Finite temperature dynamical correlation functions of the XX chain

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Outline of the talk

- Introduction: Dynamical two-point functions of quantum spin chains
- QTM formalism and thermal form-factor series
- Thermal form-factor series (TFFS) for dynamical two-point functions
- TFFS for the transverse dynamical two-point functions of the XX chain
- Long-time, large-distance asymptotics at fixed finite T
- Fredholm determinant representation
- High-temperature analysis of the transversal two-point functions

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Based on [GKKKS17] and on unpublished work with K. K. KOZLOWSKI, J. SIRKER and J. Suzuki [ARXIV:1905.04922, ARXIV:1906.03143 AND ARXIV:1908.11555]

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Long history of XX: LIEB, SCHULTZ, MATTIS 61; KATSURA; NIEMEIJER; M. SUZUKI; BAROUCH, MCCOY, TRACY, WU; BRANDT, JACOBY; CAPEL, PERK; MÜLLER, SHROCK; ITS, IZERGIN, KOREPIN, SLAVNOV; COLOMO, TOGNETTI

Quantum StatMech (of spin chains)

- Quantum (spin) chain:

$H_L \in \mathrm{End}\big(\mathbb{C}^d\big)^{\otimes L}$	Hamiltonian
L	length of chain
d	dimension of local Hilbert space
$x_j = \mathrm{id}^{\otimes(j-1)} \otimes x \otimes \mathrm{id}^{\otimes(L-j)}, \ x \in \mathrm{End}\big(\mathbb{C}^d\big)$	local operator
$j \in \{1, \dots, L\}$	lattice sites

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- Linear response theory ('Kubo theory') connects the response of a large quantum system to time-($= t$)-dependent perturbations (= experiments) with dynamical correlation functions at finite temperature T

$$\langle x_1 y_{m+1}(t) \rangle_T = \lim_{L \to \infty} \frac{\mathrm{tr}_{1,\dots,L}\left\{\mathrm{e}^{-H_L/T} x_1 \, \mathrm{e}^{\mathrm{i}tH_L} y_{m+1} \mathrm{e}^{-\mathrm{i}H_L t}\right\}}{\mathrm{tr}_{1,\dots,L}\left\{\mathrm{e}^{-H_L/T}\right\}}$$

Prime example of an integrable spin chain Hamiltonian

- The XXZ model

$$H_L(\Delta) = J \sum_{j=1}^{L} \{ \sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta \sigma_{j-1}^z \sigma_j^z \} - \frac{h}{2} \sum_{j=1}^{L} \sigma_j^z$$

$J > 0$, $h \in \mathbb{R}$, $\Delta \in \mathbb{R}$

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- Main goal of my research: Calculate

$$\left\langle \sigma_1^z \sigma_{m+1}^z(t) \right\rangle_T , \quad \left\langle \sigma_1^- \sigma_{m+1}^+(t) \right\rangle_T$$

explicitly for all values of m, t, T and h!

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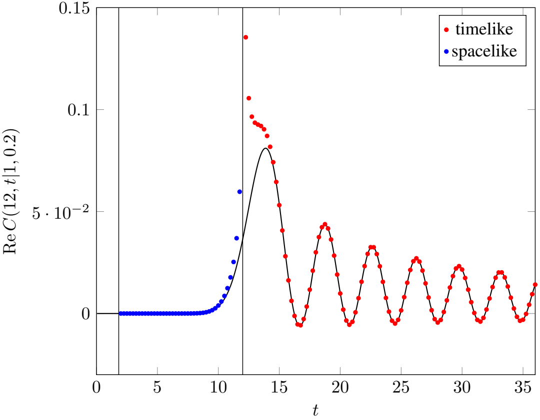
- For the XX model the longitudinal two-point functions are

$$\langle\sigma_1^z\sigma_{m+1}^z(t)\rangle_T - \langle\sigma_1^z\rangle^2 = \left[\int_{-\pi}^{\pi}\frac{\mathrm{d}p}{\pi}\frac{\mathrm{e}^{\mathrm{i}(mp-t\varepsilon(p))}}{1+\mathrm{e}^{\varepsilon(p)/T}}\right]\left[\int_{-\pi}^{\pi}\frac{\mathrm{d}p}{\pi}\frac{\mathrm{e}^{-\mathrm{i}(mp-t\varepsilon(p))}}{1+\mathrm{e}^{-\varepsilon(p)/T}}\right]$$

where $\varepsilon(p) = h - 4J\cos(p)$

Longitudinal correlation functions of XX model

- This simple expression can be analyzed numerically and asymptotically by means of the saddle point method

