Unitary matrix model, supersymmetric gauge theory, and Painlevé system

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1. Introduction

Since the discovery of the AGT correspondence [Alday-Gaiotto-Tachikawa, '09], the matrix model/gauge theory correspondence is refined and is collected new interests. A certain class of β-deformed ensembles for matrix models has been serving as integral representations of 2d (regular and irregular) conformal blocks. In accordance with the AGT correspondence, they are related to the Nekrasov (instanton) partition functions of the 4d $\mathcal{N}=2$ supersymmetric gauge theories in the omega background.

The partition functions of the β-deformed 'matrix models' depending on two integration contours C_{L} and C_{R} which directly generate the Nekrasov instanton partition function for 4d $\mathcal{N}=2$ $SU(2)$ gauge theory with N_f fundamental matters can be generically presented as

$$\begin{aligned} & Z^{(N_f)}(N_{\mathrm{L}}, N_{\mathrm{R}}) \\ & = \mathcal{N}_{(N_f)} \left(\prod_{I=1}^{N} \int_{C_I^{(N_f)}} \mathrm{d}w_I \right) \Delta(w)^{2\beta} \exp\left(\sqrt{\beta} \sum_{I=1}^{N} W^{(N_f)}(w_I) \right). \end{aligned}$$

Here, $\mathcal{N}_{(N_f)}$ is a normalization factor, $\Delta(w) = \prod_{I<J}(w_I - w_J)$ the Vandermonde determinant, $W^{(N_f)}(w)$ the potential. The integration contours are $C_I^{(N_f)} = C_{\mathrm{L}}$ for $1 \leq I \leq N_{\mathrm{L}}$ and $C_I^{(N_f)} = C_{\mathrm{R}}$ for $N_{\mathrm{L}} < I \leq N$ with $N = N_{\mathrm{L}} + N_{\mathrm{R}}$.

For example, for $N_f = 4$, the potential $W^{(N_f=4)}(w)$ is given by the three-logarithmic potential

$$W^{(4)}(w) = \alpha_1 \log(w) + \alpha_2 \log(w - q_0) + \alpha_3 \log(w - 1),$$

and the contours can be chosen as finite intervals: $C_{\rm L} = [0, q_0]$, $C_{\rm R} = [1, \infty]$. The parameter dictionary between the matrix model and the gauge theory:

$$\sqrt{\beta} N_{\rm L} = \mathfrak{a} - m_2, \qquad \sqrt{\beta} N_{\rm R} = -(\mathfrak{a} + m_1), \qquad q_0 = \mathrm{e}^{\mathrm{i}\pi\tau_0},$$

$$\alpha_1 = m_2 - m_4 + \epsilon, \qquad \alpha_2 = m_2 + m_4, \qquad \alpha_3 = m_1 + m_3,$$

$$\epsilon = \epsilon_1 + \epsilon_2, \qquad \epsilon_1 = \sqrt{\beta}, \qquad \epsilon_2 = -1/\sqrt{\beta}.$$

$\mathfrak{a}$: the vacuum expectation value of the Higgs scalar,

m_i: masses, $\tau_0 = (\theta_0/\pi) + 8\pi\mathrm{i}/g_0^2$: the (complexified) UV gauge coupling, $\epsilon_{1,2}$: the omega background parameters

Following [Mironov-Morozov, '17], we introduce their generating function

$$\begin{aligned} &\underline{Z}^{(N_f)}(N; \mu_{\rm L}, \mu_{\rm R}) \\ &= \sum_{N_{\rm L}+N_{\rm R}=N} \frac{\mu_L^{N_L}}{N_L!} \frac{\mu_R^{N_{\rm R}}}{N_{\rm R}!} Z^{(N_f)}(N_{\rm L}, N_{\rm R}) \\ &= \frac{\mathcal{N}_{(N_f)}}{N!} \int_C \mathrm{d}^N w \, \Delta^{2\beta}(w) \, \exp\left(\sqrt{\beta} \sum_{I=1}^{N} W(w_I) \right), \end{aligned}$$

where $C = \mu_{\rm L} C_{\rm L} + \mu_{\rm R} C_{\rm R}$. This is an analog of the dual Nekrasov function. We expect that at $\beta = 1$ (the self-dual omega background $\epsilon = \epsilon_1 + \epsilon_2 = 0$), this is a "tau-function"-like object. In particular, for $N_f = 2$, $\beta = 1$, we can see that this generating function is related to a unitary matrix model and is indeed a tau-function of the Painlevé III' system.

