Current distribution for a two-species particle model from first principles

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ACEMS

1+1D growth

1.2 Measurement Techniques 5

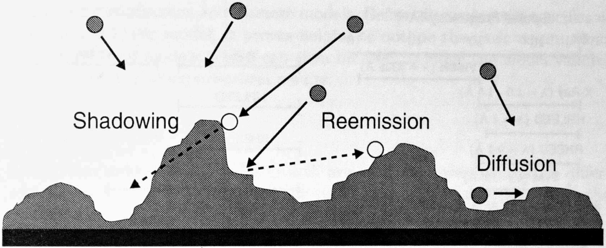


Fig. 1.3. Diagram of growth effects including diffusion, shadowing, and reemission that may affect surface morphology during thin film growth. The incident particle flux may arrive at the surface with a wide angular distribution depending on the deposition methods and parameters.

KPZ growth

- Stochastic growth normal to the surface
- Kardar-Parisi-Zhang (KPZ) 1986
- Basic object: (random) height function $h(x,t)$

KPZ equation (nonlinear stochastic PDE):

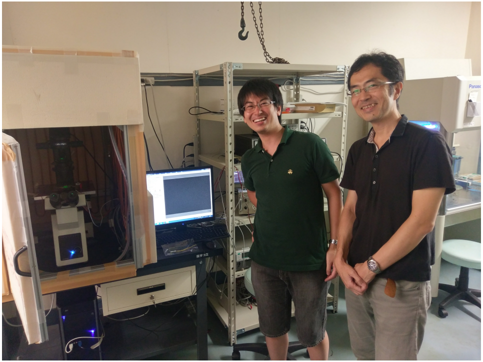
KPZ equation

$$\partial_t h(t,x) = \frac{1}{2}\partial_x^2 h(t,x) + \frac{1}{2}\left(\partial_x h(t,x)\right)^2 + \xi(t,x)$$

Claim:
Diffusion + non-linearity + space-time Gaussian white noise
correctly describes 1+1D surface growth

Realisation in liquid crystal growth: Takeuchi lab

1+1D Growth



Takeuchi and Sasamoto, Tokyo 2017

1+1D Growth

Flat

Flat initial conditions

KPZ growth

Theorem (Non-Gaussian fluctuations)

$$h \sim \nu t + c t^{1/3}, \quad t \to \infty$$

Transformation to Stochastic Heat Equation (SHE):

$$h(t,x) := \log z(t,x).$$

SHE equation

$$\partial_t z(t,x) = \frac{1}{2}\partial_x^2 z(t,x) + \xi(t,x) z(t,x)$$

Fluctuations

The Laplace transform formula for $z(t,x)$ can be written as a Fredholm determinant

Theorem (Laplace transform of SHE)

$$\mathbb{E}[e^{-\zeta z(t,0)}] = \det(I - K_\zeta)_{L^2(\mathbb{R}_+)}$$

$$K_\zeta(\eta,\eta') = \int_{\mathbb{R}} \frac{\zeta}{\zeta + e^{-\xi t^{1/2}}} \mathrm{Ai}(\xi+\eta)\mathrm{Ai}(\xi+\eta')d\xi.$$

Theorem (Fluctuations of SHE)

$$\lim_{t\to\infty} P\left(\frac{h(t,x)-t}{t^{1/3}} < s\right) = F_{\mathrm{GUE}}(s).$$

$F_{\mathrm{GUE}}(s)$ is the Tracy-Widom distribution of the Gaussian Unitary Ensemble of random matrix theory.

Universality

The Tracy-Widom distribution also appears in the asymmetric exclusion process on $\mathbb{Z}$.

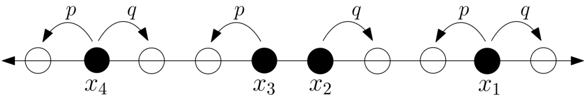


FIGURE 4. ASEP particle configuration with possible jumps and rates denoted by arrows.

Let $N_y(t)$ be the number of particles to have crossed a given site y after time t.

