Q matrix and Bäklund for quantizing integrable models

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Ipht

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Plan

- Some classical results
- Quantization of the Toda chain
- The Ruisjenaars chain

Relation with Spin Chains

- From a Heisenberg chain point of view:

$$L(u) = \begin{pmatrix} u + S^z & S^-/r \\ S^+/r & (u - S^z)/r^2 \end{pmatrix}$$

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- Obtain Toda Lax Matrix:

$$L(u) = \begin{pmatrix} u + p & e^q \\ -e^{-q} & 0 \end{pmatrix}$$

Hamiltonian

- Monodromy matrix:

$$T(u) = L_1(u) \cdots L_n(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

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- Open chain Hamiltonian:

$$A = u^N + Pu^{N-1} + Hu^{N-2} + \cdots$$

$$H = \sum_1^N \frac{p_k^2}{2} + \sum_1^{N-1} e^{q_k - q_{k+1}}$$

- Obey

$$RTT = TTR$$

relation.

classical solution of the open chain by separation of variables

- equation of motion: $\dot{L}_k = M_k L_k - L_k M_{k+1}$

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- u_k are assymptotic momentas. At $t = \pm\infty$ $p_k = u_{\sigma_k}$.

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- $$\tilde{q}_j - q_j - p_j t = \sum_{k>j} \phi_{jk} - \sum_{k<j} \phi_{kj}$$

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- $\Lambda = A(u) + D(u)$ conserved.
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- trajectories are on a hyperelliptic curve (trace Λ):

$$\mu_k + 1/\mu_k = \Lambda(u_k)$$

Hamilton Jacobi

- Set of $n-1$ inedependant Hamilton Jacobi equations:

$$2\cosh(S'(u_k)) = \Lambda(u_k)$$

-

$$S = \sum_k S_k(u_k)$$

- $\dot{\tilde{B}}(u_k) = -\mu_k + 1/\mu_k$
- $\frac{du_k}{\sqrt{\Lambda^2 - 4}} = \frac{dt}{\tilde{B}'(u_k)}$

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- $\frac{du_k}{\sqrt{\Lambda^2 - 4}} = \frac{dt}{\tilde{B}'(u_k)}$
- multiplying by u_k^j and summing over k:

$$\sum_k \int^{u_k} \frac{u^j du}{\sqrt{\Lambda^2 - 4}} = \delta_{n-1,j} t$$

Kac Mac Laughlin.

Baxter Strategy:

- triangularize L conjugate L by M_j, so that Λ unchanged.

$$M_j = \begin{pmatrix} 1 & y_j \\ . & 1 \end{pmatrix}$$

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- $L_{12} = 0$:

$$p_j = -u + \frac{x_j}{y_j} + \frac{y_{j+1}}{x_j}$$

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$$\frac{\partial W}{\partial x_j} = -p_j, \quad \frac{\partial W}{\partial y_j} = p_j'$$

- you can show:

$$\Lambda(x_j, -\frac{\partial W}{\partial x_j}) = (y_j, \frac{\partial W}{\partial y_j})$$

Preserves conserved quantites Bäcklund transform

Quantum Q

- $Q_u = e^{W_u(y,x)}$ Kernel

$$[\Lambda, Q] = 0$$

- compute the trace:

$$\Lambda Q_u = Q_{u-i} + Q_{u+i}$$

quantum analogue of hyperelliptic curve

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- in the open case there is a limit that adds a variable:

$$Q_{n,n-1} \cdots Q_{3,2} Q_{2,1} = \psi(x_1, \cdots, x_n)$$

Whittacker functions.

Gutwiller, Gaudin-P

- Bessel recursion relation:

$$-2iuQ_u = \rho^{1/2}(-Q_{u-i} + Q_{u+i})$$

- Bessel solution entire:

$$I_{iu}(\rho^{1/2}) = \sinh(\pi u) Q_{\downarrow u}$$

$$I_{-iu}(\rho^{1/2}) = \sinh(\pi u) Q_{\uparrow u}$$

- linear combination (poles at $u = k$ cancel)

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- $Q(u) = K_{-iu}(\rho^{1/2}) =$ Macdonald Function.