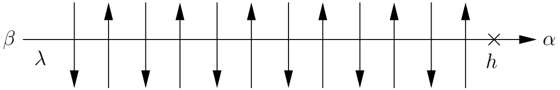


Real part of the connected longitudinal two-point function of the XX chain at $m = 12$, $T = 1$, $h = 0.2$ and $J = 1/4$ as a function of time

Reduced density matrix and form factor series

- Staggered ($\pm h_R/NT$) monodromy matrix of fundamental models

$T^\alpha_\beta(\lambda|h) =$



Every vertex represents an R-matrix. $t(\lambda|h) = T^\alpha_\alpha(\lambda|h)$ is the quantum transfer matrix with eigenvectors $|\Psi_n\rangle$ and eigenvalues $\Lambda_n(\lambda|h)$. $f(T,h) = -T\ln\Lambda_0(0|h)$

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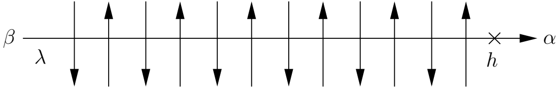

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- Reduced density matrix [GKS04] of chain segment of length $m+1$

$$\lim_{L\to\infty} \mathrm{tr}_{m+2,\dots,L} \left\{ \frac{\mathrm{e}^{-H_L/T}}{Z_L} \right\} = \lim_{N\to\infty} \frac{\langle \Psi_0 | T(0|h)^{\otimes(m+1)} | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle \Lambda_0(0|h)^{(m+1)}}$$

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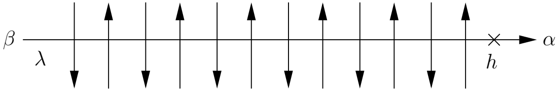

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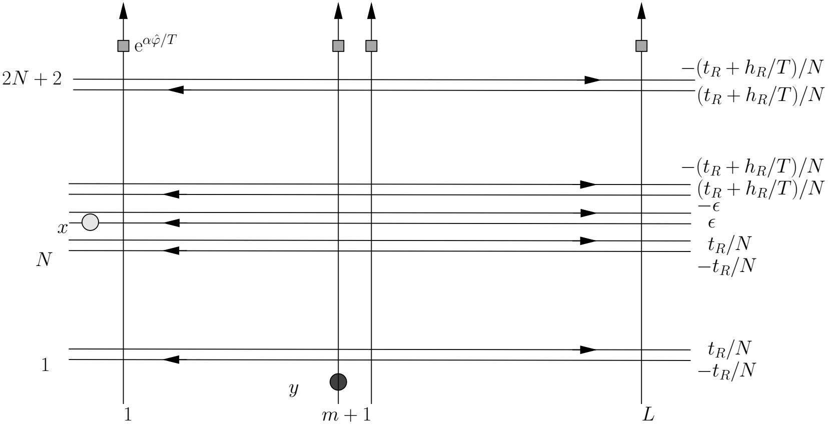
- Thermal form factor series [DGK13]

$$\langle x_1 y_{m+1}\rangle_{T,h} = \lim_{N\to+\infty}\frac{\langle\Psi_0|\mathrm{tr}\{xT(0|h)\}t(0|h)^{m-1}\,\mathrm{tr}\{yT(0|h)\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle\Lambda_0(0|h)^m}$$

$$= \lim_{N\to+\infty}\sum_n \frac{\langle\Psi_0|X(0|h)|\Psi_n\rangle\langle\Psi_n|Y(0|h)|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle\Lambda_0(0|h)\langle\Psi_n|\Psi_n\rangle\Lambda_n(0|h)}\left(\frac{\Lambda_n(0|h)}{\Lambda_0(0|h)}\right)^m$$

The dynamical case



Graphical representation of the unnormalized finite Trotter number approximant to the dynamical two-point function, h_R ‘energy scale’, $t_R = \mathrm{i}h_R t$