Let $Q_y(t) = \tau^{N_y(t)}$ with $\tau = \frac{p}{q}$ and

$$\widetilde{Q}_y(t) = \frac{Q_y(t) - Q_{y-1}(t)}{\tau - 1}.$$

Fundamental solution of ASEP

Theorem (Fundamental solution of ASEP)

$$\mathbb{E}[\widetilde{Q}_{x_1}(t)\cdots\widetilde{Q}_{x_k}(t)] = \oint\cdots\oint \prod_{1\leqslant i<j\leqslant k} \frac{z_i - z_j}{z_i - \tau z_j} \prod_{j=1}^{k} e^{-\lambda(z_j)t} \left(\frac{1+z_j/\tau}{1+z_j}\right)^{x_j+1} dz_j$$

Fluctuations of particle flow across the origin follow KPZ statistics given by the Tracy-Widom distribution:

Theorem (Fluctuations of ASEP)

$$\lim_{t\to\infty} P\left(\frac{N_0(t)-\nu t}{t^{1/3}} > -s\right) = F_{\mathrm{GUE}}(s).$$

Summary

Ingredients:

- Integrable stochastic lattice model
- Observable expressed in terms of k-fold integral
- Asympotics for large $k \rightarrow$ Fredholm determinant
- Saddle point analysis

New results:

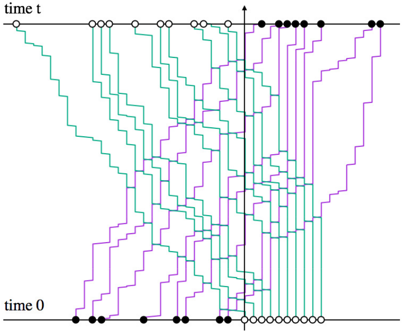
- Rank two model
- Dynamic poles in integral (from nested Bethe ansatz)
- Combination of Gaussian and GUE modes

AHR model

Introduced by Arndt-Heinzl-Rittenberg, the transition rates are

$$\begin{aligned} p: &\quad (\bullet, \cdot) \to (\cdot, \bullet) \\ p': &\quad (\cdot, \circ) \to (\circ, \cdot) \\ 1: &\quad (\bullet, \circ) \to (\circ, \bullet) \end{aligned}$$

Throughout we will take $p + p' = 1 \quad \to \quad$ factorised steady state.



Nonlinear Fluctuating Hydrodynamics

Continuity equation

$$\frac{\partial \mathbf{u}(x,t)}{\partial t} + \frac{\partial \mathbf{j}(\mathbf{u}(x,t))}{\partial x} = 0,$$

where $\mathbf{u}(x,t) = (\rho_+, \rho_-)$ and $\mathbf{j}(\mathbf{u}) = (j_+, j_-)$ given by non-linear flows

$$j_+(\mathbf{u}) = \rho_+(1 - \rho_+ - \rho_-) + 2\rho_+\rho_-,$$
$$j_-(\mathbf{u}) = -(1 - \rho_+ - \rho_-)\rho_- - 2\rho_+\rho_- .$$

Adding diffusion and noise, heuristic non-linear fluctuating hydrodynamics (NLFHD) leads to

$$P_{\text{crossing}}(t) \sim F_{\text{GUE}}(s_+)F_{\text{Gauss}}(s_-),$$

where $s_\pm$ are eigenmodes.

Aim of this talk is to rigorously prove this.

Green's function

Let $\vec{z} = \vec{x} \cup \vec{y}$ be coordinates of $+$ and $-$ particles.

Definition

The Green function satisfies the master equation

$$\frac{d}{dt} G(\vec{z}; t) = \sum_{\vec{z}'} M_{\vec{z};\vec{z}'} G(\vec{z}', t); \qquad t > 0,$$

and initial condition

$$G(\vec{x}; \vec{y}; 0) = \prod_{i=1}^{n} \delta_{x_i, x_i^0} \prod_{j=1}^{m} \delta_{y_j, y_j^0}$$

Explicit form can be determined by Bethe ansatz using boundary conditions:

$$\text{Exclusion:} \quad G(x, x; \vec{y}; t) = G(x, x+1; \vec{y}; t),$$

$$\text{Scattering:} \quad G(y; y+1; t) = q\, G(y+1; y+1; t) + p\, G(y; y; t).$$

Green's function

Initial conditions: assume $x_j^0 < y_k^0$, i.e. at $t = 0$ all $\bullet$ particles are to the left of all $\circ$ particles.

Final condition: $x_j > y_k$, i.e. at time t all $\circ$ particles are to the left of all $\bullet$ particles.