Plan of this talk

1. Introduction
2. Irregular limit to $N_f = 2$ case and unitary matrix model
3. generalized GWW model
4. Summary

2. Irregular limit to $N_f = 2$ case and unitary matrix model

Irregular limits from $N_f = 4$ to $N_f = 2$

- From $N_f = 4$ to $N_f = 3$: $m_4 \to \infty$ with $\Lambda_3 := 4 q_0 m_4$ fixed.
- From $N_f = 3$ to $N_f = 2$: $m_3 \to \infty$ with $\Lambda_2^2 := \Lambda_3 m_3$ fixed.

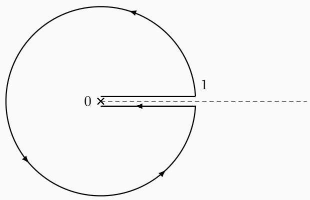
With appropriate deformation of the contours and normalizations, the generating function for $N_f = 2$ is given by

$$\underline{Z}^{(2)}(N; \mu_{\mathrm{L}}, \mu_{\mathrm{R}}) = \frac{1}{N!} \int_C \mathrm{d}^N w \, \Delta(w)^{2\beta} \exp\left(\sqrt{\beta} \sum_{I=1}^{N} W^{(2)}(w_I) \right),$$

where $C = \mu_{\mathrm{L}} C_0 + \mu_{\mathrm{R}} C_\infty$, and the potential is

$$W^{(2)}(w) = -\frac{\Lambda_2}{2} \left(w + \frac{1}{w} \right) + (2m_2 + \epsilon) \log w.$$

The integration contours C_0 (left) and C_∞ (right)



Comment: The cut runs along the positive real axis of the w-plane . When $\beta = 1$ and $2m_2$ is an integer, the cut vanishes. In this case, the integration over C_0 or C_∞ can be reduced to the integral over the unit circle. Furthemore,

$$\underline{Z}^{(2)}(N; \mu_{\mathrm{L}}, \mu_{\mathrm{R}}) = \frac{(\mu_{\mathrm{L}} + \mu_{\mathrm{R}})^N}{N!} Z_{U(N)}(M).$$

$Z_{U(N)}$: the partition function of the unitary matrix model

$$Z_{U(N)}(M) = \frac{1}{\mathrm{vol}(U(N))} \int [\mathrm{d}U] \exp\left(\mathrm{Tr}\, W(U)\right),$$

$$W(z) = -\frac{1}{2g}\left(z + \frac{1}{z}\right) + M \log z, \qquad (M \in \mathbb{Z}),$$

$[\mathrm{d}U]$: the Haar measure for the unitary matrix $U \in U(N)$,

$$\frac{1}{g} = \Lambda_2, \qquad N = -(m_1 + m_2), \qquad M = -(m_1 - m_2).$$

Instead of $\underline{Z}^{(2)}(N; \mu_{\mathrm{L}}, \mu_{\mathrm{R}})$, we will examine $Z_{U(N)}(M)$.

3. generalized GWW model

- We consider the following unitary matrix model with logarithmic potential: $U \in U(N)$,

$$Z_{U(N)}(M) := \frac{1}{\mathrm{vol}(U(N))} \int [\mathrm{d}U] \exp\Big(\mathrm{Tr}\, W(U)\Big),$$

$$W(z) = -\frac{1}{2g}\left(z + \frac{1}{z}\right) + M \log z, \qquad (M \in \mathbb{Z}).$$

Relation with the parameters of $\mathcal{N} = 2$ $N_f = 2$ $SU(2)$ gauge theory

$$\frac{1}{g} = \Lambda_2, \qquad N = -(m_1 + m_2), \qquad M = (m_2 - m_1).$$

Λ_2: dynamical scale, m_i :masses of hypermultiplets

- When there is no logarithmic term ($M = 0$), this unitary matrix model is the Gross-Witten-Wadia(GWW) model [Gross-Witten, '80], [Wadia, '79]

- When $M \neq 0$, we call this model as the generalized GWW model.

