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The q-deformed Haldane–Shastry model

from the affine Hecke algebra, via 'freezing', to exact eigenvectors

Jules Lamers

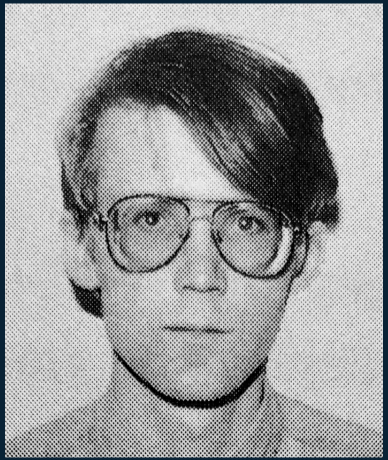
[1801.05728]

ongoing work with

V. Pasquier and **D. Serban**

building on **D. Uglov**

[hep-th/950814]



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Outline

- Recap of the **Haldane–Shastry model**
- ***q*-deformed** Haldane–Shastry model
  - Hamiltonian
  - Key properties
  - Exact highest-weight vectors
  - Overview of underlying algebraic structure
- Hamiltonian (Macdonald operator) of spin-Ruijsenaars

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Haldane–Shastry model

[Haldane '88]
[Shastry '88]

spin-1/2 chain, L sites

Hilbert space: $(\mathbb{C}^2)^{\otimes L}$

$1 - P_{ij} = \frac{1}{2}\,(1 - \vec{\sigma}_i \cdot \vec{\sigma}_j)$

exchange interactions

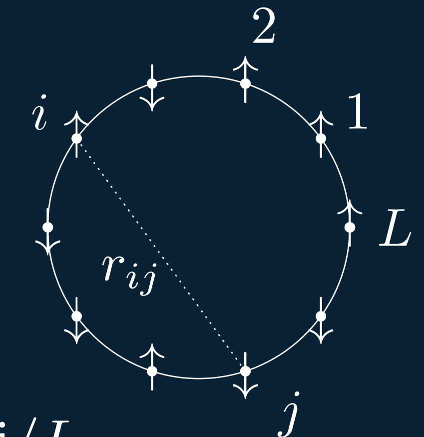
$$H_{\mathrm{HS}} = \sum_{i<j}^{L} \frac{1}{r_{ij}^2}\,(1 - P_{ij})$$

long-range
pairwise interactions

pair **potential**

$$\frac{1}{r_{ij}^2} = \mathrm{ev}_\omega \frac{z_i\, z_j}{(z_i - z_j)^2}$$

$$\mathrm{ev}_\omega : z_j \mapsto \omega^j, \ \omega = \mathrm{e}^{2\pi \mathrm{i}/L}$$



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$H_{\mathrm{HS}} = \sum_{i<j}^{L} \frac{1}{r_{ij}^2} (P_{ij} - 1)$'s key properties

- **Infinite**-dim **symmetry** algebra already **for** $L < \infty$: isotropy $\mathfrak{sl}_2 \subset Y(\mathfrak{sl}_2)$ **Yangian** invariance

- **Exact** highest-weight **wave functions**:

$$\Psi_\lambda(i_1, \cdots, i_M) = \mathrm{ev}_\omega \, \widetilde{\Psi}_\lambda(z_{i_1}, \cdots, z_{i_M}) \qquad \begin{array}{l} \mathrm{ev}_\omega : \, z_j \mapsto \omega^j \\ \omega = \mathrm{e}^{2\pi \mathrm{i}/L} \end{array}$$

$$\widetilde{\Psi}_\lambda(\vec{z}) = \underbrace{\prod_{m<n}^{M} (z_m - z_n)^2}_{\text{Vandermonde}^2} \underbrace{J_\lambda^{(\alpha=2)}(z_1, \cdots, z_M)}_{\text{Jack (zonal)}}$$

$L - 2M + 1 \geq \lambda_1 \geq \cdots \geq \lambda_M \geq 1$

- Free gas of anyons

Underlying structure

[Bernard *et al* ’93]
[Talstra Haldane ’95]

algebra
degenerate affine Hecke

$((\mathbb{C}^2)^{\otimes L} \otimes \mathbb{C}[z_1, \cdots, z_L])^{S_L}$

$\mathbb{C}[z_1, \dots, z_M]^{S_M}$

generalised model
spin-Sutherland
- Yangian invariance

‘freezing’ $\alpha \to \infty \ (\hbar \to 0)$
$\mathrm{ev}_\omega : z_j \mapsto \omega^j, \ \omega = \mathrm{e}^{2\pi \mathrm{i}/L}$

spin chain
Haldane–Shastry
- Yangian invariance
- exact h.-w. vectors

$\alpha = 2$

quantum many-body system
(trig) Sutherland
- exact eigenfunctions:
Jack $J_\lambda^{(\alpha)}(z_1, \cdots, z_M)$

q-deformed underlying structure

[Bernard *et al* '93]
[Uglov '95]
[JL Pasquier Serban]

algebra

affine Hecke

$q = \mathsf{t}_{\mathrm{Macd}}^{1/2}$ $p = \mathsf{q}_{\mathrm{Macd}}$

$\big((\mathbb{C}^2)^{\otimes L} \otimes \mathbb{C}[z_1, \cdots, z_L]\big)^{S_L}$

$\mathbb{C}[z_1, \dots, z_M]^{S_M}$

generalised model

spin-Ruijsenaars

- quantum-affine invariance

'freezing' $p \to 1 \ (\hbar \to 0)$
$\mathrm{ev}_\omega : z_j \mapsto \omega^j, \ \omega = \mathrm{e}^{2\pi \mathrm{i}/L}$

spin chain

q-Haldane–Shastry

- quantum-affine invariance
- exact h.-w. vectors

$q' = p' = q^2$
$\mathsf{t}' = (\mathsf{q}')^2$

quantum many-body system

(trig) Ruijsenaars

- exact eigenfunctions: Macdonald $P_\lambda(z_1, \cdots, z_M)$

q-deformed (chiral) Hamiltonian

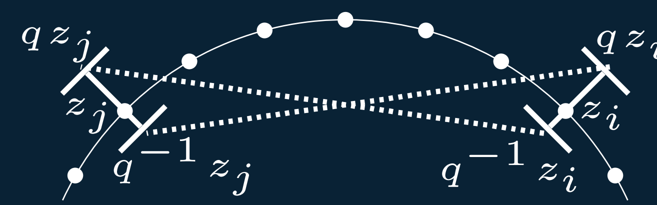
[Uglov '95]
[JL '18]

spin-1/2 chain, L sites
Hilbert space: $(\mathbb{C}^2)^{\otimes L}$

q-deformed
exchange interactions

$$H = \frac{[L]_q}{L} \sum_{i<j}^{L} \mathrm{ev}_\omega\, V(z_i, z_j)\, S_{[ij]}$$

long-range
pairwise form



point-split potential

$$V(z_i, z_j) = \frac{z_i\, z_j}{(q\, z_i - q^{-1} z_j)(q^{-1} z_i - q\, z_j)}$$

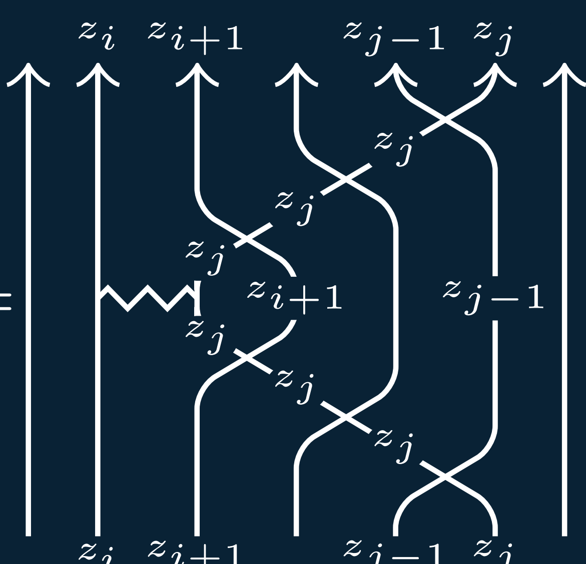
$$\mathrm{ev}_\omega :\ z_j \mapsto \omega^j,\ \ \omega = \mathrm{e}^{2\pi\mathrm{i}/L}$$

q-deformed long-range interactions

The long-range exchange

$$1 - P_{ij} = \underbrace{P_{j,j-1} \cdots P_{i+1,i+2}}_{\text{transport}} \underbrace{(1 - P_{i,i+1})}_{\text{nearest-neighbour exchange}} \underbrace{P_{i+1,i+2} \cdots P_{j-1,j}}_{\text{transport}}$$

is deformed to

$S_{[i,j]} =$ 

 $= \check{R}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(q-q^{-1})u}{q\,u-q^{-1}} & \frac{u-1}{q\,u-q^{-1}} & 0 \\ 0 & \frac{u-1}{q\,u-q^{-1}} & \frac{q-q^{-1}}{q\,u-q^{-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

 $= (q^{-1} - q)\, \check{R}'(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

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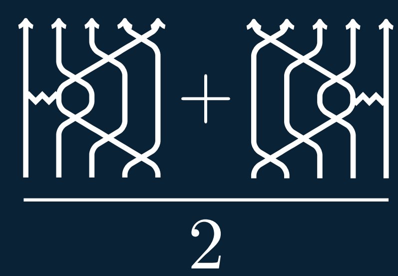
Key properties

- **Quantum-affine symmetry** at finite size [Bernard *et al* ’93]
- **Multi-spin interactions** [Uglov ’95]
- **Spectrum** is **real** if $q \in \mathbb{R}$ [JL ’18]
  for **‘full’** Hamiltonian also if $q \in S^1$ [JL Pasquier Serban]

$$H^{\mathrm{full}} = \frac{[L]_q}{L} \sum_{i<j}^{L} \mathrm{ev}_\omega \, V(z_i, z_j)$$



- **Exact** highest-weight **wave functions** [JL Pasquier Serban]

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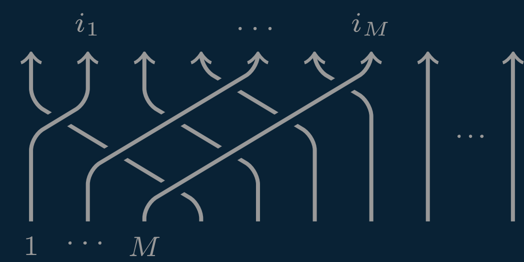
Exact highest-weight wave functions

[JL Pasquier Serban]



$$\mathrm{ev}_\omega : z_j \mapsto \omega^j$$

$$\omega = \mathrm{e}^{2\pi \mathrm{i}/L}$$

Hecke $T_i = -\dfrac{q\, z_i - q^{-1} z_{i+1}}{z_i - z_{i+1}} (1 - s_i) + q$

$$\Psi_\lambda(i_1, \cdots, i_M) = \mathrm{ev}_\omega \, T_{[i_1,\cdots,i_M]} \, \widetilde{\Psi}_\lambda(z_1, \cdots, z_M)$$

$$\widetilde{\Psi}_\lambda(\vec{z}) = \underbrace{\prod_{m<n}^{M} (q\, z_m - q^{-1} z_n)(q^{-1} z_m - q\, z_n)}_{\text{symmetric square of } q\text{-Vandermonde}} \; \underbrace{P_\lambda(\vec{z})}_{\substack{\text{Macdonald} \\ (q\text{-zonal})}}$$

$$L - 2M + 1 \geq \lambda_1 \geq \cdots \geq \lambda_M \geq 1$$

$q' = p' = q^2$

$\mathsf{t}' = (\mathsf{q}')^2$

Spin-Ruijsenaars model

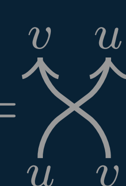
Physical space: $\big((\mathbb{C}^2)^{\otimes L} \otimes \mathbb{C}[z_1, \cdots, z_L]\big)^{S_L}$ [Bernard *et al* '93]

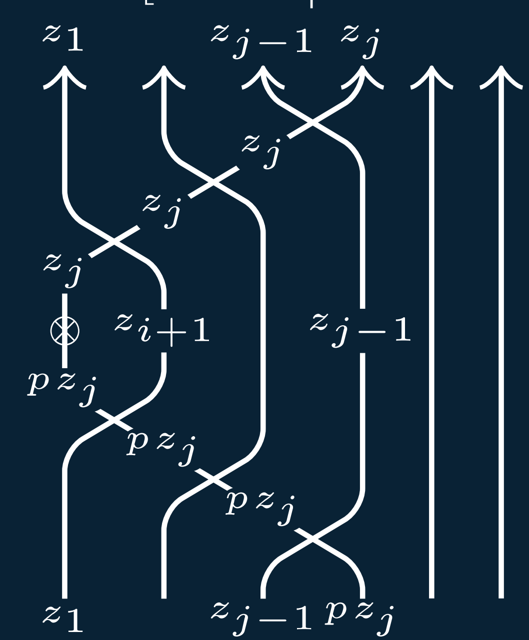
$s_i\, \check{R}_{i,i+1}(z_i/z_{i+1})$

[JL Pasquier Serban]

Spin-Macdonald:

$$\widetilde{H} = Y_1 + \cdots + Y_L = \sum_{j=1}^{L} \prod_{k(\neq j)}^{L} \frac{q\, z_j - q^{-1} z_k}{z_j - z_k}$$

$\check{R}(u/v) =$ 



Monodromy matrix:

$$\widetilde{T}_0(u) = P_{0L}\, \check{R}_{0L}(uY_L) \cdots P_{01}\, \check{R}_{01}(uY_1)$$ [Bernard *et al* '93]

Conclusion
Summary

- **Haldane–Shastry** model can be ***q*-deformed**
  - guiding property:
    quantum-affine symmetry already at finite length
  - **pairwise form**, accounting for **multi-spin** interactions
  - from **explicit spin-Macdonald–Ruijsenaars** operators
- **Exact spectrum** at finite size
  - **highest-weight vectors** feature **Macdonalds**

$$q' = p' = q^2$$
$$\mathrm{t}' = (\mathrm{q}')^2$$

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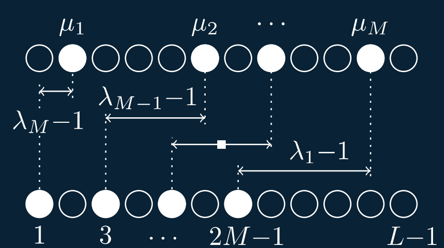
Conclusion
Future directions

- More detail
  - explicit (level $c = 0$) quantum-affine action
  - (conformal?) field theory describing low energy
  - q root of unity
- Generalizations
  - higher rank, super case, type BC, etc
  - q-deformed Inozemtsev spin chain?
  - XYZ-like version??

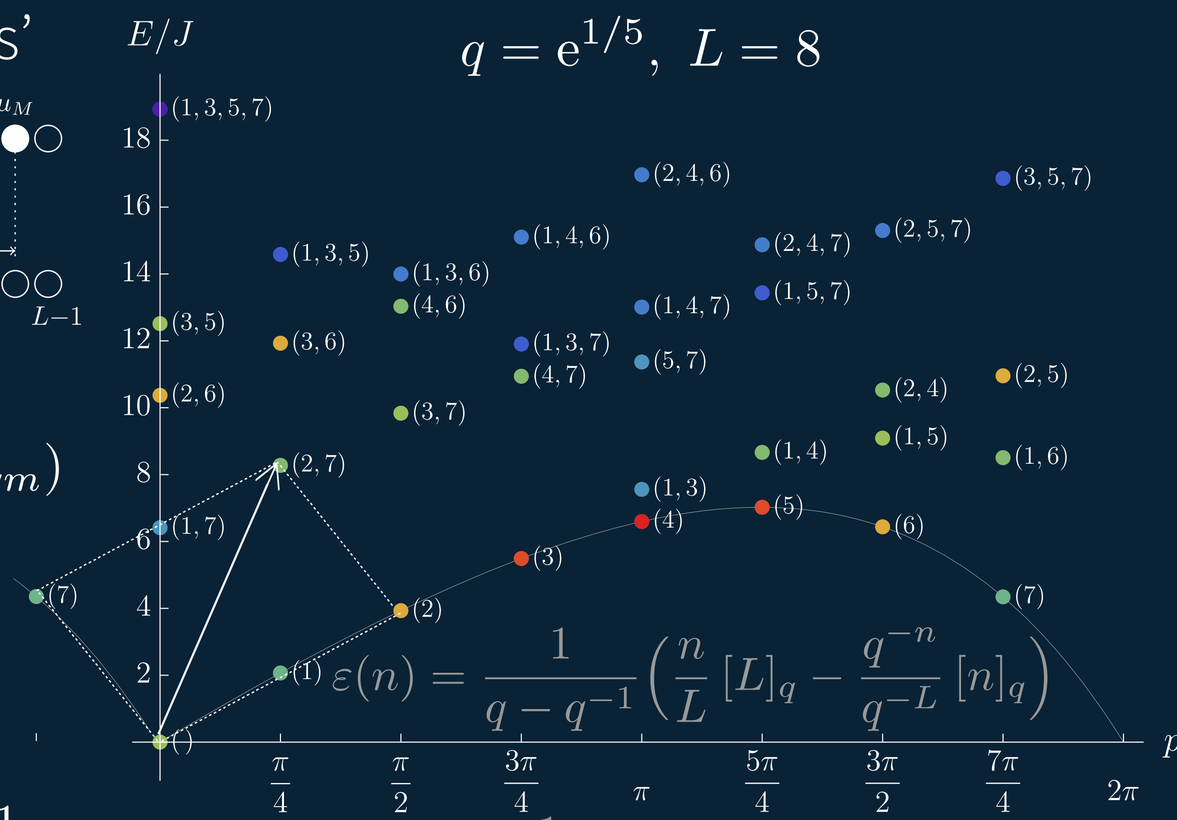
Bonus material
Spectrum at finite length

pass to 'motifs'



$$E(\mu) = \sum_{m=1}^{M} \varepsilon(\mu_m)$$

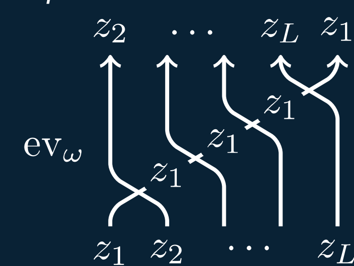
addition is allowed iff $\mu_{m+1} > \mu_m + 1$



$$\varepsilon(n) = \frac{1}{q-q^{-1}}\left(\frac{n}{L}\,[L]_q - \frac{q^{-n}}{q^{-L}}\,[n]_q\right)$$

$$\varepsilon^{\text{full}}(n) = \frac{1}{2}\,[n]_q\,[L-n]_q$$

q-translation



q-momentum

$$K(\mu) = \frac{2\pi}{L}\sum_{m=1}^{M} \mu_m \mod 2\pi$$

Bonus material

Exact highest-weight vectors revisited

In the M-particle sector: via **XXZ Bethe vectors**

[JL Pasquier Serban]

$$|\Psi_\lambda\rangle = \mathrm{cst} \times \sum_{i_1<\cdots<i_M}^{L} \mathrm{ev}_\omega \widetilde{\Phi}_\lambda(z_{i_1}, \cdots, z_{i_M}) \, B(z_{i_1};\vec{z}) \cdots B(z_{i_M};\vec{z}) \, |\uparrow \cdots \uparrow\rangle$$

$$\widetilde{\Phi}_\lambda(z_1, \cdots, z_M) = \underbrace{\prod_{m<n}^{M} (z_m - z_n)^2}_{\text{as for HS}} \, \underbrace{P_\lambda(z_1, \cdots, z_M)}_{\substack{\text{Macdonald} \\ (q\text{-zonal})}}$$

$q' = p' = q^2$
$\mathrm{t}' = (\mathrm{q}')^2$

$L - 2M + 1 \geq \lambda_1 \geq \cdots \geq \lambda_M \geq 1$