\frac{|H|}{t}\right)^{5/2}}$$

$$-t^2\frac{4}{15\pi}(-z)^{5/2}$$

$$\frac{1}{\pi}\sqrt{-b}\,\theta(-b)$$

$$= e^{-\frac{4}{15\pi}\frac{|H|^{5/2}}{\sqrt{t}}}$$

$t \ll 1$

far tail same for small and large time !

large time

left tail H/t<0

$$P(H,t) \sim \exp\left(-t^2\, \Phi_-(\frac{H}{t})\right)$$

$$\Phi_-(z) \simeq_{z\to-\infty} \frac{4}{15\pi}(-z)^{5/2} + O(z^2) + \dots$$

1st cumulant   2nd cum

$\Phi_-(z)$?

large time

left tail H/t<0

$$P(H,t) \sim \exp\left(-t^2\,\Phi_-(\frac{H}{t})\right)$$

$$\Phi_-(z) \simeq_{z\to-\infty} \frac{4}{15\pi}(-z)^{5/2} + O(z^2) + \dots$$

1st cumulant   2nd cum

$\Phi_-(z)$?

1) Non-local Painleve eq   1703.03310, Sasarov, Meerson, Prolhac

2) Coulomb gas (edge)   1803.05887, AK,PLD,Corwin, Ghosal, Tsai

3) Stochastic Airy Operator   1809.03410, Tsai   rigorous

4) Cumulant expansion   1802.08618, AK,PLD
1808.07710 AK,PLD, Prolhac

Also 1811.00509, AK,PLD   4 methods (CG,SAO,Painleve,cumulants)

$$L = t\sum_i \phi(u + t^{-\frac{2}{3}} a_i)$$

1802.03273, 1810.07129, Corwin, Ghosal $|H|^{5/2}$rigorous

$\Phi_-(z)$ via the Coulomb gas at the edge

rescaled APP $a_i = b_i t^{\frac{2}{3}}$ empirical measure $\mu_t(b) = t^{-1} \sum_{i=1}^{+\infty} \delta_{a_i t^{-2/3}}(b)$

$$\Phi_-(z) = \lim_{t \to +\infty} \frac{-1}{t^2} \log \left\langle \exp\left(- t^2 \int db \mu_t(b)(b-z)_+ \right) \right\rangle$$

$+ ..$

$\Phi_-(z)$ via the Coulomb gas at the edge

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$$\Phi_-(z) = \lim_{t \to +\infty} \frac{-1}{t^2} \log \left\langle \exp\left(-t^2 \int db \mu_t(b)(b-z)_+ \right) \right\rangle$$

LDP for AiryPP

$$\mathrm{Prob}(\mu_t \simeq \mu) \sim \exp(-t^2 I_{\mathrm{Ai}}(\mu))$$

$+ ..$

$\Phi_-(z)$ via the Coulomb gas at the edge

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LDP for AiryPP => variational problem

$$\mathrm{Prob}(\mu_t \simeq \mu) \sim \exp(-t^2 I_{\mathrm{Ai}}(\mu))$$

$$\Phi_-(z) = \min_{\mu} \Sigma(\mu)$$

$$\Sigma(\mu) = \int db\, \mu(b)(b-z)_+ + I_{\mathrm{Ai}}(\mu)$$

$$\int db\, (\mu(b) - \mu_{\mathrm{Ai}}(b)) = 0$$

$$\mu_{\mathrm{Ai}}(b) = \frac{\sqrt{-b}}{\pi} \theta(-b)$$

+..

$\Phi_-(z)$ via the Coulomb gas at the edge

rescaled APP $a_i = b_i t^{\frac{2}{3}}$ empirical measure $\mu_t(b) = t^{-1} \sum_{i=1}^{+\infty} \delta_{a_i t^{-2/3}}(b)$

$$\Phi_-(z) = \lim_{t\to+\infty} \frac{-1}{t^2} \log \left\langle \exp\left(- t^2 \int db \mu_t(b)(b-z)_+ \right) \right\rangle$$

LDP for AiryPP => variational problem

$$\mathrm{Prob}(\mu_t \simeq \mu) \sim \exp(-t^2 I_{\mathrm{Ai}}(\mu))$$

$$\Phi_-(z) = \min_{\mu} \Sigma(\mu)$$

↑ obtained from LDP for GUE

$$\Sigma(\mu) = \int db\, \mu(b)(b-z)_+ + I_{\mathrm{Ai}}(\mu)$$

taking edge limit $\lambda_i = 2 + \frac{a_i}{N^{2/3}}$

$$\int db\, (\mu(b) - \mu_{\mathrm{Ai}}(b)) = 0$$

$$P[\{\lambda_i\}] \propto \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-\frac{N}{4}\sum_i \lambda_i^2}$$

$$\mu_{\mathrm{Ai}}(b) = \frac{\sqrt{-b}}{\pi}\theta(-b)$$

$$\mathrm{Prob}(\Lambda_N \simeq \Lambda) \sim e^{-N^2 I_2(\Lambda)}$$

$$I_{\mathrm{Ai}}(\mu) = -\int db_1 db_2 \log|b_1 - b_2| \prod_{j=1}^{2} (\mu(b_j) - \mu_{\mathrm{Ai}}(b_j)) \; +..$$

$$\Phi_-(z) = \min_\mu \Sigma(\mu) \qquad \Sigma(\mu) = \int db\, \mu(b)(b-z)_+ + I_{\mathrm{Ai}}(\mu)$$

$$\int db\, (\mu(b) - \mu_{\mathrm{Ai}}(b)) = 0$$

$$I_{\mathrm{Ai}}(\mu) = -\int db_1 db_2 \log|b_1 - b_2| \prod_{j=1}^{2} (\mu(b_j) - \mu_{\mathrm{Ai}}(b_j))$$

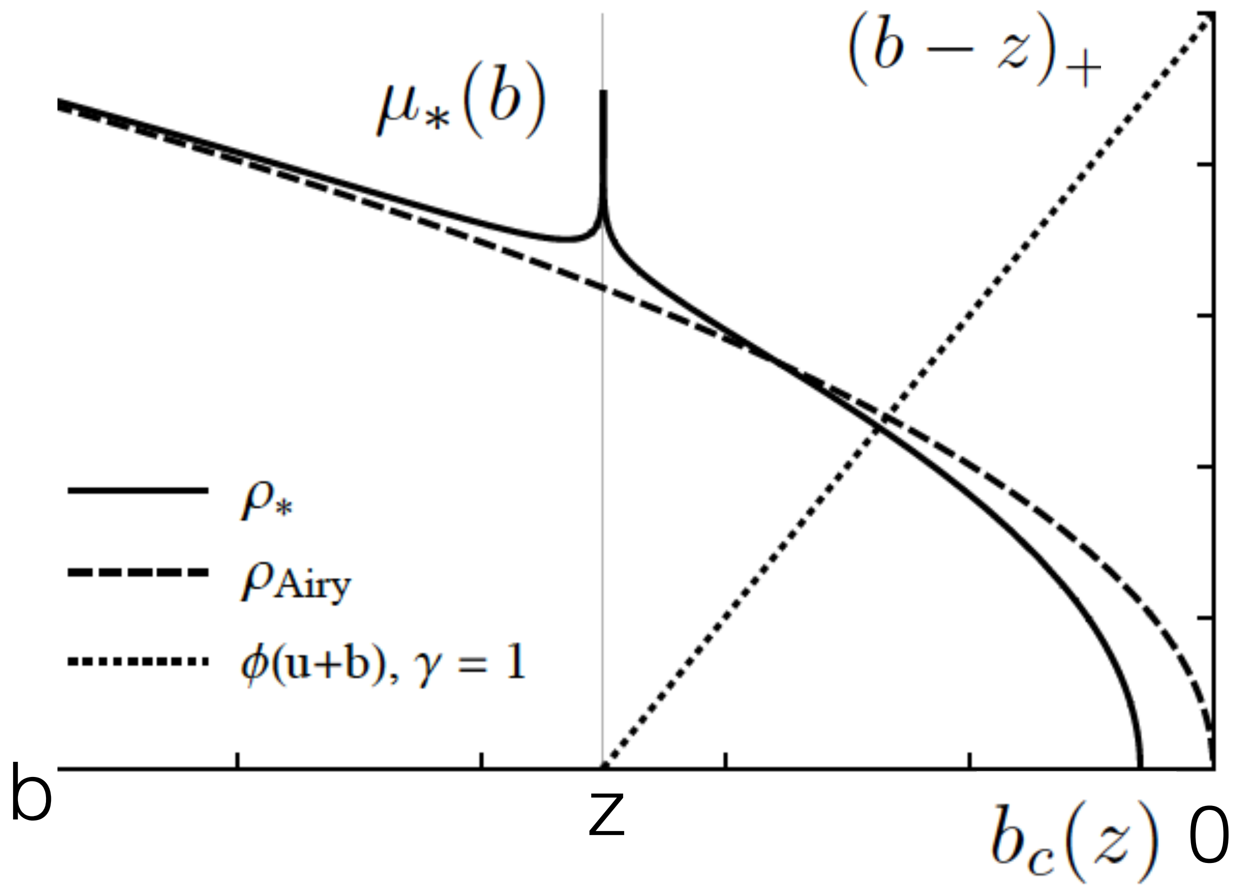
$$\mu_{\mathrm{Ai}}(b) = \frac{\sqrt{-b}}{\pi}\theta(-b)$$