Result 1: Thermal form factor series for dynamical correlation functions

THEOREM: [GKKKS17] Consider any fundamental integrable lattice model with $U(1)$ symmetric R-matrix R, $[R(\lambda,\mu), \Theta(\alpha) \otimes \Theta(\alpha)] = 0$, and Hamiltonian

$$H_L = \sum_{j=1}^{L} \left[h_R \partial_\lambda (PR)_{j-1,j}(\lambda, 0) \big|_{\lambda=0} - \alpha \Theta_j'(0) \Theta_j^{-1}(0) \right]$$

with $h_R, \alpha \in \mathbb{C}$ appropriately. Let $\kappa = \alpha / T$

$$T_a(\lambda|\kappa) = \Theta_a(\kappa) R^{t_1}_{\overline{2N+2},a}(\nu_{2N+2}, \lambda) R_{a,\overline{2N+1}}(\lambda, \nu_{2N+1}) \dots R^{t_1}_{\overline{2},a}(\nu_2, \lambda) R_{a,\overline{1}}(\lambda, \nu_1)$$

the associated staggered and inhomogeneous monodromy matrix, $t(\lambda|\kappa) = \mathrm{tr}\{T(\lambda|\kappa)\}$ the inhomogeneous quantum transfer matrix of the system with eigenvalues $\Lambda_n(\lambda|\kappa)$ and eigenstates $|\Psi_n\rangle$, and $X(\lambda|\kappa) = \mathrm{tr}\{xT(\lambda|\kappa)\}$, $Y(\lambda|\kappa) = \mathrm{tr}\{yT(\lambda|\kappa)\}$.

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Then there exists a Trotter decomposition $\{\nu_j\}_{j=1}^{2N+2}$ such that

$$\langle x_1 y_{m+1}(t)\rangle_T = \lim_{N\to\infty}\lim_{\varepsilon\to 0} \mathrm{e}^{\mathrm{i}t\alpha s(x)} \sum_n \frac{\langle\Psi_0|X(\varepsilon|\kappa)|\Psi_n\rangle\langle\Psi_n|Y(\varepsilon|\kappa)|\Psi_0\rangle}{\Lambda_n(\varepsilon|\kappa)\langle\Psi_0|\Psi_0\rangle\Lambda_0(\varepsilon|\kappa)\langle\Psi_n|\Psi_n\rangle} \times \left(\frac{\Lambda_n\big(\frac{t_R}{N}\big|\kappa\big)\Lambda_0\big(-\frac{t_R}{N}\big|\kappa\big)}{\Lambda_0\big(\frac{t_R}{N}\big|\kappa\big)\Lambda_n\big(-\frac{t_R}{N}\big|\kappa\big)}\right)^{\frac{N}{2}} \left(\frac{\Lambda_n(0|\kappa)}{\Lambda_0(0|\kappa)}\right)^m$$

Here $t_R = \mathrm{i}h_R t$ and s is such that $\Theta(\alpha)x\Theta(-\alpha) = \mathrm{e}^{\alpha s}x$

Evaluation of the thermal form factor series

In oder to evaluate the thermal form factor series we need to

1. evaluate all non-vanishing amplitudes

$$A_n(\lambda|\kappa) = \frac{\langle \Psi_0 | X(\lambda|\kappa) | \Psi_n \rangle \langle \Psi_n | Y(\lambda|\kappa) | \Psi_0 \rangle}{\Lambda_n(\lambda|\kappa) \langle \Psi_0 | \Psi_0 \rangle \Lambda_0(\lambda|\kappa) \langle \Psi_n | \Psi_n \rangle}$$

2. evaluate all corresponding eigenvalue ratios

$$\rho_n(0|\kappa) = \frac{\Lambda_n(0|\kappa)}{\Lambda_0(0|\kappa)} = \mathrm{e}^{-\frac{1}{\xi_n} + \mathrm{i}\varphi_n}$$

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REMARKS:

1. Summation and Trotter limit do not commute
2. Setting $t = 0$ in the form factor series we recover the known form factor series of the static correlation functions [DGK 13]
3. In case of the longitudinal two-point functions of the XX chain we obtain the well-known double-integral representation directly from the thermal form factor series (TFFS)

Functions characterizing the XX chain

- One-particle momentum and energy

$$p(\lambda) = -\mathrm{i} \ln\bigl(-\mathrm{i}\,\mathrm{th}(\lambda)\bigr)\,, \quad \varepsilon(\lambda) = h + 2Jp'(\lambda)$$