- The integration over the unitary matrix U reduces into the integration over the eigenvalues:

$$Z_{U(N)} = \frac{1}{N!} \left(\prod_{i=1}^{N} \oint \frac{\mathrm{d}z_i}{2\pi\mathrm{i}\, z_i} \right) \Delta(z)\Delta(z^{-1}) \exp\left(\sum_{i=1}^{N} W(z_i) \right),$$

$$\Delta(z) = \prod_{1 \le i < j \le N} (z_i - z_j), \qquad \Delta(z^{-1}) = \prod_{1 \le i < j \le N} (z_i^{-1} - z_j^{-1}).$$

• The partition function can be written as a determinant of an $N \times N$ matrix

$$Z_{U(N)} = (-1)^{NM} \det \Big(I_{M+j-i}(1/g) \Big)_{1 \le i,j \le N}$$

$I_\nu(z)$: modified Bessel function of the first kind

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n} .$$

• This determinant representation is the exact result and contains all perturbative and non-perturbative effect in the coupling constant g.

• (Comment) Except for the first phase factor $(-1)^{NM}$, the range of M can be analytically continued from integers to any complex numbers through this determinant representation.

- In order to extract the perturbative properties of the partition function and the genus expansion of the free energy, we use the method of orthogonal polynomials and small g-expansion.

3.1 Method of orthogonal polynomials

- Method of orthogonal polynomials[Bessis, '79], [Itzykson-Zuber, '80].

- The case of (symmetric) unitary matrix models (which include the GWW model) [Periwal-Schevitz, '90] $\longrightarrow$ extend their results to the case of $M \neq 0$.

- $p_n(z)$ and $\tilde{p}_n(1/z)$ $(n \geq 0)$: monic polynomials

$$p_n(z) = z^n + \sum_{k=0}^{n-1} A_k^{(n)} z^k, \qquad \tilde{p}_n(1/z) = z^{-n} + \sum_{k=0}^{n-1} B_k^{(n)} z^{-k},$$

- Orthogonality condition$\rightarrow$ normalization constants h_n

$$\oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \mathrm{e}^{W(z)} p_n(z) \tilde{p}_m(1/z) = h_n \, \delta_{n,m}.$$

- The partition function can be written in terms of the constant terms $A_n := p_n(0) = A_n^{(0)}$, $B_n := \tilde{p}_n(0) = B_n^{(0)}$:

$$Z_{U(N)} = h_0^N \prod_{n=1}^{N-1} (1 - A_n B_n)^{N-n}.$$

$$h_n = (-1)^{nM} K_M^{(n)}, \qquad A_n = \frac{K_{M+1}^{(n)}}{K_M^{(n)}}, \qquad B_n = \frac{K_{M-1}^{(n)}}{K_M^{(n)}},$$

$$K_\nu^{(n)} := \det\Big(I_{\nu+j-i}(1/g) \Big)_{1 \leq i,j \leq n}, \qquad (\nu \in \mathbb{C}).$$

3.1.1 Recursion relations for the constant terms and the discrete Painlevé equation

- string equations(recursion relations for the constant terms):

$$A_{n+1} = -A_{n-1} + \frac{2ngA_n}{1 - A_nB_n}, \qquad B_{n+1} = -B_{n-1} + \frac{2ngB_n}{1 - A_nB_n},$$

$$A_nB_{n+1} - A_{n+1}B_n = 2Mg.$$

- The initial conditions:

$$A_0 = B_0 = 1, \qquad A_1 = \frac{I_{M+1}(1/g)}{I_M(1/g)}, \qquad B_1 = \frac{I_{M-1}(1/g)}{I_M(1/g)}.$$

• Relation with the discrete Painlevé equations

Let

$$X_n(M) := \frac{A_{n+1}(M)}{A_n(M)}, \qquad Y_n(M) := \frac{B_{n+1}(M)}{B_n(M)}.$$

The string equations become the discrete Painlevé equations for X_n and Y_n.

d-P($(2A_1)^{(1)}$ / $D_6^{(1)}$): $(2A_1)^{(1)}$: symmetry; $D_6^{(1)}$: rational surface

[Fokas-Grammaticos-Ramani, '93], [Nijhoff-Satsuma-Kajiwara-Grammaticos-Ramani, '96]

$$\frac{z_n}{1+x_{n+1}x_n} + \frac{z_{n-1}}{1+x_n x_{n-1}} = \mathfrak{t}\left(-x_n + \frac{1}{x_n}\right) + z_n + \mu,$$

$$z_n = an + b.$$

In this case, $\mathfrak{t} = 1/(2g)$, $a = b = 1$;

$\mu = n - M$ for $x_n = X_n$; $\mu = n + M$ for $x_n = Y_n$.