$$\Phi_-(z) = \min_\mu \Sigma(\mu) \qquad \Sigma(\mu) = \int db\, \mu(b)(b-z)_+ + I_{\rm Ai}(\mu)$$

$$\int db\, (\mu(b) - \mu_{\rm Ai}(b)) = 0$$

$$I_{\rm Ai}(\mu) = -\int db_1 db_2 \log|b_1 - b_2| \prod_{j=1}^{2} (\mu(b_j) - \mu_{\rm Ai}(b_j))$$

$$\mu_{\rm Ai}(b) = \frac{\sqrt{-b}}{\pi}\theta(-b)$$



$$\mu_*(b) = \left(\frac{\sqrt{|b-b_c|}}{\pi} + \frac{1}{2\pi^2}\log\frac{\sqrt{|b-b_c|} - \frac{\pi}{2}b_c}{\sqrt{|b-b_c|} + \frac{\pi}{2}b_c}\right)\theta_{b<b_c}$$

$$b_c(z) = \frac{-2}{\pi^2}\left(\sqrt{1-\pi^2 z} - 1\right)$$

$$\Phi_-(z) = \min_\mu \Sigma(\mu) \qquad \Sigma(\mu) = \int db\, \mu(b)(b-z)_+ + I_{\mathrm{Ai}}(\mu)$$

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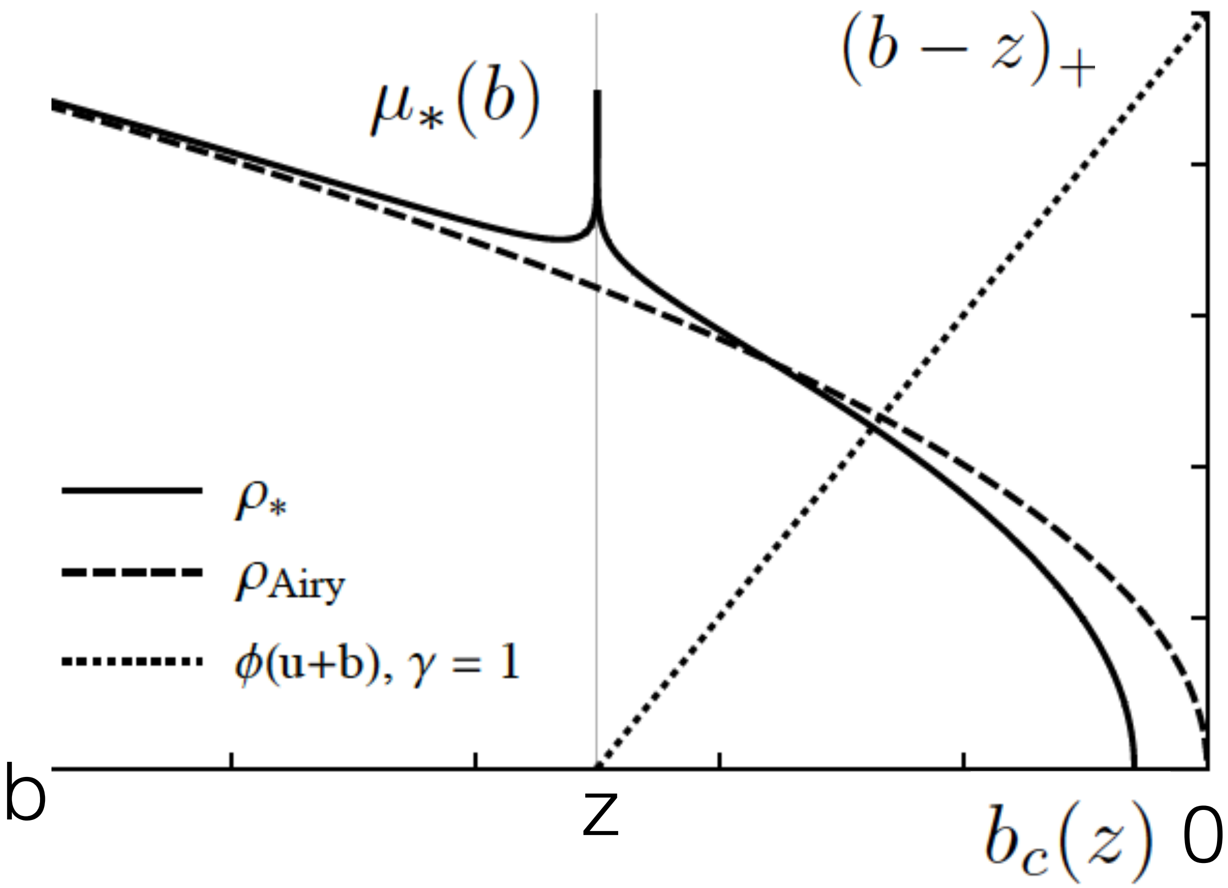
$$\int db\, (\mu(b) - \mu_{\mathrm{Ai}}(b)) = 0$$

$$\mu_{\mathrm{Ai}}(b) = \frac{\sqrt{-b}}{\pi}\theta(-b)$$



$$\mu_*(b) = \left(\frac{\sqrt{|b-b_c|}}{\pi} + \frac{1}{2\pi^2}\log\frac{\sqrt{|b-b_c|} - \frac{\pi}{2}b_c}{\sqrt{|b-b_c|} + \frac{\pi}{2}b_c}\right)\theta_{b<b_c}$$