- Fundamental strip, Fermi rapidities and momenta

$$\mathcal{S} = \left\{\lambda \in \mathbb{C} \middle| -\frac{\pi}{4} \le \mathrm{Im}\,\lambda < \frac{3\pi}{4}\right\}$$

The one-particle energy ε has precisely two roots

$$\lambda_F^\pm = \frac{\mathrm{i}\pi}{4} \pm z_F\,, \quad z_F = \frac{1}{2}\,\mathrm{arch}\left(\frac{4J}{h}\right)$$

in $\mathcal{S}$. These roots are called the Fermi rapidities. The value

$$p_F = p(\lambda_F^-) = \arccos\left(\frac{h}{4J}\right)$$

of the momentum function evaluated at the left Fermi rapidity is the Fermi momentum

Functions and contours characterizing the XX chain

- Square of a generalized Cauchy determinant

$$\mathcal{D}\big(\{x_j\}_{j=1}^{n_h}, \{y_k\}_{k=1}^{n_p}\big) = \frac{\big[\prod_{1\le j<k\le n_h} \mathrm{sh}^2(x_j - x_k)\big]\big[\prod_{1\le j<k\le n_p} \mathrm{sh}^2(y_j - y_k)\big]}{\prod_{j=1}^{n_h}\prod_{k=1}^{n_p} \mathrm{sh}^2(x_j - y_k)}$$

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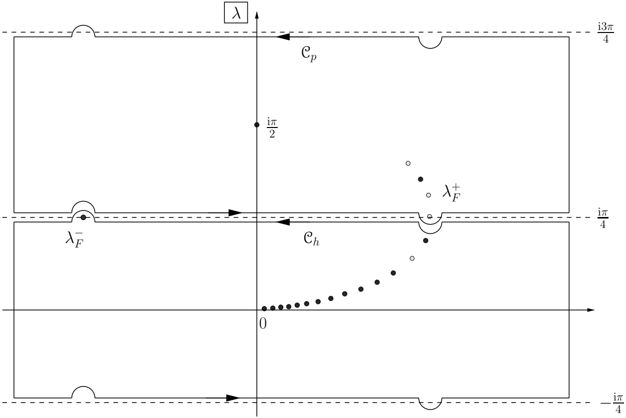
- Particle and hole contour $\mathcal{C}_p$ and $\mathcal{C}_h$



Functions characterizing the XX chain

- An auxiliary function

$$z(\lambda) = \frac{1}{2\pi i} \ln\left[\operatorname{cth}\left(\frac{\varepsilon(\lambda)}{2T}\right)\right]$$

- The functions Φ_p and Φ_h:

$$\Phi_h(x) = \frac{i p'(x)}{2} \exp\left\{ i \int_{\mathcal{C}_h} d\lambda\, p'(\lambda) z(\lambda) \frac{\operatorname{sh}(x+\lambda)}{\operatorname{sh}(x-\lambda)} \right\}$$

for all $x \in \mathcal{S} \setminus \mathcal{C}_h$, and

$$\Phi_p(x) = \frac{i p'(x)}{2} \exp\left\{ -i \int_{\mathcal{C}_p} d\lambda\, p'(\lambda) z(\lambda) \frac{\operatorname{sh}(x+\lambda)}{\operatorname{sh}(x-\lambda)} \right\}$$

for all $x \in \mathcal{S} \setminus \mathcal{C}_p$

- An amplitude

$$\mathcal{A}(T,h) = \exp\left\{ - \int_{\mathcal{C}'_h \subset \mathcal{C}_h} d\lambda\, z(\lambda) \int_{\mathcal{C}_h} d\mu\, z(\mu) \operatorname{cth}'(\lambda - \mu) \right\}$$

Result 2: TFFS for transverse dynamical two-point functions of the XX chain

- THEOREM: [GKKKS17] The transverse dynamical two-point functions of the XX chain have the series representation