Then

$$G(x_j, y_k, t) = \oint \prod_{j=1}^{n} \mathrm{d}\, z_j \prod_{k=1}^{m} \mathrm{d}\, w_k \, \mathbf{e}^{\Lambda t} \prod_{k=1}^{m} \prod_{j=1}^{n} \frac{1}{q z_j + p w_k}$$
$$\times \det \left(\left(\frac{z_j - 1}{z_i - 1} \right)^{j-1} z_i^{x_j} \right) z_j^{-x_j^0 - 1}$$
$$\times \det \left(\left(\frac{w_k - 1}{w_\ell - 1} \right)^{m-k} w_\ell^{-y_k} \right) w_k^{y_k^0 - 1},$$

with all contours around the origin, and with eigenvalue

$$\Lambda = p \sum_{j=1}^{n} (z_j^{-1} - 1) + q \sum_{k=1}^{m} (w_k^{-1} - 1).$$

Current distribution: step initial condition

Given the following step initial condition

$$\cdots\cdots\overbrace{\bullet\,\bullet\cdots\bullet\,\bullet}^{n\text{times}}\overbrace{\circ\,\circ\cdots\circ\,\circ}^{m\text{times}}\cdots\cdots,$$

Then

$$\mathbb{P}(x_1(t)\geqslant s)=\sum_{x_1=s}^{\infty}\cdots\sum_{x_n=x_{n-1}+1}^{\infty}\sum_{y_1=-\infty}^{-n}\cdots\sum_{y_n=y_{n-1}+1}^{-1}G(\{x_j\};\{y_k\};t),$$

Proposition

$$\mathbb{P}(x_1(t)\geqslant 0)=$$

$$\oint\prod_{j=1}^{n}dz_j\prod_{k=1}^{m}dw_k\,\mathbf{e}^{\wedge}\mathbf{t}\,\frac{\prod\limits_{1\leqslant i<j\leqslant n}(z_i-z_j)\prod\limits_{1\leqslant k<l\leqslant m}(w_l-w_k)\prod\limits_{j=1}^{n}z_j^{n-j+s}\prod\limits_{k=1}^{m}w_k^{k-1}}{\prod\limits_{j=1}^{n}(z_j-1)^{n+1-j}\prod\limits_{k=1}^{m}(w_k-1)^{k}\prod\limits_{j=1}^{n}\prod\limits_{k=1}^{m}\left(qz_j+pw_k\right)},$$

with all contours around the origin.

Current distribution

- $e^{\Lambda t}$ produces an essential singularity at origin: $\Lambda = p\sum_{j=1}^{n}(z_j^{-1}-1) + q\sum_{k=1}^{m}(w_k^{-1}-1)$.
- Deform w-contours to lie around poles other than the origin
- Only (simple) poles at $w = 1$ give nonzero contribution

After evaluating the residues in w, we get

Proposition

$$\mathbb{P}(x_1(t) \geqslant 0) = \oint \prod_{j=1}^{n} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}}\, \mathbf{e}^{\tilde{\Lambda} t} \frac{\prod_{1\leqslant i<j\leqslant n}(z_i - z_j)\prod_{j=1}^{n} z_j^{n-j}}{\prod_{j=1}^{n}(z_j-1)^{n+1-j}\prod_{j=1}^{n}\left(qz_j+p\right)^m}$$

TASEP limit

Corollary

When $n = m$ and $p = q = 1/2$ we retrieve the same distribution as for the single species TASEP under step initial condition, i.e.

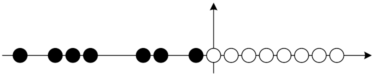
$$P_{n,n,1}(t) = \frac{1}{n!} \oint \prod_{j=1}^{n} dx_j \; e^{\mathcal{E}t} \frac{\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)^2}{\prod_{j=1}^{n} (x_j - 1)^n},$$

where the contours are still around the origin and $\mathcal{E} = \sum_{j=1}^{n} (x_j^{-1} - 1)$.

This is made explicit by symmetrising and the simple change of variable $z_j = x_j/(2 - x_j)$.

From known analysis (Tracy-Widom) this probability converges to the GUE distribution as $n, t \to \infty$.