$$b_c(z) = \frac{-2}{\pi^2}(\sqrt{1-\pi^2 z} - 1)$$

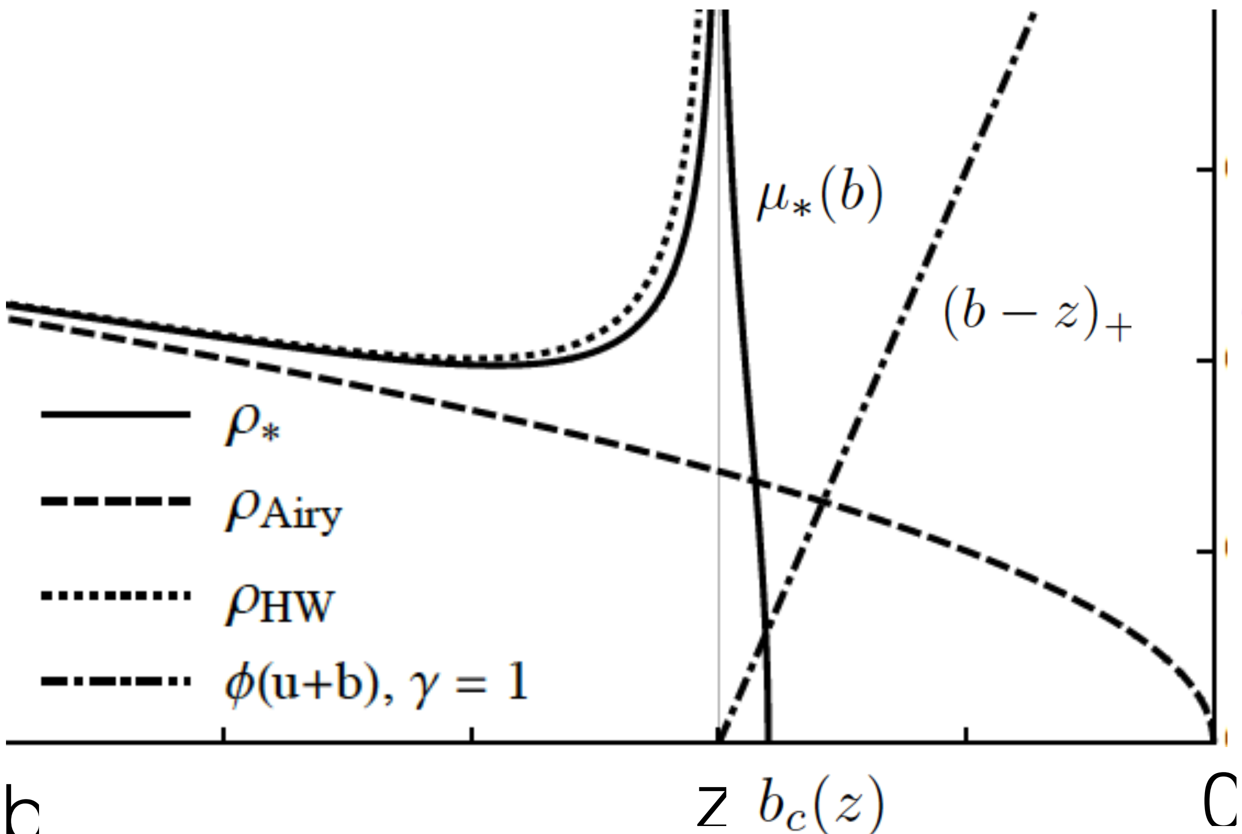
$$\Phi_-(z) = \Sigma(\mu^*) = \frac{4}{15\pi^6}[(1-\pi^2 z)^{5/2} - 1] + \frac{2z}{3\pi^4} - \frac{z^2}{2\pi^2}$$

$$\Phi_-(z) = \min_\mu \Sigma(\mu) \qquad \Sigma(\mu) = \int db\, \mu(b)(b-z)_+ + I_{\rm Ai}(\mu)$$

$$\int db\,(\mu(b) - \mu_{\rm Ai}(b)) = 0$$

$$I_{\rm Ai}(\mu) = -\int db_1 db_2 \log|b_1 - b_2| \prod_{j=1}^{2} (\mu(b_j) - \mu_{\rm Ai}(b_j))$$

$$\mu_{\rm Ai}(b) = \frac{\sqrt{-b}}{\pi}\theta(-b)$$



$$\mu_*(b) = \left(\frac{\sqrt{|b-b_c|}}{\pi} + \frac{1}{2\pi^2}\log\frac{\sqrt{|b-b_c|} - \frac{\pi}{2}b_c}{\sqrt{|b-b_c|} + \frac{\pi}{2}b_c}\right)\theta_{b<b_c}$$

$$b_c(z) = \frac{-2}{\pi^2}(\sqrt{1-\pi^2 z} - 1)$$

$z \to 0$ $\quad \frac{|z|^3}{12}$ hard wall Dean,Majumdar

$$\Phi_-(z) = \Sigma(\mu^*) = \frac{4}{15\pi^6}[(1-\pi^2 z)^{5/2} - 1] + \frac{2z}{3\pi^4} - \frac{z^2}{2\pi^2}$$

$t \ll 1$ short time large deviations

$$|H| = O(1) \gg t^{1/4}$$

$$P(H,t) \sim \exp\left(-\frac{\Phi(H)}{\sqrt{t}}\right)$$

droplet, flat, Brownien

1/2space x>0

$$\partial_x h|_{x=0} = A$$

$t \ll 1$ short time large deviations

$|H| = O(1) \gg t^{1/4}$

$$P(H,t) \sim \exp\left(-\frac{\Phi(H)}{\sqrt{t}}\right)$$

droplet, flat, Brownien

$$\Phi(H) \simeq \begin{cases} c_-|H|^{5/2} & H \to -\infty \\ c\,H^2 & H \simeq 0 \\ c_+|H|^{3/2} & H \to +\infty \end{cases}$$

1/2space x>0

$$\partial_x h|_{x=0} = A$$

c_- droplet, Brownien $\frac{4}{15\pi}$

flat $\frac{8}{15\pi}$

1/2 space drop $\frac{2}{15\pi}$

c_+ droplet, flat $\frac{4}{3}$

Brownien $\frac{2}{3}$

c perturb theo

exact solutions -> exact $\Phi(H)$

weak noise theory/optimal fluctuation
-> exact tails, diff. eq. numerical

$$z > 0$$

$$\left\langle \exp\left(-\frac{z}{\sqrt{t}} e^{H}\right) \right\rangle_{\mathrm{KPZ}} \sim \exp\left(-\frac{\psi(z)}{\sqrt{t}}\right) \longleftarrow \left\langle \prod_{i=1}^{+\infty} \frac{1}{1 + z e^{t^{1/3} a_i}} \right\rangle_{\mathrm{Airy}} \qquad e^{-st^{1/3}} \equiv z$$

$$\left\langle \exp(-\frac{z}{\sqrt{t}}e^H) \right\rangle_{\mathrm{KPZ}} \overset{z>0}{\sim} \exp(-\frac{\psi(z)}{\sqrt{t}}) \longleftarrow \left\langle \prod_{i=1}^{+\infty} \frac{1}{1+ze^{t^{1/3}a_i}} \right\rangle_{\mathrm{Airy}} \quad e^{-st^{1/3}} \equiv z$$

$$||$$

$$\int dH P(H,t) e^{-\frac{z}{\sqrt{t}}e^H}$$

$$\downarrow$$

saddle point

$$\sim \exp\left(-\frac{\Phi(H)}{\sqrt{t}}\right)$$

$$\psi(z) = \min_{H\in\mathbb{R}}(ze^H + \Phi(H))$$

$$\Phi(H) = \max_{z\in I}(\psi(z) - ze^H)$$

$\psi(z)$ strictly convex

$$\left\langle \exp(-\frac{z}{\sqrt{t}} e^{H}) \right\rangle_{\mathrm{KPZ}} \overset{z>0}{\sim} \exp(-\frac{\psi(z)}{\sqrt{t}}) \longleftarrow \left\langle \prod_{i=1}^{+\infty} \frac{1}{1 + z e^{t^{1/3} a_i}} \right\rangle_{\mathrm{Airy}} \qquad e^{-st^{1/3}} \equiv z$$

$$\log(l.h.s) = \log \langle \exp(-\sum_i \log(1 + z e^{t^{1/3} a_i})) \rangle_{\mathrm{Ai}} \qquad a \to a t^{-1/3}$$

$$= -\frac{1}{\sqrt{t}} \psi(z) + O(1)$$

higher cumulant

first cumulant

$$\psi(z) = \int da \log(1 + z e^{a}) \rho_\infty(a)$$

droplet $\rho_\infty(a) = \frac{\sqrt{-a}}{\pi} \theta(-a)$

Brownian Lambert functions

$$\left\langle \exp\left(-\frac{z}{\sqrt{t}}e^{H}\right)\right\rangle_{\mathrm{KPZ}} \overset{z>0}{\sim} \exp\left(-\frac{\psi(z)}{\sqrt{t}}\right)$$