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = (-1)^m \mathcal{F}(m) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n-1)!} \prod_{j=1}^{n} \int_{\mathcal{C}_h} \frac{\mathrm{d}x_j}{\pi \mathrm{i}} \frac{\Phi_p(x_j) \mathrm{e}^{\mathrm{i}(mp(x_j) - t\varepsilon(x_j))}}{1 - \mathrm{e}^{\varepsilon(x_j)/T}}$$
$$\times \prod_{k=1}^{n-1} \int_{\mathcal{C}_p} \frac{\mathrm{d}y_k}{\pi \mathrm{i}} \frac{\mathrm{e}^{-\mathrm{i}(mp(y_k) - t\varepsilon(y_k))}}{\Phi_h(y_k)\big(1 - \mathrm{e}^{-\varepsilon(y_k)/T}\big)} \mathcal{D}\big(\{x_j\}_{j=1}^n, \{y_k\}_{k=1}^{n-1}\big)$$

where

$$\mathcal{F}(m) = \mathrm{e}^{-\mathrm{i}mp_F} \mathcal{A}(T,h) \exp\left\{ -m \int_{\mathcal{C}_h} \frac{\mathrm{d}\lambda}{2\pi} p'(\lambda) \ln \left| \mathrm{cth}\left(\frac{\varepsilon(\lambda)}{2T} \right) \right| \right\}$$

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- This is now a series over classes of excitations

- It is just a function to be further studied. We claim that it is manifestly different from the series obtained by [IIKS 93] and rather appropriate for numerical and asymptotic analysis

Result 3: Spacelike asmyptotics for fixed T

- THEOREM: [GKS19] In the spacelike regime $m > 4Jt$ our TFFS for the transversal two-point function is absolutely convergent and determines the late-time long-distance asymptotics

$$t \to +\infty, \quad m \to +\infty \quad \text{at any fixed ratio } \alpha = \frac{m}{4Jt} > 1$$

of the transverse dynamical correlation function of the XX chain

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = C(T,h)(-1)^m \exp\left\{ -m \int_{\mathcal{C}} \frac{d\lambda}{2\pi} p'(\lambda) \ln \left| \operatorname{cth}\left(\frac{\varepsilon(\lambda)}{2T} \right) \right| \right\} \times \left(1 + \mathcal{O}(t^{-\infty})\right)$$

where

$$C(T,h) = \frac{2T\mathcal{A}(T,h)\Phi_p(\lambda_F^-)}{\varepsilon'(\lambda_F^-)}$$

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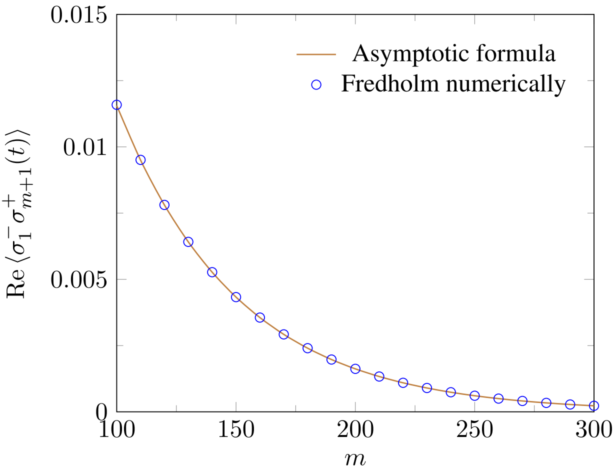
REMARKS:

1. The ‘constant’ $C(T,h)$ was heretofore unknown
2. Obtained by direct asymptotic analysis of the series

Comparison with numerical result



Real part of $\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle$ as a function of m for $T/J = 0.05$, $h/J = 0.1$ and $Jt = 10$ evaluated numerically from the Fredholm determinant representation (dots) and from the asymptotic formula

Functions for Fredholm determinant

- Define the following functions

$$\varphi(x,y) = \frac{\mathrm{e}^{y-x}}{\mathrm{sh}(y-x)}$$

$$V_h(x) = \frac{\Phi_p(x)\mathrm{e}^{\mathrm{i}(mp(x)-t\varepsilon(x))}}{\pi\mathrm{i}\bigl(\mathrm{e}^{\varepsilon(x)/T}-1\bigr)}\,, \qquad V_p(y) = \frac{\mathrm{e}^{-\mathrm{i}(mp(y)-t\varepsilon(y))}}{\pi\mathrm{i}\Phi_h(y)\bigl(1-\mathrm{e}^{-\varepsilon(y)/T}\bigr)}$$