Step-Bernoulli condition



Let the distance among positive particles be independently distributed with parameter ρ',

Proposition

The total exchange probability $P_{n,m,\rho}(t)$ *with Bernoulli initial data is given by*

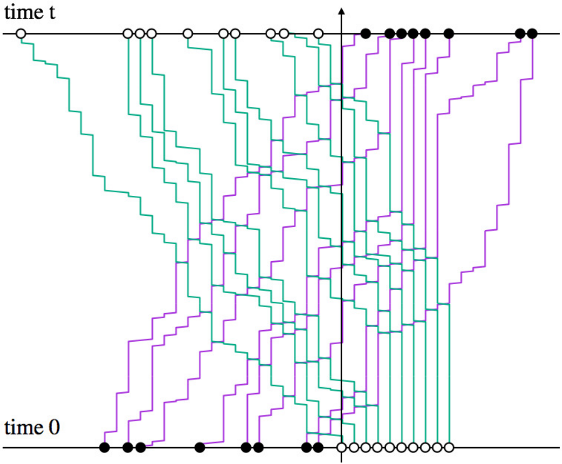
$$P_{n,m,\rho}(t) = \oint \prod_{j=1}^{n} dz_j \prod_{k=1}^{m} dw_k \, e^{\Lambda_{n,m} t} \times$$

$$\frac{\rho^n \prod\limits_{1\leqslant i<j\leqslant n} (z_i - z_j) \prod\limits_{1\leqslant k<l\leqslant m} (w_l - w_k) \prod\limits_{j=1}^{n} z_j^{n-j} \prod\limits_{k=1}^{m} w_k^{k-1}}{\prod\limits_{j=1}^{n} (z_j - 1)^{n+1-j} (1 - \rho' z_j) \prod\limits_{k=1}^{m} (w_k - 1)^k \prod\limits_{j=1}^{n} \prod\limits_{k=1}^{m} \left(q z_j + p w_k \right)},$$

with all contours around the origin.

The w-contours can be readily evaluated if $n > m$ but not when $n < m$

Exchange



Asymptotics

Non-linear fluctuating hydrodynamics (KPZ formalism) suggests a scaling limit of the form

$$n = j_1 \mathbf{t} + \alpha \mathbf{t}^{1/3} + \beta \mathbf{t}^{1/2},$$
$$m = j_2 \mathbf{t} + \gamma \mathbf{t}^{1/3} + \delta \mathbf{t}^{1/2},$$

where $j_{1,2}$, α, β, γ, δ are known functions of ρ', and $n < m$.

Need to analyse

$$P_{n,m,\rho}(t) = \underbrace{\oint \dots \oint}_{n \times m} \text{factorised integrand}$$

where n, m, t are large.

Trick: Convert to Fredholm determinant:

$$P_{n,m,\rho}(t) = \det(\mathbb{I} - AB)_{m \times m} = \det(\mathbb{I} - BA)_{L^2(\mathbb{R})},$$

where n, m, t all occur as parameters in BA.

Example

After symmetrising and deforming contours, P involves integrals of the form

$$\mathcal{I} = \frac{1}{m!}\oint_{0,1,-1} \mathrm{d}^m w \, \frac{e^{\tilde{\Lambda}_m t} \prod_{1 \leqslant k < \ell \leqslant m} (w_\ell - w_k)^2}{\prod_{k=1}^m w_k^m (w_k - 1)^m \prod_{k=1}^m (\frac{1}{2}(1 + 1/w_k))^n},$$

where $\tilde{\Lambda}_m = \frac{1}{2}\sum_{k=1}^m (w_k - 1)$.

This expression can be written as

$$\boxed{\mathcal{I} = \frac{1}{m!}\oint_{0,1,-1} \frac{\mathrm{d}^m w}{\prod_{k=1}^m w_k} \frac{\prod_{1 \leqslant k \neq \ell \leqslant m} (1 - w_k/w_\ell)}{\prod_{k,\ell=1}^m (1 - a_\ell/w_k)} \prod_{k=1}^m \frac{g_{n,m}(w_k, 0)}{g_{n,m}(a_k, 0)}}$$

with

$$g_{n,m}(w, x) = \left(\frac{w}{w+1}\right)^n w^{x-m} e^{wt/2},$$

and $a_k = 1, \quad 1 \leqslant k \leqslant m$.