$$\log(l.h.s) = \log\left\langle \exp\left(-\sum_i \log\left(1+ze^{t^{1/3}a_i}\right)\right)\right\rangle_{\mathrm{Ai}} \qquad a \to at^{-1/3}$$

$$= -\frac{1}{\sqrt{t}}\psi(z) + O(1)$$

higher cumulant

first cumulant

droplet $\rho_\infty(a) = \frac{\sqrt{-a}}{\pi}\theta(-a)$

$$\psi(z) = \int da \log(1+ze^{a})\rho_\infty(a)$$

Brownian Lambert functions

$-\Psi(z) \propto$	$\int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\mathrm{Li}_2(-zk^2e^{-k^2})$	$\int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\mathrm{Li}_2(-ze^{-k^2})$	$\int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\mathrm{Li}_2(-z\frac{e^{-k^2}}{k^2})$
Full space		Droplet	Brownian Flat
Half space	Droplet $(A=\infty)$	Droplet $(A=0)$ Brownian $(A=\infty)$	Brownian $(A=0)$ Flat $(A=0)$

$$\mathrm{Li}_p(x) = \sum_{k=1}^{+\infty}\frac{z^k}{k^p}$$

Exact short-time height distribution in 1D KPZ equation and edge fermions at high temperature

PLD, S. Majumdar, A. Rosso, G. Schehr,
Phys. Rev. Lett. 117 070403 (2016).

The rate function of the droplet IC is given by

- For $H \leq H_c = \log \zeta(\frac{3}{2})$

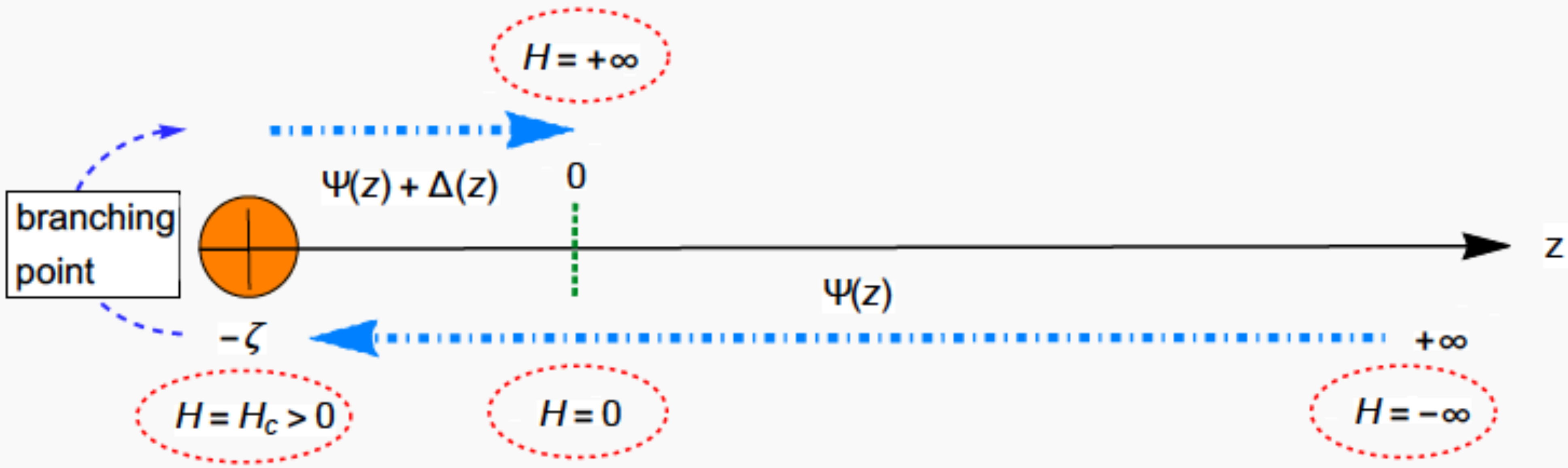
$$\Phi(H) = -\frac{1}{\sqrt{4\pi}} \min_{z \in [-1,+\infty[} [z e^H + \mathrm{Li}_{5/2}(-z)]$$

- for $H \geq H_c$

$$\Phi(H) = -\frac{1}{\sqrt{4\pi}} \min_{z \in [-1,0[} [z e^H + \mathrm{Li}_{5/2}(-z) - \frac{8\sqrt{\pi}}{3}[-\log(-z)]^{3/2}]$$

$\Phi(H)$ is analytic, the left tail is $\Phi(H) \simeq_{H \to -\infty} \frac{4}{15\pi}|H|^{5/2}$ and the right tail is $\Phi(H) \simeq_{H \to +\infty} \frac{4}{3} H^{3/2}$.