$$\Omega = \int_{\mathcal{C}_h} \mathrm{d}x\, V_h(x)$$

$$E_h(x) = \int_{\mathcal{C}_h} \mathrm{d}y\, V_h(y)\varphi(y,x)$$

$$V(x,y) = \int_{\mathcal{C}_h} \mathrm{d}z\, V_h(z)\varphi(z,x)\varphi(z,y)$$

- And the integral operators $\widehat{V}$ and $\widehat{P}$ with respect to the contour $\mathcal{C}_p$

$$\widehat{V}f(x) = \int_{\mathcal{C}_p} \mathrm{d}y\, V_p(y)V(x,y)f(y)\,,$$

$$\widehat{P}f(x) = \frac{E_h(x)}{\Omega}\int_{\mathcal{C}_p} \mathrm{d}y\, V_p(y)E_h(y)f(y)\,.$$

Result 4: A Fredholm determinant representation

- THEOREM: The transversal correlation functions of the XX chain admit the Fredholm determinant representation

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = (-1)^m \mathcal{F}(m) \Omega(m,t) \det_{\mathcal{C}_p} \bigl(\mathrm{id} + \widehat{V} - \widehat{P}\bigr)$$

Here $\widehat{V}$ is an integrable integral operator

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  1. The Fredholm determinant representation is suitable for a numerical analysis [GKSS19]

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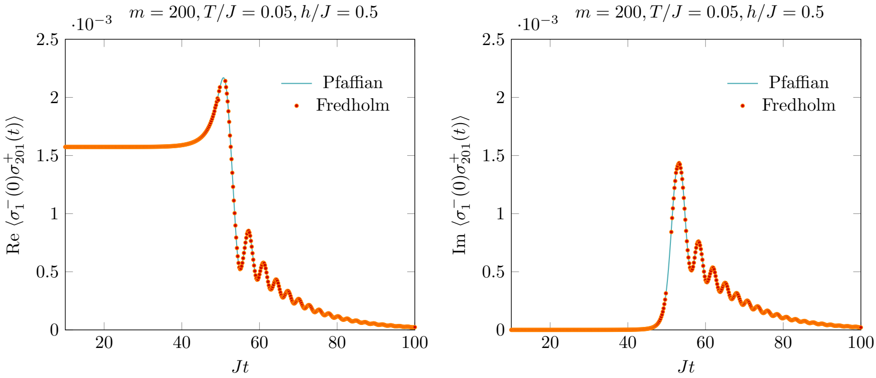
- REMARKS:
  1. The Fredholm determinant representation is suitable for a numerical analysis [GKSS19]
  2. Comparing with the asymptotic behaviour of the correlation function in the spacelike regime $m > 4Jt$ we see that

$$\det_{\mathcal{C}_p}\bigl(\mathrm{id} + \widehat{V} - \widehat{P}\bigr) \sim 1 + \mathcal{O}(t^{-\infty})$$

     meaning that the Fredholm determinant collects the higher-order corrections to the main asymptotics

Numerical evaluation of Fredholm determinant



Real part (left) and imaginary part (right) of $\langle\sigma_1^-\sigma_{201}^+(t)\rangle$. Comparison between standard numerical method (Pfaffian method) and numerical evaluation of the Fredholm determinant (dots)

Comparison with IIKS 93

- Its, Izergin, Korepin and Slavnov in 1993 obtained a different Fredholm determinant representation

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = (-1)^m \Big[\det_{\mathcal{C}} \big(\mathrm{id} + \widehat{W} + \widehat{Q} \big) - \det_{\mathcal{C}} \big(\mathrm{id} + \widehat{W} \big) \Big]$$

where $\widehat{W}$ is an integrable operator acting on functions on
$\mathcal{C} = [-\infty - \mathrm{i}\pi/4, +\infty - \mathrm{i}\pi/4] \cup [+\infty + \mathrm{i}\pi/4, -\infty + \mathrm{i}\pi/4]$ with kernel

$$W(\lambda, \mu) = \frac{\mathrm{ch}(\lambda) H(\lambda) - \mathrm{ch}(\mu) H(\mu)}{\mathrm{sh}(\lambda - \mu)} \frac{\mathrm{e}^{g(\mu)}}{\pi (1 + \mathrm{e}^{\varepsilon(\mu)/T})}$$

$$H(\lambda) = \mathrm{v.p.} \int_{\mathcal{C}} \frac{\mathrm{d}\mu}{\pi} \frac{\mathrm{e}^{-g(\mu)}}{\mathrm{ch}(\mu) \, \mathrm{sh}(\mu - \lambda)}$$

and $\widehat{Q}$ is a 1d projector acting as

$$\widehat{Q} \varphi(\lambda) = \frac{1}{\mathrm{ch}(\lambda)} \int_{\mathcal{C}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \frac{\mathrm{e}^{g(\mu)} \varphi(\mu)}{\mathrm{sh}(\mu)(1 + \mathrm{e}^{\varepsilon(\mu)/T})}$$