General case

- $iu \to \Lambda(u) = \prod_{k=1}^{N}(-2i)(u - v_k)$
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- TQ equation:

$$\Lambda(u)Q_u = \rho^{1/2}((-)^N Q_{u-i} + Q_{u+i})$$

Operator equation $\to$ Eigenvalue equation,

General case

- Solve recursion relation:

$$Q_{\uparrow}(u;u_j) = (\rho^{1/2}/2)^{iu} \prod_j \Gamma(\frac{u - v_j}{i})$$

$$Q_{\downarrow}(u;u_j) = (\rho^{1/2}/2)^{-iu} \prod_j \Gamma(\frac{v_j - u}{i})$$

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- right hand side depends on v_i roots of Λ.

$$\begin{aligned} \mu_{\uparrow}(u) &= \mu_{\uparrow}(u+i) + (-)^N \rho \frac{\mu_{\uparrow}(u-i)}{\Lambda(u)\Lambda(u-i)} \\ \mu_{\downarrow}(u) &= \mu_{\downarrow}(u-i) + (-)^N \rho \frac{\mu_{\downarrow}(u+i)}{\Lambda(u)\Lambda(u+i)} \end{aligned} \quad (1)$$

- have continuous fraction representation.

$$r_{n+1} = 1 + \frac{(-)^n \rho^{1/2}}{\Lambda_n \Lambda_{n-1} r_n}$$

quantum wronskien

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- then for two linearly independant solutions a_n, b_n, one has

$$w_n = a_n b_{n+1} - b_n a_{n+1}$$

is constant.

quantum wronskien

- The quantum wronskien

$$W_\mu(u) = \mu_\uparrow(u)\mu_\downarrow(u-i) - (-)^N \rho \frac{\mu_\uparrow(u-i)\mu_\downarrow(u)}{\Lambda(u)\Lambda(u-i)}$$

can be represented as an infinite determinant (so called Hill determinant).

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- $$W_\mu(u) = \prod_j \frac{\sinh \pi(u-u_k)}{\sinh \pi(u-v_k)}$$

- Numerator=Bethe roots, Denominator=transfer matrix roots.

- look for linear combination:

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- Bethe équations results from cancellation of poles:

$$\frac{Q_{\uparrow}}{Q_{\downarrow}}(u_k) = \xi$$

- Koszlowski-Teschner (after Nekrasov Shatashvili) How to get rid of v_k? We see that in the Hill determinant, the zeros nearly coincide with the poles, so, we try to modify the method so as to make them coincide exactly?

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- modify $\Lambda \to \Lambda_0$ assume Bethe roots coincide with zeros of Λ_0:v_k:

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- zero order Bethe équations from cancellation of poles:

$$\xi = \rho^{2iu_k} \prod_j \frac{\Gamma(\frac{u_k - u_j}{i})}{\Gamma(\frac{u_j - u_k}{i})}, \ \forall k$$

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$$2\pi i n_k = -\log \xi + 2iu_k \log(\rho) + \sum_j \log(\frac{\Gamma(\frac{u_k - u_j}{i})}{\Gamma(\frac{u_j - u_k}{i})}), \ \forall k$$

coincide exactly with Sutherland. Although the approach is entierly different.

- This cannot be exact, then improve Q:

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- get equation for ν: Same right hand size as Gutzwiller with $\Lambda \to \Lambda_0$

$$\frac{\Lambda}{\Lambda_0}\nu_\uparrow(u) = \nu_\uparrow(u+i) + (-)^N \rho \frac{\nu_\uparrow(u-i)}{\Lambda_0(u)\Lambda_0(u-i)}$$

$$\frac{\Lambda}{\Lambda_0}\nu_\downarrow(u) = \nu_\downarrow(u-i) + (-)^N \rho \frac{\nu_\downarrow(u+i)}{\Lambda_0(u)\Lambda_0(u+i)}$$

- can be solved as a series in ρ for both Λ and ν.

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- by construction $\nu_\uparrow, \nu_\downarrow$ have their poles at $u_k + im$ and $u_k - im$ with $m \geq 1$, we can solve for:

$$K * (\log(\nu_\uparrow(u)\nu_\downarrow(u-i))) = \log(\nu_\uparrow(u-i)\nu_\downarrow(u))$$

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- set $Y = \nu_\uparrow(u-i)\nu_\downarrow(u)$, then above equality becomes:

$$\log(Y(u)) = K * \log(1 + \frac{(-)^N \rho Y(u)}{\Lambda_0(u)\Lambda_0(u-i)})$$

- K is the Szegö kernel:

$$K(u) = \frac{1}{2\pi}\frac{1}{1+u^2}$$

- So we get modified Bethe equations:

$$2\pi i n_k = -\log\xi + 2iu_k\log(\rho) + \sum_j \log(\frac{\Gamma(\frac{u_k-u_j}{i})}{\Gamma(\frac{u_j-u_k}{i})})$$

$$+\int\frac{dv}{2\pi}(\frac{1}{u_k-v+i}+\frac{1}{u_k-v})\log(1+\frac{(-)^N\rho Y(v)}{\Lambda_0(v)\Lambda_0(v-i)})$$

The q-Toda chain.