- For simplicity, we assume $\mathrm{Re}\, g > 0$.

The small g-expansion (asymptotic expansion) of the modified Bessel functions

$$I_\nu(1/g) \sim \left(\frac{g}{2\pi}\right)^{1/2} \mathrm{e}^{(1/g)} \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\Big(1 + \underbrace{O(\mathrm{e}^{-2/g})}_{\text{instanton contribution}}\Big) C_k(\nu) \left(\frac{g}{2}\right)^k\right],$$

$$C_k(\nu) = \frac{\Gamma(\nu + k + (1/2))}{\Gamma(\nu - k + (1/2))}.$$

• A truncation of the initial conditions which ignore the contribution from the instantons: $A_0 = B_0 = 1$,

$$A_1(M) = \frac{\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} C_k(M+1) \left(\frac{g}{2}\right)^k}{\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} C_k(M) \left(\frac{g}{2}\right)^k}, \qquad B_1(M) = A_1(-M).$$

• This truncation is consistent with the string equations. In particular, the following condition holds:

$$A_0 B_1 - A_1 B_0 = 2Mg.$$

- Expansion coeffients of the small-g expansion

$$A_n(M) = \sum_{k=0}^{\infty} \frac{A_{n,k}(M)}{k!} \left(\frac{g}{2}\right)^k, \qquad B_n(M) = \sum_{k=0}^{\infty} \frac{A_{n,k}(-M)}{k!} \left(\frac{g}{2}\right)^k.$$

- Using the recursion relations

$$A_{n+1} = -A_{n-1} + \frac{2ngA_n}{1 - A_nB_n}, \qquad B_{n+1} = -B_{n-1} + \frac{2ngB_n}{1 - A_nB_n},$$

$A_{n,k}$ are determined for $0 \leq k \leq 11$.

- (Comment) The perturbative solution of the recursion relations takes the form of $A_n = 1 + \cdots$, $B_n = 1 + \cdots$. If A_n and B_n are $O(g^\ell)$, $g/(1 - A_nB_n)$ is $O(g^{\ell-1})$. Hence A_{n+1} and B_{n+1} become $O(g^{\ell-1})$.
The order in g decreases by one for each iteration in n.

3.2 Properties of the generalized GWW model

We explain the following propertes of the generalized GWW model:

1. free energy,
2. relation with the tau-function of the Painlevé III system,
3. the dougle scaling limit and the Argyres-Douglas point.

3.2.1 The genus expansion of the free energy

- The genus expansion of the free energy (keeping M finite):

$$\mathcal{F} = \log Z_{U(N)}(M) = \sum_{k=0}^{\infty} \mathcal{F}_k(S) g^{2k-2}, \qquad S = Ng.$$

- By extrapolating the results of the small g-expansion, $\mathcal{F}_k$ are determined up to 3rd order: (green part: discrepancy from the Gaussian Hermitian matrix model)

$$\mathcal{F}_0(S) = (1/2)S^2(\log S - (3/2)) + S,$$

$$\mathcal{F}_1(S) = -(1/12)\log S + (1/8)\big((2M)^2 - 1\big)\log(1-S),$$

$$\mathcal{F}_2(S) = -(1/240)S^{-2} + (1/384)\big((2M)^2 - 1\big)\big((2M)^2 - 3^2\big)\frac{S}{(1-S)^3},$$

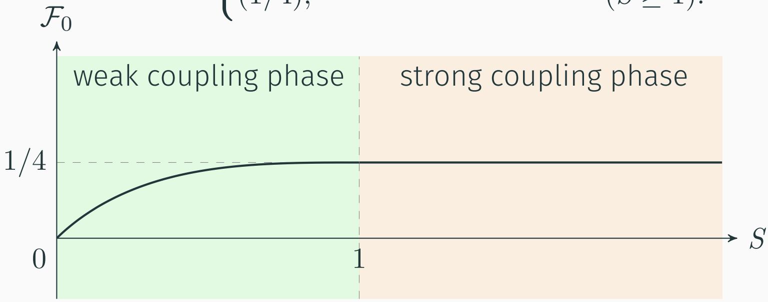
$$\mathcal{F}_3(S) = (1/10008)S^{-4} - (1/15360)\big((2M)^2 - 1\big)\big((2M)^2 - 3^2\big) \times \Big[3\big((2M)^2 - 5^2\big) + 2\big((2M)^2 - 15\big)S\Big]\frac{S}{(1-S)^6}.$$