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- We interpret our Fred-det-rep as a ‘resummation’ a la Borodin-Okounkov that is more suitable for the long-time, large-distance asymptotic analysis

Result 5: Hight-T analysis by means of matrix Riemann-Hilbert problem

- THEOREM: [GKS19, arXiv:1905.04922] In the high-T limit the transverse dynamical correlation function of the XX-chain behaves as

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = \frac{1}{2}\left(-\frac{J}{T}\right)^m \exp\left\{ \mathrm{i}c\tau - \frac{\tau^2}{4} + \int_0^\tau \mathrm{d}\tau' \, u_m(\tau') \right\}$$

$$\times \frac{Q_{m+1}(-\mathrm{i})P'_m(-\mathrm{i}) - P_m(-\mathrm{i})Q'_{m+1}(-\mathrm{i})}{\left(Q_{m+1}(-\mathrm{i})P'_m(-\mathrm{i}) - P_m(-\mathrm{i})Q'_{m+1}(-\mathrm{i})\right)\big|_{\tau=0}} \left(1 + \mathcal{O}(T^{-2})\right)$$

where $\tau = -4J\left(t - \frac{\mathrm{i}}{2T}\right)$, $c = \frac{h}{4J}$

$$u_m(\tau) = \frac{\mathrm{i}}{2}\left[c_m^{(m-1)} - \gamma_m \left\{ F_m(0)\left(P_m(0) - Q'_{m+1}(0)\right) + G_{m+1}(0)P'_m(0) \right\} \right]$$

and were P_m and Q_{m+1} are polynomical with time-dependent coefficients obaying fully explixit linear equations

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$$u_m(\tau) = \frac{\mathrm{i}}{2}\left[c_m^{(m-1)} - \gamma_m\left\{F_m(0)(P_m(0) - Q_{m+1}'(0)) + G_{m+1}(0)P_m'(0)\right\}\right]$$

and were P_m and Q_{m+1} are polynomical with time-dependent coefficients obaying fully explixit linear equations

- COROLLARY: (Generalizing Brandt & Jacoby 1976) In the high-T limit the transverse auto-correlation function of the XX-chain behaves as

$$\langle \sigma_1^- \sigma_1^+(t) \rangle_T = \frac{1}{2} \mathrm{e}^{-\mathrm{i}h(t-\mathrm{i}/(2T)) - 4J^2(t-\mathrm{i}/(2T))^2} \left(1 + \mathcal{O}(T^{-2})\right)$$

Construction of the polynomials

- $$P_m(z) = z^m + \sum_{k=0}^{m-1} c_m^{(k)} z^k \,, \quad Q_{m+1}(z) = z^{m+1} + \sum_{k=0}^{m} d_{m+1}^{(k)} z^k$$

  where the coefficients satisfy the linear equations

  $$\sum_{k=0}^{m-1} A_k^l(m) c_m^{(k)} = -A_m^l(m) \,, \qquad \sum_{k=0}^{m} A_k^n(m) d_{m+1}^{(k)} = -A_{m+1}^n(m)$$

  $l = 0, \dots, m-1$; $n = 0, \dots, m$.

- Introducing the column vectors

  $$\mathbf{C}_i = (A_k^0(m), \dots, A_k^{m-1}(m))^t \,, \quad \mathbf{D}_j = (A_l^0(m), \dots, A_l^m(m))^t$$

  for $i = 0, \dots, m$, $j = 0, \dots, m+1$, we obtain by Cramer's rule

  $$c_m^{(n)} = -\frac{\det(\mathbf{C}_0, \dots, \mathbf{C}_{n-1}, \mathbf{C}_m, \mathbf{C}_{n+1}, \dots, \mathbf{C}_{m-1})}{\det_{i,j=0,\dots,m-1}\bigl(A_j^i(m)\bigr)} \,,$$

  $$d_{m+1}^{(n)} = -\frac{\det(\mathbf{D}_0, \dots, \mathbf{D}_{n-1}, \mathbf{D}_{m+1}, \mathbf{D}_{n+1}, \dots, \mathbf{D}_m)}{\det_{i,j=0,\dots,m}\bigl(A_j^i(m)\bigr)}$$

Construction of the polynomials

- The $A_k^l(m)$ are finite sums of modified Bessel functions $\mathrm{I}_n(x)$:

$$A_k^l(m) = -\frac{\mathrm{i}^{m-k-l}\mathrm{e}^{\mathrm{i}c\tau}}{c}\Big\{\mathrm{I}_{m-k-l-1}(-\tau) + \big(1+\mathrm{i}c(m-k-l-1)\big)\Big[\mathrm{e}^{-\tau} - \sum_{n=k+l+1-m}^{m-k-l-1}\mathrm{I}_n(-\tau)\Big] + \mathrm{i}c\tau\big(\mathrm{I}_{m-k-l-1}(-\tau)+\mathrm{I}_{m-k-l}(-\tau)\big)\Big\}$$

for $m-k-l-1 \geq 0$, and

$$A_k^l(m) = -\frac{\mathrm{i}^{m-k-l}\mathrm{e}^{\mathrm{i}c\tau}}{c}\Big\{\mathrm{I}_{m-k-l-1}(-\tau) + \big(1+\mathrm{i}c(m-k-l-1)\big)\Big[\mathrm{e}^{-\tau} + \sum_{n=m-k-l}^{k+l-m}\mathrm{I}_n(-\tau)\Big] + \mathrm{i}c\tau\big(\mathrm{I}_{m-k-l-1}(-\tau)+\mathrm{I}_{m-k-l}(-\tau)\big)\Big\}$$

for $m-k-l-1 < 0$

Construction of the function u_m

- For the computation of u_m one can use the formulae

$$\gamma_m^{-1} = \sum_{k=0}^{m} A_k^m(m) c_m^{(k)}$$

$$F_m(0) = \sum_{k=0}^{m} A_k^0(m+1) c_m^{(k)}, \quad G_{m+1}(0) = \sum_{k=0}^{m+1} A_k^0(m+1) d_{m+1}^{(k)}$$

$$P_m(0) = c_m^{(0)}, \quad P'_m(0) = c_m^{(1)}, \quad Q'_{m+1}(0) = d_{m+1}^{(1)}$$

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$$P_m(0) = c_m^{(0)}, \quad P'_m(0) = c_m^{(1)}, \quad Q'_{m+1}(0) = d_{m+1}^{(1)}$$

- An explicit example beyond Perk & Capel 1977

$$\left\langle \sigma_1^- \sigma_4^+(t) \right\rangle_{T,h=0} \sim 16 \left(\frac{J}{T} \right)^3 \exp\left\{ -4J^2 t^2 \right\} \frac{\mathrm{I}_1(-4Jt)}{(4Jt)^5}$$
$$\times \left\{ -(4Jt)^2 \, \mathrm{I}_0(-4Jt)^2 - 4Jt \, \mathrm{I}_0(-4Jt) \, \mathrm{I}_1(-4Jt) + \left((4Jt)^2 + 2 \right) \mathrm{I}_1(-4Jt)^2 \right\}$$

Summary and outlook

1. We have devised a thermal form factor series expansion for dynamical correlation functions of fundamental integrable lattice models
2. We have applied it to the transversal two-point function of the XX chain
3. The resulting series can be directly used for the long-time, large-distance asymptotic analysis of the correlation function at fixed T and in the ‘spacelike regime’
4. The series can be summed up into a Fredholm determinant representation pertaining to an integrable integral operator
5. The latter is efficient for a numerical computation of the correlation function
6. The correlation functions can be expressed in terms of the solution of a matrix Riemann-Hilbert problem associated with the integrable operator
7. This matrix Riemann-Hilbert problem has been used for the high-temperature asymptotic analysis at any fixed space-time point, extending classical work of Brandt & Jacoby 76, Perk & Capel 77

Future work: Extend all of this to XXZ!