- pause The q-Toda chain is the XXZ version of Toda.
- The expression of the Lax matrix is:

$$L_1(z) = \frac{i}{\sqrt{zX_1}} \begin{pmatrix} 1 - zX_1 & \epsilon x_1 \\ \epsilon z X_1 x_1^{-1} & . \end{pmatrix}$$

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- $x_1 = e^{2i\pi q_1/\omega_2}$, X_k acts by translating q_k: $q_k \to q_k + i\omega_1$, $q = e^{2i\pi\omega_1/\omega_2}$. dual quantities exchange $\omega_1, \ \omega_2$
- Commuting Weyl pairs:

$$X_1 x_1 = q x_1 X_1$$

$$\tilde{X}_1 \tilde{x}_1 = \tilde{q} x_1 \tilde{X}_1$$

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Requests to find a solution

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- Modular invarient $\tilde{\Lambda}Q$ satisfyed with ω_1, ω_2 exchanged.

$$\tilde{\Lambda}(u)Q_u = \rho^{1/2}((-)^N Q_{u-i} + Q_{u+i})$$

- Behavior at infinity:

$$|Q(u)| \sim e^{-\pi N(\frac{1}{\omega_1}+\frac{1}{\omega_2})/2}$$

Modularity

- Now, Hill determinant becomes elliptic function

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- Connection between two "modular transform" hyperelliptic curves?, apparently meaningless for ω_1, ω_2 real? (τ imaginary).

- due to modularity, for $\omega_1/\omega_2 = p/r$ a fraction, some highly nontrivial algebraic relations relate Λ and $\tilde{\Lambda}$ spectra due to $r\omega_1 = p\omega_2$.

$$\tilde{W}_\mu(u) = \prod_j \frac{\tilde{\theta}(\pi(u-u_k))}{\tilde{\theta}(\pi(u-v_k))}$$

Solution of TQ

- The Sutherland approximation for Q^0 must be Modular. Use invarient Γ Function:

$$\frac{\Gamma_q(u+i\omega_1)}{\Gamma_q(u)} = -2i\sinh_{\omega_2}(u)$$

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- then improve Q_0 by assuming factorization:

$$Q_\uparrow(u) = Q^0_\uparrow(u)\nu_\uparrow(u)\tilde{\nu}_\uparrow(u)\ ,\ Q_\downarrow(u) = Q^0_\downarrow(u)\nu_\downarrow(u)\tilde{\nu}_\downarrow(u),$$

Kashaev Sergeev

Table: $\omega_1/\omega_2 = i, \eta = 0$

n	root	H	$\lvert H - H_{KS}\rvert$
0	.3535533905	4.594358809 i	$2\ 10^{-18}$
-1	.612117375	-13.878304778+6.1612962432443 i	$5\ 10^{-15}$
-2	.79079992	-31.32504489969-12.153338942 i	$3\ 10^{-14}$
-3	.935447530	-33.7154767687 4-54.1710567 i	$3\ 10^{-17}$

I compare the first four energies H obtained for $\omega_1/\omega_2 = i, \eta = 0$, at fifth order with those of Kashaev and Sergeev (table 3 and 4 of their paper). For the three first values $n = 0, -1, -2, -3$ of the Bethe equations

Sciarappa

Table: $\omega_1/\omega_2 = 2^{-1/2}, \eta = 0$

n	root	H
0	.462871608964	2.460524271907
-1	.680791907983	3.598470877254
-2	.844632649750	4.4628893132238

I compare the energies H obtained for $\omega_1/\omega_2 = 2^{-1/2}, \eta = 0$, at third order in ρ, with those of table 5 of Sciarappa (Sc) . The three first values $n = 0, -1, -2$ are in agreement to his precision 10^{-10}.

Sciarappa

Table: $\omega_1 = 2^{-1/2}, \omega_2 = 1, \eta = \log(3)/8\pi$

n	root	H
0	.4354731597837	2.752848101914
-1	.6178613438775	3.883834678235
-2	.7553058969907	4.746028853867

The simplest problem

- Two particles, $\omega_1/\omega_2 = 1$.

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- Two particles, $\omega_1/\omega_2 = 1$.
- Entire solution of L_2 on the reals of:

$$\phi(x+1) + \phi(x-1) + 2\cos(2\pi x)\phi(x) = E\phi(x)$$

  Ground state was obtained independently by Marino and Kashaev-Sergeev.
- quantum =classical motion on curve:

$$2\cos(2\pi y) + 2\cos(2\pi x) = E$$

Conclusions

- Wave functions?
- q a root of unity?