Analytic continuation

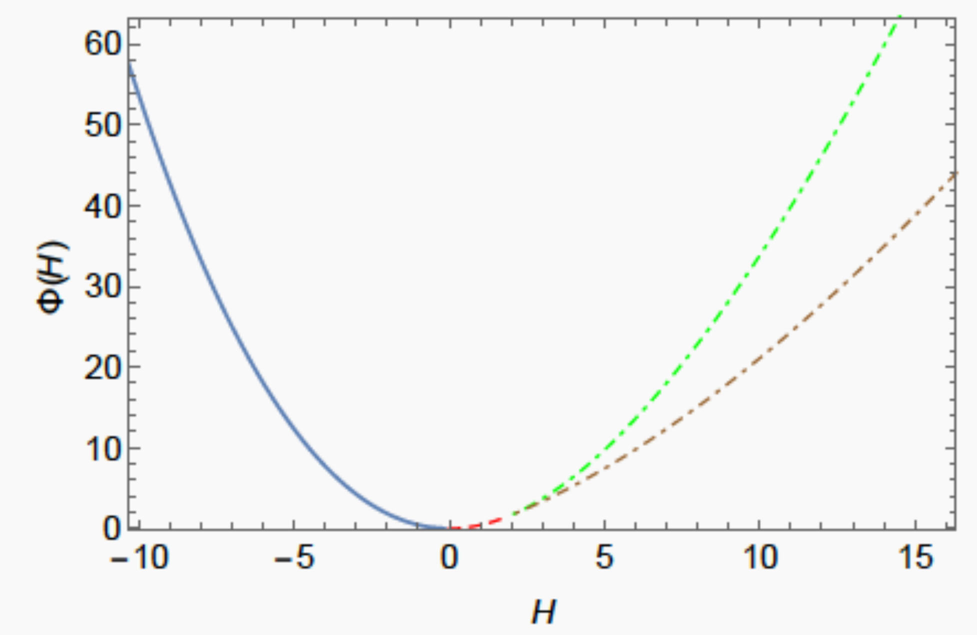


The branching point is the one of $\Psi(z)$

Exact short-time height distribution for the Brownian IC

A. Krajenbrink, PLD, Phys. Rev. E 96, 020102 (2017)

$$P(H,t) \sim \exp\left(-\frac{\Phi(H)}{\sqrt{t}}\right)$$



Singularity and dynamical phase transition

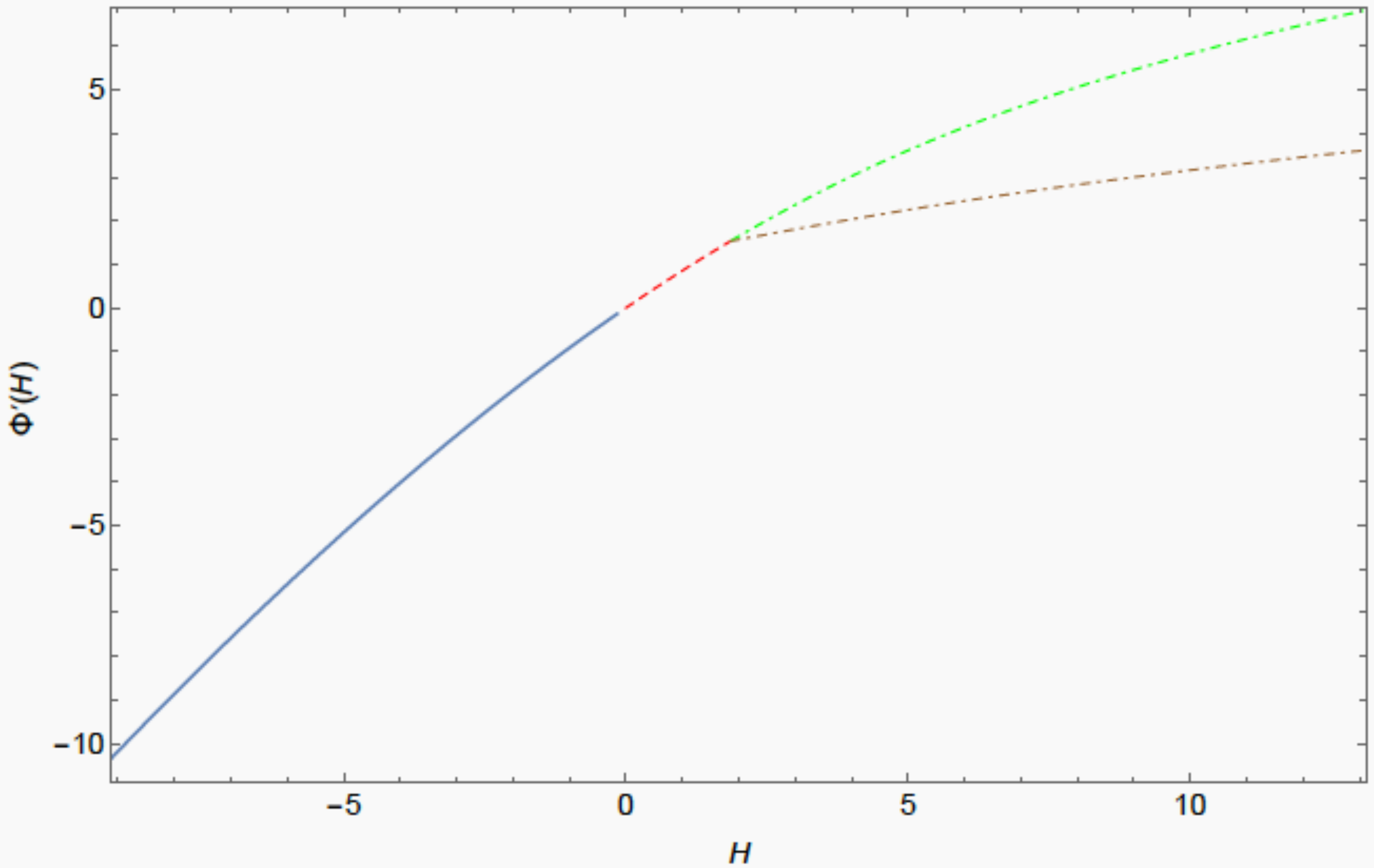


Figure: The function $\Phi'(H)$. The blue line corresponds to the $H < 0$ solution, the red line to the first continuation for $0 < H < H_c$, the green line to the analytic branch $H_c < H$ and the brown line to the non-analytic branch for $H_c < H$. Note the singularity for the brown line.

$\boxed{\text{z is in}}$ $I_1 = [0, +\infty],\ I_2 = [0, e^{-1}],\ I_3 =]0, e^{-1}]$

$\boxed{\text{H is in}}$ $J_1 = [-\infty, H_c(0)],\ J_2 = [H_c(0), H_{c2}(0)],\ J_3 = [H_{c2}(0), +\infty]$

$$H_c(0) = 0$$

$$H_{c2}(0) = 2\ln(2e - \Psi_0'(e^{-1})) - 1$$

$$\simeq 1.85316$$

z is in $I_1 = [0, +\infty],\ I_2 = [0, e^{-1}],\ I_3 =]0, e^{-1}]$

H is in $J_1 = [-\infty, H_c(0)],\ J_2 = [H_c(0), H_{c2}(0)],\ J_3 = [H_{c2}(0), +\infty]$

relation between H and z in these intervals

$$H_c(0) = 0$$

$$H_{c2}(0) = 2\ln(2e - \Psi_0'(e^{-1})) - 1 \simeq 1.85316$$

$$e^H = z\Psi'(z)^2 \quad \text{for} \quad z \in I_1 \text{ and } H \in J_1$$

$$e^H = z[\Psi'(z) + \Delta_0'(z)]^2 \quad \text{for } z \in I_2 \text{ and } H \in J_2$$

For $z \in I_3$ and $H \in J_3$ there are two distinct relations

$$e^H = z[\Psi'(z) + \Delta'_{-1}(z)]^2 \qquad (analytic)$$

$$e^H = z[\Psi'(z) + \frac{\Delta'_{-1}(z) + \Delta_0'(z)}{2}]^2 \ (non\ analytic)$$

$$\Delta_0(z) = \frac{4}{3}[\tilde{w}^2 - W_0(-ze^{\tilde{w}^2})]^{\frac{3}{2}} - 4[\tilde{w}^2 - W_0(-ze^{\tilde{w}^2})]^{\frac{1}{2}} + 2\tilde{w}\ \ln(\frac{\tilde{w} + [\tilde{w}^2 - W_0(-ze^{\tilde{w}^2})]^{\frac{1}{2}}}{|\tilde{w} - [\tilde{w}^2 - W_0(-ze^{\tilde{w}^2})]^{\frac{1}{2}}|})$$

$$\Delta_{-1}(z) = \frac{4}{3}[\tilde{w}^2 - W_{-1}(-ze^{\tilde{w}^2})]^{\frac{3}{2}} - 4[\tilde{w}^2 - W_{-1}(-ze^{\tilde{w}^2})]^{\frac{1}{2}} + 2\tilde{w}\ \ln(\frac{\tilde{w} + [\tilde{w}^2 - W_{-1}(-ze^{\tilde{w}^2})]^{\frac{1}{2}}}{|\tilde{w} - [\tilde{w}^2 - W_{-1}(-ze^{\tilde{w}^2})]^{\frac{1}{2}}|}) \qquad (27)$$

$\boxed{\text{z is in}}$ $I_1=[0,+\infty],\ I_2=[0,e^{-1}],\ I_3=]0,e^{-1}]$

$\boxed{\text{H is in}}$ $J_1=[-\infty,H_c(0)],\ J_2=[H_c(0),H_{c2}(0)],\ J_3=[H_{c2}(0),+\infty]$

$\boxed{\text{relation between } H \text{ and } z \text{ in these intervals}}$

$$H_c(0)=0$$

$$H_{c2}(0)=2\ln(2e-\Psi_0'(e^{-1}))-1$$

$$\simeq 1.85316$$

$$e^H=z\Psi'(z)^2 \quad \text{for} \quad z\in I_1 \text{ and } H\in J_1$$

$$e^H=z[\Psi'(z)+\Delta_0'(z)]^2 \quad \text{for } z\in I_2 \text{ and } H\in J_2$$

For $z\in I_3$ and $H\in J_3$ there are two distinct relations

$$e^H=z[\Psi'(z)+\Delta_{-1}'(z)]^2 \qquad (analytic)$$

$$e^H=z[\Psi'(z)+\frac{\Delta_{-1}'(z)+\Delta_0'(z)}{2}]^2\ (non\ analytic)$$

$$\Delta_0(z)=\frac{4}{3}[\tilde{w}^2-W_0(-ze^{\tilde{w}^2})]^{\frac{3}{2}}-4[\tilde{w}^2-W_0(-ze^{\tilde{w}^2})]^{\frac{1}{2}}$$
$$+2\tilde{w}\ln(\frac{\tilde{w}+[\tilde{w}^2-W_0(-ze^{\tilde{w}^2})]^{\frac{1}{2}}}{|\tilde{w}-[\tilde{w}^2-W_0(-ze^{\tilde{w}^2})]^{\frac{1}{2}}|})$$

$\boxed{\text{relation between } \Phi(H) \text{ and } z}$

$$\Delta_{-1}(z)=\frac{4}{3}[\tilde{w}^2-W_{-1}(-ze^{\tilde{w}^2})]^{\frac{3}{2}}-4[\tilde{w}^2-W_{-1}(-ze^{\tilde{w}^2})]^{\frac{1}{2}}$$
$$+2\tilde{w}\ln(\frac{\tilde{w}+[\tilde{w}^2-W_{-1}(-ze^{\tilde{w}^2})]^{\frac{1}{2}}}{|\tilde{w}-[\tilde{w}^2-W_{-1}(-ze^{\tilde{w}^2})]^{\frac{1}{2}}|}) \qquad (27)$$

$$\Phi(H)=\Psi(z)-2z\Psi'(z) \quad \text{for } z\in I_1$$

$$\Phi(H)=\Psi(z)-2z\Psi'(z)+\frac{4}{3}[-W_0(-z)]^{\frac{3}{2}} \text{ for } z\in I_2$$

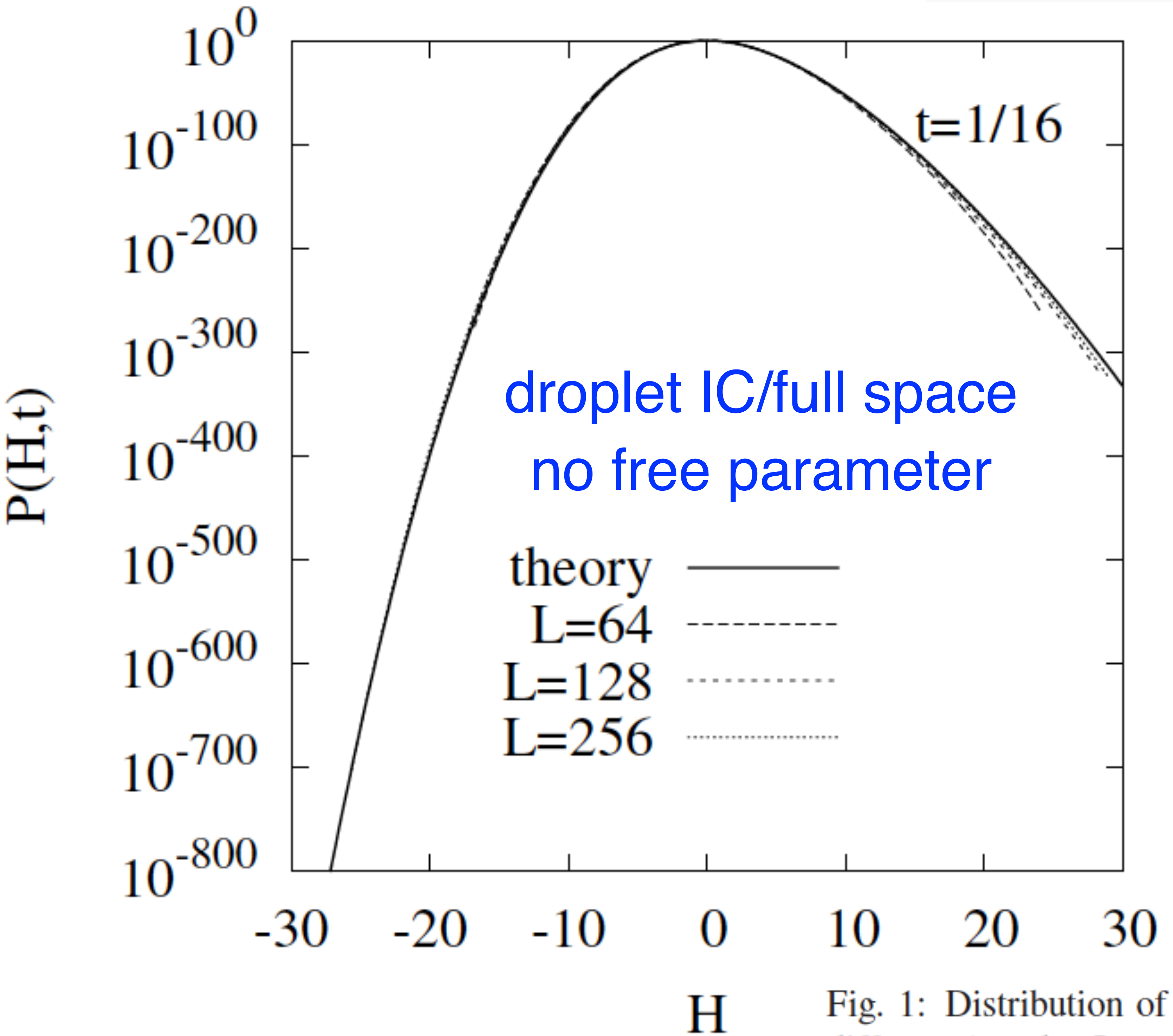
For $z\in I_3$ there exist two branches for $\Phi(H)$

$$\Phi(H)=\Psi(z)-2z\Psi'(z)+\frac{4}{3}[-W_{-1}(-z)]^{\frac{3}{2}} \quad (analytic)$$

$$\Phi(H)=\Psi(z)-2z\Psi'(z)+\frac{2}{3}[-W_0(-z)]^{\frac{3}{2}}+\frac{2}{3}[-W_{-1}(-z)]^{\frac{3}{2}}$$

Numerical Simulations

A. K. Hartmann, P. Le Doussal, S. N. Majumdar, A. Rosso, G. Schehr. "High-precision simulation of the height distribution for the KPZ equation" arXiv:1802.02106



directed polymer

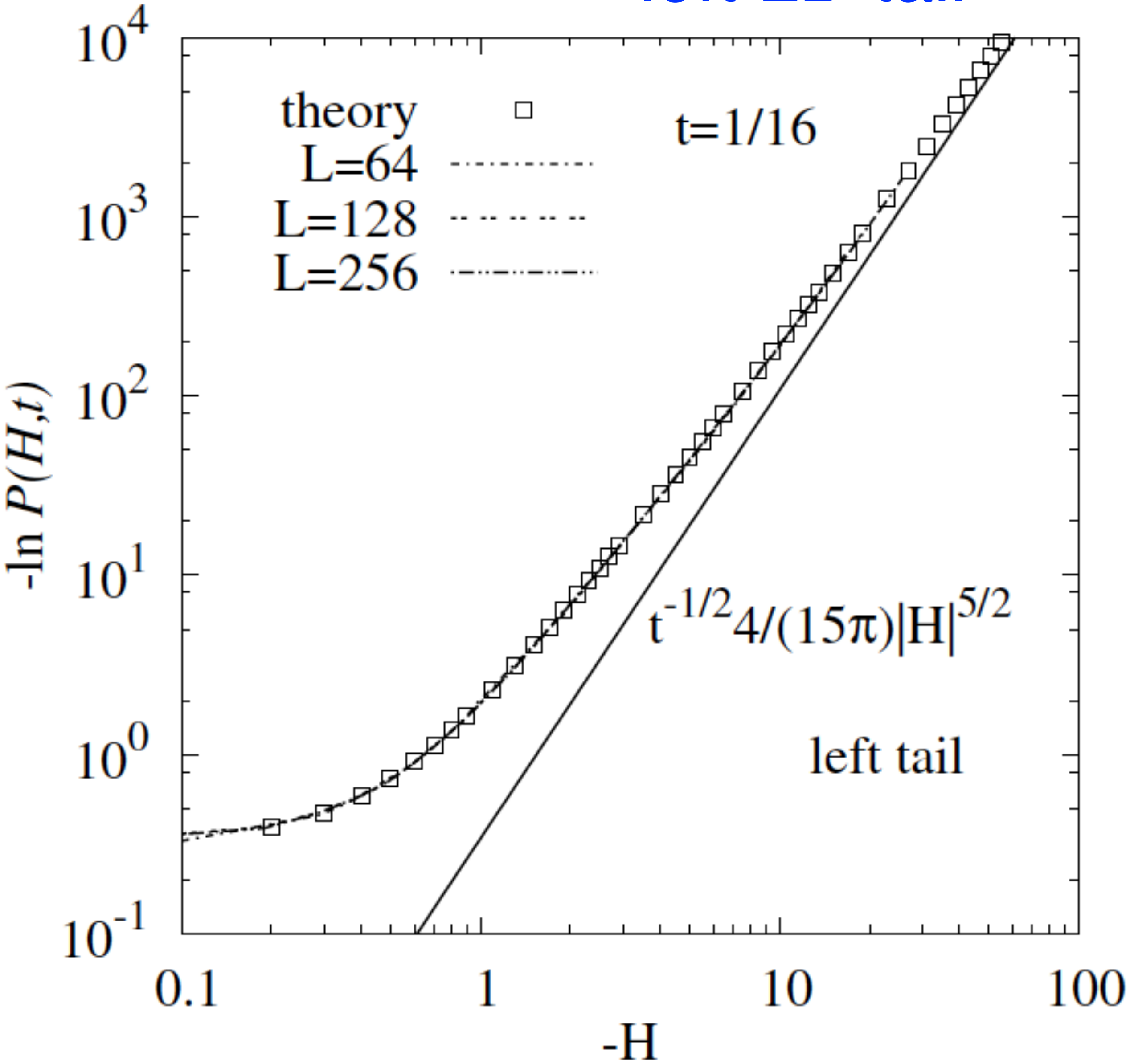
square lattice T >>1

=> KPZ equation

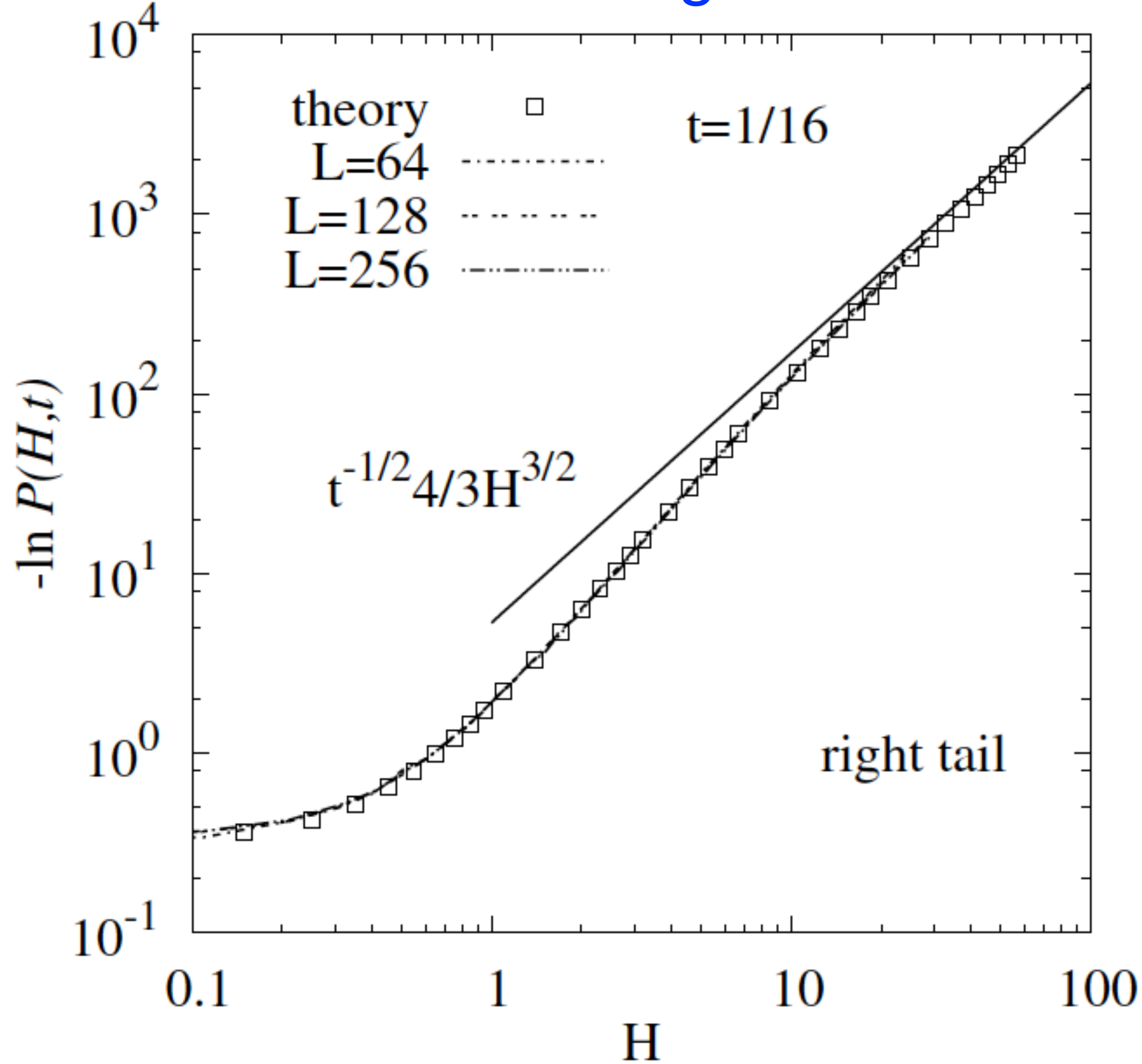
importance sampling

Fig. 1: Distribution of $P(H,t)$ for a short time $t = 1/16$ for three different lengths $L = 64$, $L = 128$ and $L = 256$. The solid line indicates the analytical result in Eq. (2) obtained in Ref. [21]. The agreement between numerical and analytical results is extremely good (on the left tail, down to values of the order 10^{-800}).

left LD tail

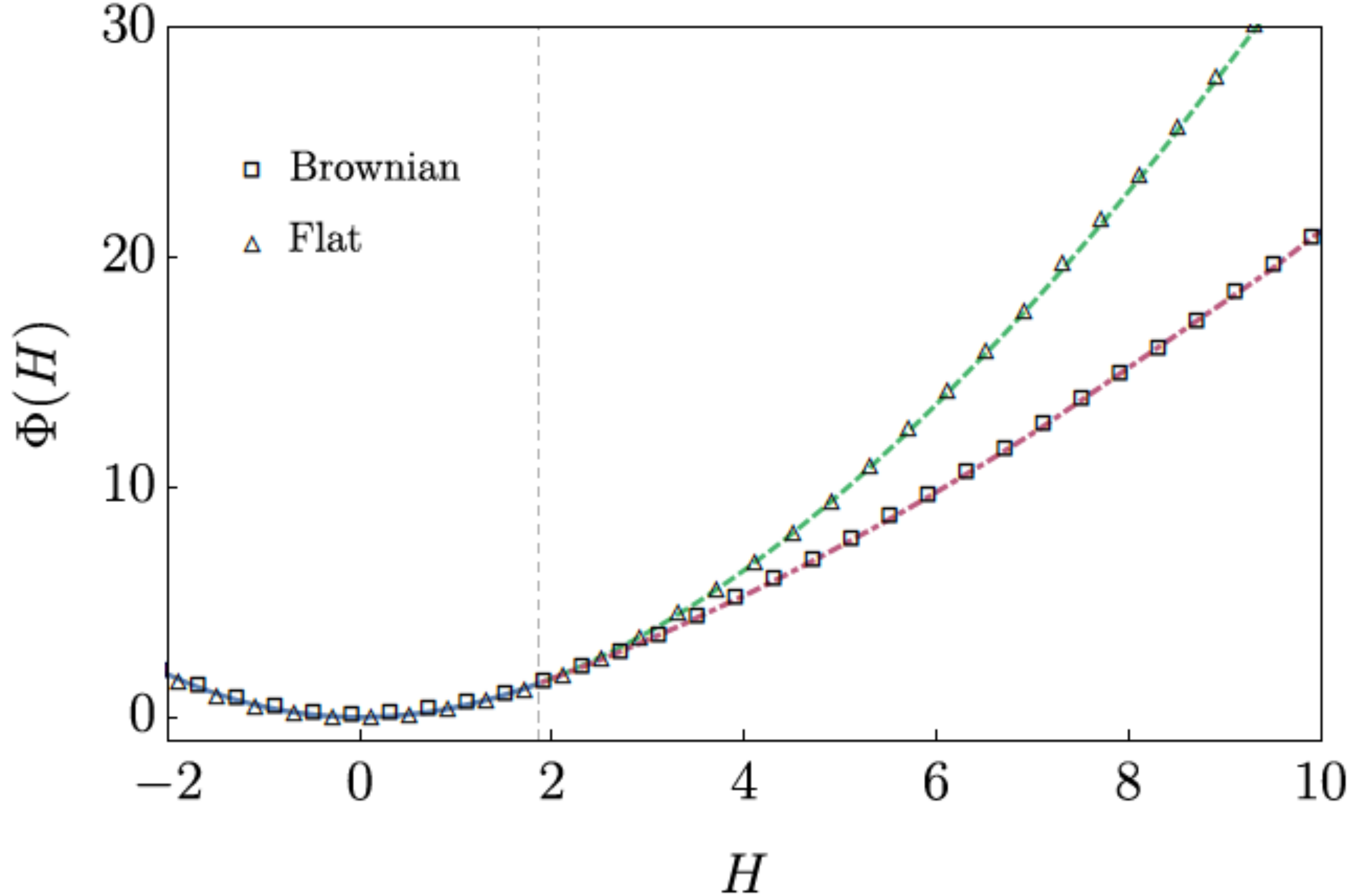


right LD tail

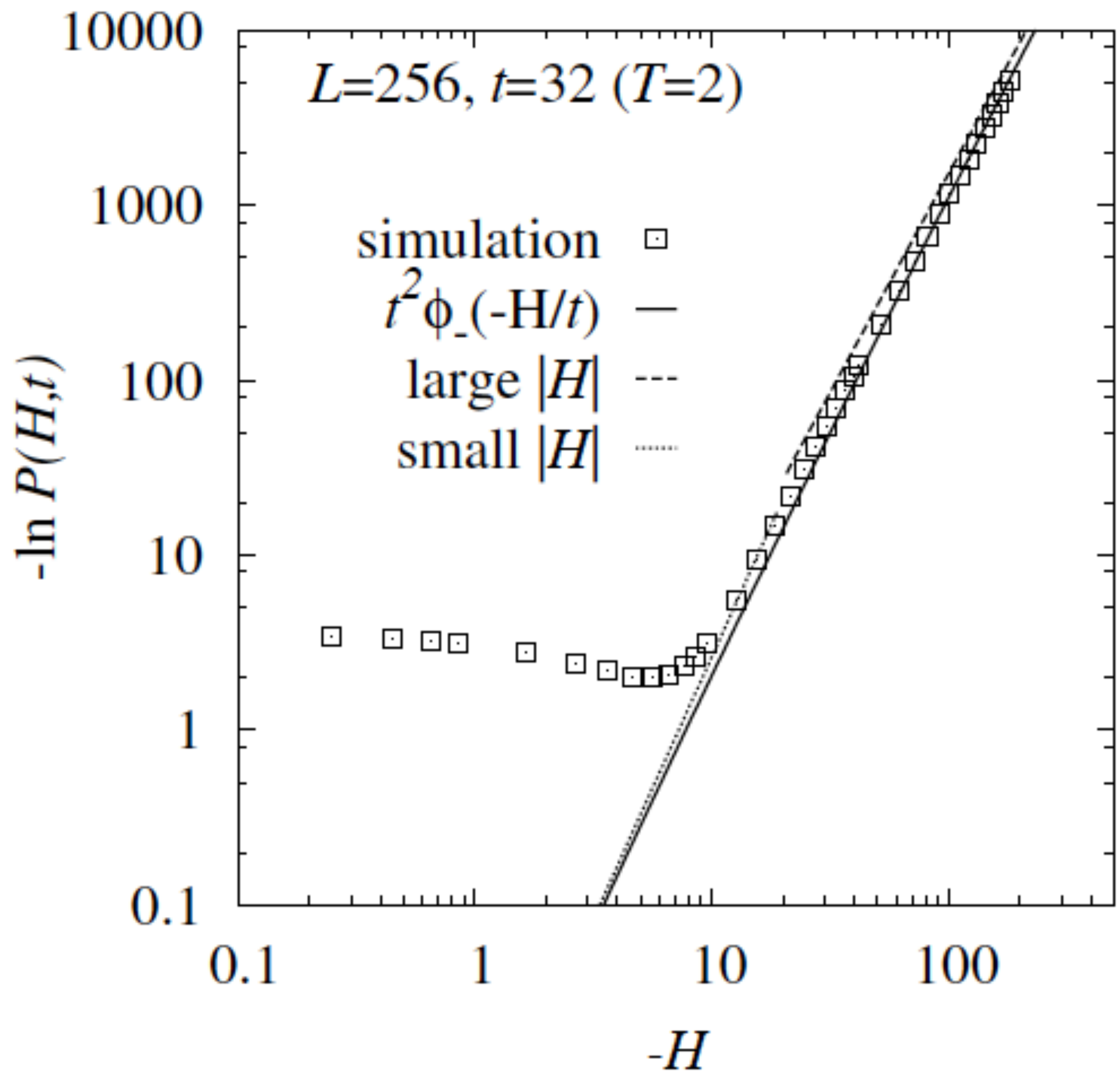


Brownian and flat IC/full space

Hartmann, AK, PLD in preparation



$$\Phi_{\mathrm{Brownian}}(H) = \Phi_{\mathbf{NA}}(H) \qquad \Phi_{\mathrm{flat}}(H) = 2^{-\frac{3}{2}}\Phi_{\mathbf{A}}(2H)$$



later times

droplet IC/full space

Fig. 5: Logarithm of the left tail of $P(H,t)$ for longer time ($t = 32$) and for the longest length $L = 256$, shown in double-logarithmic scale. The solid line shows the analytical prediction of Eq. (6). The broken line shows the resulting limiting power-law: $|H|^3/(12t)$ for very large H, and $\frac{4}{15\pi}|H|^{5/2}/\sqrt{t}$ for moderate large H.

for the KPZ equation

1) large time large deviation (left) $P(H,t) \sim \exp\left(-t^2\,\Phi_-(\frac{H}{t})\right)$

droplet IC obtained $\Phi_-(z)$

how to obtain general IC ? $\Phi_-^{\rm Brown}(z) =?= \Phi_-^{\rm drop}(z)$

$\Phi_-^{\rm flat}(z) =?= 2\,\Phi_-^{\rm drop}(z)$

A=-1/2, GOE $\Phi_-^{1/2{\rm sp,drop}}(z) = \frac{1}{2}\Phi_-^{\rm fullsp,drop}(z)$

can one bypass exact solutions ?

2) short time large deviation $P(H,t) \sim \exp\left(-\frac{\Phi(H)}{\sqrt{t}}\right)$ obtained $\Phi(H)$

systematic expansion up to O(t^3)

connection to Weak Noise Theory ? integrability? multi-point?

far tails large |H| seem same large t and small t ? process?