- Comparison with the free energy of the GWW model($M=0$)

The GWW model in the large N limit exhibits the 3rd order phase transition at $S=1$.

(The 3rd derivative of the planar free energy $\mathcal{F}_0'''$ is discontinuous at $S=1$).

$$\mathcal{F}_0^{(M=0)}(S) = \begin{cases} (1/2)S^2(\log S-(3/2))+S, & (0\le S\le 1), \\ (1/4), & (S\ge 1). \end{cases}$$



By setting $M = 0$, our result correctly reduces to the free energy of the GWW model in the weak coupling phase ($0 \leq S \leq 1$).

$$\mathcal{F}_0(S) = (1/2)S^2(\log S - (3/2)) + S,$$

$$\mathcal{F}_1(S) = -(1/12)\log S + (1/8)\big((2M)^2 - 1\big)\log(1 - S),$$

$$\mathcal{F}_2(S) = -(1/240)S^{-2} + (1/384)\big((2M)^2 - 1\big)\big((2M)^2 - 3^2\big)\frac{S}{(1-S)^3},$$

$$\mathcal{F}_3(S) = (1/10008)S^{-4} - (1/15360)\big((2M)^2 - 1\big)\big((2M)^2 - 3^2\big) \times \Big[3\big((2M)^2 - 5^2\big) + 2\big((2M)^2 - 15\big)S\Big]\frac{S}{(1-S)^6}.$$

3.2.2 Partition function as a tau function of Painlevé III

- Painlevé III' equation ($q = q(s)$; $\alpha, \beta, \gamma, \delta$ are parameters)

$$\frac{\mathrm{d}^2 q}{\mathrm{d}s^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}s}\right)^2 - \frac{1}{s}\left(\frac{\mathrm{d}q}{\mathrm{d}s}\right) + \frac{q^2}{4\,s^2}(\gamma\, q + \alpha) + \frac{\beta}{4\,s} + \frac{\delta}{4\,q}.$$

- When $\gamma\delta \neq 0$, this is equivalent to the following Hamiltonian system [Okamoto, '79]. ($q = q(s)$, $p = p(s)$):

$$\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{\partial H_{\mathrm{III}'}}{\partial p}, \qquad \frac{\mathrm{d}p}{\mathrm{d}s} = -\frac{\partial H_{\mathrm{III}'}}{\partial q},$$

$$H_{\mathrm{III}'}(s) = \frac{1}{s}\left[q^2 p^2 - (q^2 + v_2 q - s)p + \frac{1}{2}(v_1 + v_2)q\right].$$

v_1, v_2: parameters.

$$\alpha = -4\,v_1, \qquad \beta = 4(v_2 + 1), \qquad \gamma = 4, \qquad \delta = -4.$$

(Without loss of generality, we can set $\gamma = -\delta = 4$ by rescaling of the time s and variables.)

- PIII' vs PIII: by a canonical transformation from $(q(s), p(s))$ to $(y(\mathfrak{t}), p_y(\mathfrak{t}))$

$$q(s) = \mathfrak{t}\, y(\mathfrak{t}), \qquad p(s) = \frac{p_y(\mathfrak{t})}{\mathfrak{t}},$$

$$s = \mathfrak{t}^2, \qquad H_{\mathrm{III}'}(s) = \frac{1}{2\mathfrak{t}} \left(H_{\mathrm{III}}(\mathfrak{t}) + \frac{y(\mathfrak{t}) p_y(\mathfrak{t})}{\mathfrak{t}} \right),$$

$$\mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}H_{\mathrm{III}'} \wedge \mathrm{d}s = \mathrm{d}p_y \wedge \mathrm{d}y - \mathrm{d}H_{\mathrm{III}} \wedge \mathrm{d}\mathfrak{t}.$$

the Painlevé III' equation turns into the Painlevé III equation.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\mathfrak{t}^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}\mathfrak{t}}\right)^2 - \frac{1}{\mathfrak{t}}\frac{\mathrm{d}y}{\mathrm{d}\mathfrak{t}} + \frac{1}{\mathfrak{t}}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}.$$

- The tau-function of the PIII' equation

$$H_{\mathrm{III}'}(s) = \frac{\mathrm{d}}{\mathrm{d}s} \log \tau(s).$$

- By using one of the theorems in [Forrester-Witte, '02], we can show that the partition function of the generalized GWW model is a tau-function of PIII':

$$\tau(s) = s^{(1/2)MN} Z_{U(N)}(M), \qquad s = \frac{1}{4g^2}.$$

Here,

$$v_1 = M + N = -2m_1, \qquad v_2 = -M + N = -2m_2.$$

3.2.3 Double scaling limit and Argyres-Douglas point

- Argyres-Douglas point [Argyres-Douglas, '95]: a fixed point in the moduli space of the supersymmetric gauge theory which exhibits the $\mathcal{N} = 2$ superconformal symmetries.

- Let us consider an AD point(H_1 AD point) [Argyres-Plesser-Seiberg-Witten, '95]. It can be obtained by fine-tuning the parameters of the $N_f = 2$ $SU(2)$ gauge theory.

- We show that the double scaling limit of the matrix model can be related to the limit to the AD fixed point in the gauge theory.

the double scaling limit of the unitary matrix model which corresponds to the $N_f = 2$ $SU(2)$ gauge theory

- the size of the matrix goes to infinity: $N \to \infty$,
- send the 't Hooft coupling $S = Ng$ to its critical value ($S_c = 1$): $S \to 1$
- keep the following quantity finite:

$$\kappa := \frac{1}{N} \frac{1}{(1-S)^{(1/2)(2-\gamma_{\rm st})}}.$$

($\gamma_{st} = -1$: susceptibility of the matrix model).

• Let us introduce the scaling variable a by $a := 1/N^{1/3}$. ($N \to \infty \iff a \to 0$.)

• The 't Hooft coupling

$$S = Ng = 1 - ca^2, \qquad (c = \kappa^{-2/3}).$$

• (Assumption) In the $N \to \infty$ limit, the discrete variables f_n ($n = 1, 2, \cdots, N$) turn into a continuous function of $x = n/N$:

$$f_n \to f\left(\frac{n}{N}\right) \equiv f(x), \qquad x \equiv \frac{n}{N}, \qquad (0 \leq x \leq 1).$$

• A coordinate transformation from the variable x to "time" variable t

$$ng = \frac{n}{N} Ng = Sx = 1 - \frac{1}{2} a^2 t.$$

• The term $A_n B_n$ in the partition function

$$Z_{U(N)}(M) = h_0^N \prod_{n=1}^{N-1} (1 - A_n B_n)^{N-n}$$

approaches to its critical value ($= 0$) by

$$A_n B_n = a^2\, u(t).$$

• The double scaling limit of the string equations gives the Painlevé II equation

$$u'' = \frac{(u')^2}{2\,u} + u^2 - \frac{1}{2} t\, u - \frac{M^2}{2\,u}$$

(This form of the PII equation can be obtained by the Flaschka-Newell Lax pair [Flaschka-Newell, '80].)

• Relations between parameters of the matrix model and those of the gauge theory:

$$\frac{1}{g} = \Lambda_2, \qquad N = -(m_1 + m_2), \qquad M = (m_2 - m_1).$$

• In the double scaling limit, $N = 1/a^3$,

$$S = Ng = -\frac{(m_1 + m_2)}{\Lambda_2} = 1 - ca^2, \qquad Mg = \frac{(m_2 - m_1)}{\Lambda_2} = SMa^3.$$

• The double scaling limit turns into the following limit of the mass parameters

$$m_1 = -\frac{1}{2}\Lambda_2(1-ca^2)(1+Ma^3), \qquad m_2 = -\frac{1}{2}\Lambda_2(1-ca^2)(1-Ma^3).$$

The spectral curve of the matrix model coincides with the Seiberg-Witten curve of the $SU(2)$ $N_f = 2$ gauge theory (the Gaiotto form (frist realization)) :

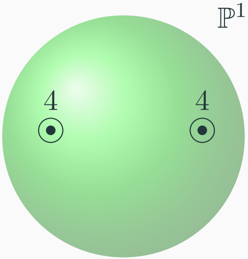
$$y^2 = \frac{{\Lambda_2}^2}{16 z^4} \left(1 + \frac{8\, m_2}{\Lambda_2} z + \frac{16\, \mathfrak{u}}{{\Lambda_2}^2} z^2 + \frac{8\, m_1}{\Lambda_2} z^3 + z^4 \right).$$

$\lambda_{\mathrm{SW}} = y(z) \mathrm{d}z$: the SW differential (1-form)

$\mathfrak{u} = \langle \mathrm{Tr}\, \phi^2 \rangle$: Coulomb moduli

- The pole structure of the quadratic differential

$$\phi = \lambda_{\rm SW}^2 = y^2 (\mathrm{d}z)^2$$
$$= \frac{{\Lambda_2}^2}{16 z^4} \left(1 + \frac{8\, m_2}{\Lambda_2} z + \frac{16\, u}{{\Lambda_2}^2} z^2 + \frac{8\, m_1}{\Lambda_2} z^3 + z^4 \right) (\mathrm{d}z)^2 .$$



$N_f = 2$ (first realization)

- a quartic pole at $z = 0$
- another quartic pole at $z = \infty$

• At the critical values

$$m_{1,c} = m_{2,c} = -\frac{1}{2}\Lambda_2, \qquad \mathfrak{u}_c = \frac{3}{8}{\Lambda_2}^2,$$

the SW curve degenerate to

$$y^2 = \frac{{\Lambda_2}^2}{16\, z^2}(z-1)^4.$$

(the critical value of $z:\ z_c = 1$)

• (Assumption) The variable z and the parameter $\mathfrak{u}$ approach to their critical values as follows:

$$z = 1 - 2a\tilde{z}, \qquad \mathfrak{u} = {\Lambda_2}^2\left(\frac{3}{8} - \frac{1}{2}a^2 c + a^4 \tilde{u}\right).$$

The SW curve behaves as

$$y^2 = {\Lambda_2}^2 \left(\tilde{z}^4 + c\,\tilde{z}^2 + M\tilde{z} + \tilde{u} \right) a^4 + O(a^5).$$

Let

$$y = \Lambda_2\, a^2\, \tilde{y},$$

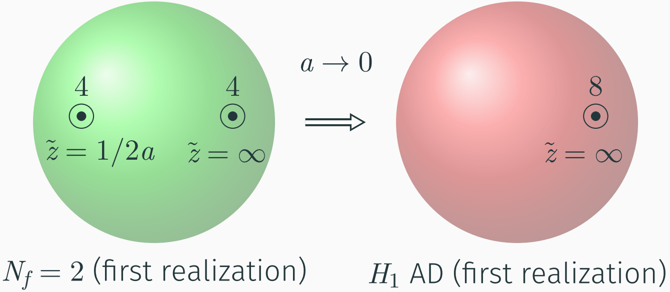
then the SW curve of the $N_f = 2$ theory reduces to the SW curve of the H_1 AD theory (first realization).

$$\tilde{y}^2 = \tilde{z}^4 + c\,\tilde{z}^2 + M\tilde{z} + \tilde{u}$$

- The quadratic differential

$$\tilde{\phi} := \tilde{y}^2(\mathrm{d}\tilde{z})^2 = \left(\tilde{z}^4 + c\,\tilde{z}^2 + M\tilde{z} + \tilde{u}\right)(\mathrm{d}\tilde{z})^2.$$

has an 8th order pole at $\tilde{z} = \infty$.



Two singularities merge into one singularity in the limit.

4. Summary

We have studied some properties of the generalized GWW model. In partiular, we have explained

1) relation with the discrete Painlevé equation

2) the genus expansion of the free energy(in the weak coupling phase)

3) the partition function as the tau-function of Painlevé III' system,

4) the correspondence between the double scaling limit of the matrix model, and the limit to the Argyres-Douglas fixed point of the gauge theory

- Future directions

  a) The free energy in the strong coupling phase,

  b) The double scaling limit of the free energy,

  c) The double scaling limit of the partition function,

  d) Other N_f cases,

  e) etc.