Assuming first that $|a_k| < 1$, the integral $\mathcal{I}$ can be written as a Fredholm determinant

$$\mathcal{I} = \det(\mathbb{I} - K(x, y)).$$

Example

The kernel

$$K(x,y) = \sum_{k=0}^{m-1} \phi_k(x)\psi_k(y),$$

with

$$\phi_k(x) = \oint_1 \mathrm{d}w \frac{1}{w^{x+1}} \left(\frac{1+w}{w}\right)^n \left(\frac{w}{1-w}\right)^{k+1} e^{-wt/2},$$

$$\psi_k(x) = \oint_{0,-1} \mathrm{d}w\, w^{x-2} \left(\frac{w}{1+w}\right)^n \left(\frac{1-w}{w}\right)^k e^{wt/2},$$

in which the pole at $w = 1$ is separated from the poles at $w = -1, 0$.

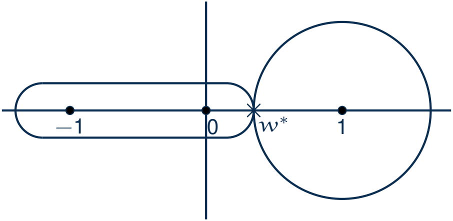
We rewrite this integral

$$\boxed{\phi_k(x) = \oint_1 \frac{\mathrm{d}w}{2\pi i} e^{f(w)t}}$$

with $k \sim m$ and

$$f(w)t = n \log\left(\frac{1+w}{w}\right) + (m - \kappa_1 t^{1/3} \log\left(\frac{w}{1-w}\right) - \xi_1 t^{1/3} \log(w) - \frac{1}{2} w t,$$

In the scaling limit, $f(w)$ has the double saddle point at $w^* = \rho'/2$:



Contours for $\phi_k(x)$, around 1, and $\psi_k(x)$; joint saddle at $w^* = \rho'/2$.

$$f(w)t = f(w^*)t - c_1(w - w^*)t^{1/3} + c_2(w - w^*)^3 t + \dots$$

Define the scaling variable v by

$$w - w^* = \frac{iv}{t^{1/3}} c_3$$

The function $f(w)$ behaves asymptotically as

$$f(w)t = f(w^*)t - iv^3 - iv\Big(\kappa + \xi\Big) + \mathcal{O}(t^{-1/6}).$$

This leads to

$$\phi_{m-t^{1/3}\kappa}(t^{1/3}\xi) \sim e^{f(w^*)t}\,\mathrm{Ai}\left(\kappa+\xi\right).$$

The scaled kernel therefore has the long time limit

$$K_t(\xi,\zeta) \to \int_0^\infty \mathrm{Ai}\left(\kappa+\xi\right)\mathrm{Ai}\left(\kappa+\zeta\right)\mathrm{d}\,\kappa := A(\xi,\zeta) \qquad \text{as } t\to\infty.$$

Proposition

The functions ϕ_k and ψ_k behave asymptotically as Airy functions. In the scaling limit the integral $\mathcal{I}$ therefore satisfies

$$\mathcal{I} \sim \det(1-K(\xi t^{1/3},\zeta t^{1/3}) = \det(1-K_t(\xi,\zeta)),$$

such that

$$\lim_{t\to\infty} K_t(\xi,\zeta) = \int_0^\infty Ai(\kappa+\xi)Ai(\kappa+\zeta)\,\mathrm{d}\,\kappa.$$

Non-Gaussian distribution function

Nested poles

The crossing probability involves mixed integrals of the form

$$\mathcal{I}_z \times \mathcal{I}_w := \frac{1}{n!}\oint_1 d^n z \, \frac{e^{\Lambda_n t} \prod_{1\leqslant i<j\leqslant n}(z_i - z_j)^2 \prod_{j=1}^n (1-\rho' z_j)}{\prod_{j=1}^n (z_j - 1)^n (\frac{1}{2}(1+z_j))^m} \times$$

$$\frac{1}{m!}\oint_{0,\pm 1,-\rho'} d^m w \, e^{\tilde{\Lambda}_m t} \frac{\prod_{1\leqslant k<l\leqslant m}(w_l - w_k)^2}{\prod_{k=1}^m w_k^{m-n}(w_k - 1)^m} \times$$

$$\prod_{k=1}^m \frac{1+1/\rho'}{1+w_k/\rho'} \prod_{j=1}^{n-1}\prod_{k=1}^m \frac{1+z_j}{1+z_j w_k}.$$

The integral $\mathcal{I}_w$ now depends on $\{z_j\}$. The Fredholm kernel is

$$K(x,y) = \sum_{k=0}^{m-1} \phi_k(x)\psi_k(y),$$

and

$$\phi_k(x) = \oint_1 dw \, \frac{1+w/\rho'}{w^{x+1}(w-1)} \prod_{j=1}^{n-1} \frac{1+z_j w}{w} \left(\frac{w}{1-w}\right)^k e^{-wt/2}.$$

Dominant contribution

Consider the contributions arising from the **combined** poles at $z_j = 1$ and $w_k = 1$ in $\mathcal{I}_z \times \mathcal{I}_w$

$$\oint_1 \mathrm{d}^n z \, \frac{\prod_{1 \leqslant i < j \leqslant n} (z_i - z_j)^2}{\prod_{j=1}^{n} (z_j - 1)^n} \times \oint_1 \mathrm{d}^m w \, \frac{\prod_{1 \leqslant k < l \leqslant m} (w_l - w_k)^2}{\prod_{k=1}^{m} (w_k - 1)^m} \prod_{j=1}^{n-1} \prod_{k=1}^{m} \frac{1 + z_j}{1 + z_j w_k}.$$

- The mixed factors in ϕ_k expand near $z = 1$

$$\frac{1+z}{1+zw} = \frac{2}{w+1} + \mathcal{O}((z-1)(w-1)).$$

- For the combined poles, we can set $z_j = 1$ in $\phi_k(x)$
- $\Rightarrow$ the z and w integration factorises

Final result

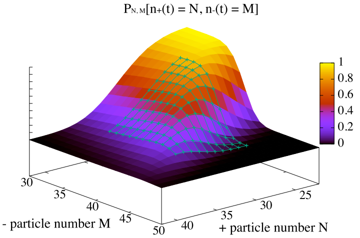
The z-integration produces a Gaussian and the w-integration a GUE distribution.

Theorem

In the appropriate scaling limit

$$\lim_{t\to\infty} P_{n,m,\rho}(t) = F_{\mathrm{GUE}}(s_+)F_{\mathrm{Gauss}}(s_-),$$

$$s_-(n,m;t) =: \frac{1}{c_2 t^{1/3}}\Big((1+\rho)n - (3-\rho)m + \tfrac{1}{2}(1-\rho)(1-(1-\rho)^2/4)t\Big),$$
$$s_+(n,m;t) =: \frac{1}{c_g t^{1/2}}\Big(-2(2-\rho)n + 2\rho m + (2-\rho)(1-\rho)\rho t\Big),$$



Conclusion

- (First) proof of Nonlinear Fluctuating Hydrodynamics for a two-component mixture
- Using integrability
- Mix of Gaussian and KPZ modes
- Dynamic poles in integrand

Universality in Nagel-Schreckenberg model

Joint work with Andreas Schadschneider, **Johannes Schmidt**, Gunter Schütz.

NaSch definition:

1. Acceleration:
$$v_n \to \min\left(v_n + 1, v_{\max}\right)$$
2. Deceleration: If $v_n > d_n$ i.e.
$$v_n \to \min\left(v_n, d_n\right)$$
3. Randomisation:
$$v_n \xrightarrow{p_s} \max\left(v_n - 1, 0\right) \; .$$
4. Vehicle movement:
$$x_n \to x_n + v_n$$

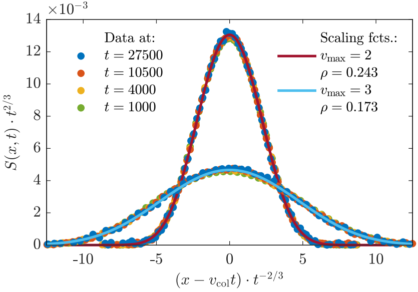
At $v_{\max} = 1$ this is equivalent to (discrete time) TASEP.

Does KPZ universality survive when $v_{\max} > 1$?

Structure function

$$S(x,t) = \langle u(x,t)u(0,0)\rangle \simeq t^{-2/3} f_{PS}\left(t^{-2/3}(x - vt)\right)$$

with the Prähofer-Spohn scaling function f_{PS}.



Integrated current

$$J_t = \int_0^t [j(0,s) - j(\rho)]ds - \int_0^{v_{col}t} u(x,0)dx$$

$$\mathcal{P}(J,t) \simeq t^{-1/3} F_{BR}\left(-J \cdot t^{-1/3}\right)$$

with the Baik-Rains scaling function $F_{BR}(x)$.

For $v_{max} = 3$: