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Diagonalization of the Heun-Askey-Wilson operator, Leonard pairs and the algebraic Bethe ansatz

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In collaboration with Rodrigo Pimenta, arXiv:1909.02464

An inspiring example : the quantum harmonic oscillator revisited...

• **Point 1 : Two hidden AW algebras.**

Let $a, a^\dagger$ satisfy the Heisenberg algebra. Consider :

$$\mathsf{H} = a^\dagger a + \frac{1}{2}\ , \quad \mathsf{X} = \frac{1}{\sqrt{2}}(a^\dagger + a)\ , \quad \mathsf{P} = \frac{i}{\sqrt{2}}(a^\dagger - a)\ .$$

→ The pair (X, H) satisfies the simplest **Askey-Wilson relations** :

$$\left[\mathsf{X}, \left[\mathsf{X}, \mathsf{H}\right]\right] \;=\; -1\ , \quad \left[\mathsf{H}, \left[\mathsf{H}, \mathsf{X}\right]\right] = \mathsf{X}\ .$$

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⇒ **The pair (X, H) generates a specialization of the Askey-Wilson algebra with generators A, A^*** [Zhedanov,'91] (*several related works in maths...*)

$$\big[\mathsf{A}, \big[\mathsf{A}, \mathsf{A}^*\big]_q\big]_{q^{-1}} = \rho\, \mathsf{A}^* + \omega\, \mathsf{A} + \eta \mathcal{I}\ ,$$
$$\big[\mathsf{A}^*, \big[\mathsf{A}^*, \mathsf{A}\big]_q\big]_{q^{-1}} = \rho\, \mathsf{A} + \omega\, \mathsf{A}^* + \eta^* \mathcal{I}$$

with $\big[X, Y\big]_q = qXY - q^{-1}YX.$

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$\rightarrow$ The pair $(\mathsf{H}, i\mathsf{P})$ satisfies **Askey-Wilson relations** with different structure constants :

$$\big[\mathsf{H}, \big[\mathsf{H}, i\mathsf{P}\big]\big] \quad = \quad i\mathsf{P} \ , \quad \big[i\mathsf{P}, \big[i\mathsf{P}, \mathsf{H}\big]\big] = 1 \ .$$

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⇒ The pair $(\mathsf{H}, i\mathsf{P})$ generates a specialization of the Askey-Wilson algebra with generators $\mathsf{A}, \mathsf{A}^\diamond$.

$$i\mathsf{P} = [\mathsf{X}, \mathsf{H}] \quad \rightarrow \quad \mathsf{A}^\diamond = \frac{i}{\sqrt{\rho}}[\mathsf{A}^*, \mathsf{A}]_q + \frac{i\omega}{(q - q^{-1})\sqrt{\rho}} .$$

The pair $(\mathsf{A}, \mathsf{A}^\diamond)$ satisfies Askey-Wilson relations with different structure constants :

$$\big[\mathsf{A}, \big[\mathsf{A}, \mathsf{A}^\diamond\big]_q\big]_{q^{-1}} = \rho\,\mathsf{A}^\diamond + \frac{i(q-q^{-1})}{\sqrt{\rho}}\eta\,\mathsf{A} - \frac{i\sqrt{\rho}}{(q-q^{-1})}\omega\mathcal{I} ,$$

$$\big[\mathsf{A}^\diamond, \big[\mathsf{A}^\diamond, \mathsf{A}\big]_q\big]_{q^{-1}} = \rho\,\mathsf{A} + \frac{i(q-q^{-1})}{\sqrt{\rho}}\eta\,\mathsf{A}^\diamond + \eta^*\mathcal{I} .$$

So if we forget about the Heisenberg algebra construction, one has the parallel :

→ The **triple** $(\mathsf{X}, \mathsf{H}, i\mathsf{P})$ satisfies the specialization of **two AW algebras** :

$$\begin{aligned} [\mathsf{X}, [\mathsf{X}, \mathsf{H}]] &= -1 \ , \quad [\mathsf{H}, [\mathsf{H}, \mathsf{X}]] = \mathsf{X} \ , \\ [\mathsf{H}, [\mathsf{H}, i\mathsf{P}]] &= i\mathsf{P} \ , \quad [i\mathsf{P}, [i\mathsf{P}, \mathsf{H}]] = 1 \ . \end{aligned}$$

with $i\mathsf{P} = [\mathsf{X}, \mathsf{H}]$.

→ The **triple** $(\mathsf{A}, \mathsf{A}^*, \mathsf{A}^\diamond)$ satisfies **two AW algebras** :

$$\begin{aligned} \big[\mathsf{A}, [\mathsf{A}, \mathsf{A}^*]_q\big]_{q^{-1}} &= \rho\, \mathsf{A}^* + \omega\, \mathsf{A} + \eta \mathcal{I} \ , \\ \big[\mathsf{A}^*, [\mathsf{A}^*, \mathsf{A}]_q\big]_{q^{-1}} &= \rho\, \mathsf{A} + \omega\, \mathsf{A}^* + \eta^* \mathcal{I} \end{aligned}$$

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with

$$\mathsf{A}^\diamond = \frac{i}{\sqrt{\rho}} [\mathsf{A}^*, \mathsf{A}]_q + \frac{i\omega}{(q-q^{-1})\sqrt{\rho}} \ .$$

• **Point 2 : Representations and the bispectral problem.** In the literature, in order to study the **spectral problem** for H **two bases** are usually introduced : $\{|x\rangle\}$ and $\{|n\rangle\}$. Introduce

$$\langle x| = e^{-x^2/2}\langle\theta_x| \text{ and } |n\rangle = (2^n n!)^{-1/2}|\theta_n^*\rangle .$$

⇒ The **transition coefficients** $\langle\theta_x|\theta_n^*\rangle$ solve a bispectral problem which reads as a 2nd-order differential equation and a 3-term recurrence relation given by :

$$\langle\theta_x|\mathsf{H}|\theta_n^*\rangle = (n+\frac{1}{2})\langle\theta_x|\theta_n^*\rangle \;\Leftrightarrow\; \left(2x - \frac{d}{dx}\right)\frac{d}{dx}H_n(x) = 2nH_n(x), \quad \underbrace{H_n(x)}_{\text{Hermite polyn.}} = \langle\theta_x|\theta_n^*\rangle$$

$$\langle\theta_x|\mathsf{X}|\theta_n^*\rangle = \langle\theta_x|\theta_{n-1}^*\rangle + 2n\langle\theta_x|\theta_{n+1}^*\rangle \quad \Leftrightarrow \quad 2xH_n(x) = H_{n-1}(x) + 2nH_{n-1}(x)\ .$$

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$\Rightarrow$ **For the pair A, A^* $\exists$ a more general bispectral problem**.

Askey-Wilson orthogonal polynomials $\{P_n(x)|n=0,1,...\}$ satisfy a bispectral problem : a 3-term recurrence relation and a 2nd order q-diff. equation

$$b_nP_{n+1}(x) + c_nP_{n-1}(x) + a_nP_n(x) = xP_n(x) \quad \Rightarrow \langle x|\mathsf{A}|\theta_n^*\rangle$$
$$\phi(z)T^+P_n(x) + \phi(z^{-1})T^-P_n(x) + \mu(x)P_n(x) = \lambda_nP_n(x) \quad \Rightarrow \langle x|\mathsf{A}^*|\theta_n^*\rangle$$

where $T^{\pm}f(z) = f(q^{\pm 2}z)$ and $x = z + z^{-1}$. $\quad \Rightarrow P_n(x) = \langle x|\theta_n^*\rangle$

- **Point 3 : Analytical Bethe ansatz.**

The **transition coefficients** $Q_n(x) = 2^{-n}\langle\theta_x|\theta_n^*\rangle = 2^{-n}H_n(x)$ can be viewed as the **Q-polynomial** satisfying the T-Q relation :

$$2xQ_n'(x) - Q_n''(x) = 2nQ_n(x) \qquad \text{where} \qquad Q_n(x) = \prod_{i=1}^{n}(x - x_i)$$

and the zeroes $\{x_i|i = 1, ..., n\}$ satisfy the set of Bethe ansatz equations

$$\frac{Q_n''(x_i)}{Q_n'(x_i)} = 2x_i \quad \Leftrightarrow \quad \sum_{j=1, j\neq i}^{n} \frac{1}{x_i - x_j} = x_i \quad \text{for all} \quad i = 1, ..., n,$$

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$$2xQ'_n(x) - Q''_n(x) = 2nQ_n(x) \qquad \text{where} \qquad Q_n(x) = \prod_{i=1}^{n}(x - x_i)$$

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$\Rightarrow$ **For $\mathbf{A}^*$ $\exists$ T-Q relation** $\quad \pi(\mathbf{A}^*)P_n(x) = \lambda_n P_n(x)$

$$\Leftrightarrow \phi(z)T^+P_n(x) + \phi(z^{-1})T^-P_n(x) + \mu(x)P_n(x) = \lambda_n P_n(x)$$

where $T^\pm f(z) = f(q^{\pm 2}z)$ and $x = z + z^{-1}$.

Zeroes of Askey-Wilson polynomials $P_n(x) \leftrightarrow$ **Bethe roots** of Q-polynomial

$\Rightarrow$ Interpret the Q-polynomial $(Q(x) \equiv P_n(x) = \langle x|\theta_n^*\rangle)$ as transition coeff.

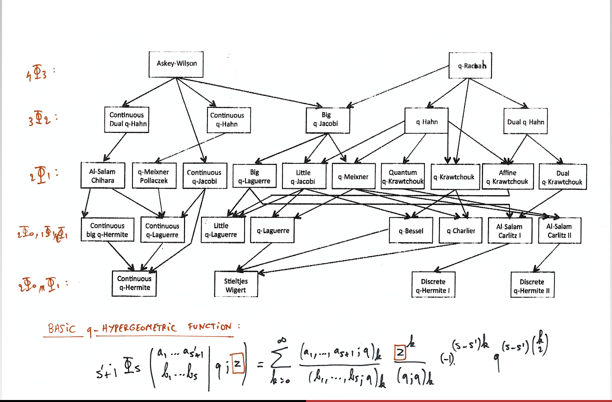
• **Point 4 : A generalized spectral problem...**

Observation : For the **quantum harmonic oscillator**, the diagonalization of the combination :

$$\mathsf{I}_{ho} = \kappa \mathsf{X} + \kappa^* \mathsf{H} + \kappa_+ i \mathsf{P}$$

is given in terms of Hermite polynomials $H_n(x)$.

Hermite polynomials at the bottom of the Askey-scheme !



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Based on the correspondence $(\mathsf{X}, \mathsf{H}, i\mathsf{P}) \longrightarrow (\mathsf{A}, \mathsf{A}^*, \mathsf{A}^\diamond)$

$\Rightarrow$ **For the pair A, A^*, define the combination :**

$$\mathsf{I} = \kappa \mathsf{A} + \kappa^* \mathsf{A}^* \ + \kappa_+ \left[\mathsf{A}, \mathsf{A}^*\right]_q + \kappa_- \left[\mathsf{A}^*, \mathsf{A}\right]_q$$

where A, A^* satisfy the **Askey-Wilson algebra**.

What is the analog of the previous construction ?

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• **Eigenvalues/states ? T-Q relation ? bispectral problem ?** Interpretation of the **Q-polynomial** as a transition coefficient between eigenbases of $\mathsf{A}, \mathsf{A}^*, \mathsf{I}$? Inhomogeneous T-Q relations : derivation from the algebra ?

• Examples of **integrable models** that can be diagonalized in this framework ? Generalizations ?

Problem 1 : Let A, A^* be generators of the **Askey-Wilson algebra**. For infinite or finite dimensional representations, solve the spectral problem for

$$\mathsf{I} = \kappa \mathsf{A} + \kappa^* \mathsf{A}^* \ + \kappa_+ \left[\mathsf{A}, \mathsf{A}^*\right]_q + \kappa_- \left[\mathsf{A}^*, \mathsf{A}\right]_q \ .$$

(i) For finite dim. reps : eigenstates/eigenvalues ?
(ii) Analog of the bispectral problem and related TQ equation ? Polynomial solutions ?

$\rightarrow$ *Motivated by works of Nomura, Grünbaum, Van Diejen, Vinet, Zhedanov, Takemura, Tsujimoto, Terwilliger, Huang, Post, DeBie, etc...*

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Problem 2 : T-Q relations with an **inhomogeneous term** arises recently in quantum integrable spin chains with **open boundary conditions** :

$$a(z)T^+Q(z) + b(z)T^-Q(z) + \underbrace{\Delta(z)}_{\text{inhomogeneous term}} = \Lambda(z)Q(z) \ .$$

(iii) Construction of the Q-polynomial as a transition coefficient ?
(iv) Inhomogeneous term : conjecture → proof based on representation theory ?

→ *Motivated by works of Lazarescu-Pasquier, Cao-Wang-Yang, Belliard-Crampé-Pimenta-Slavnov, Kitanine-Maillet-Niccoli-Terras,...*

Main results 1

Diagonalization of the HAW operator via ABA + Leonard pairs

Let $(\bar{\pi}, \bar{V})$ be a finite dim. vector space such that $\bar{\pi}(\mathsf{A}), \bar{\pi}(\mathsf{A}^*)$ are (i) diagonalizable ; (ii) $\bar{V}$ is irreducible. Then $\bar{\pi}(\mathsf{A}), \bar{\pi}(\mathsf{A}^*)$ is called a Leonard pair [Terwilliger et al.,'03].

- The **HAW operator** $\bar{\pi}(\mathsf{I})$ with

$$\mathsf{I} = \kappa \mathsf{A} + \kappa^* \mathsf{A}^* \ + \kappa_+ \left[\mathsf{A}, \mathsf{A}^*\right]_q + \kappa_- \left[\mathsf{A}^*, \mathsf{A}\right]_q$$

is **diagonalized** for :

→ Special cases ('a'=sp) $\kappa^* = \kappa_\pm = 0$ or $\kappa = \kappa_\pm = 0$;
→ Diagonal case ('a'=d) $\kappa_\pm = 0,\ \kappa, \kappa^* \neq 0$;
→ Generic case ('a'=g) $\kappa, \kappa^*, \kappa_\pm \neq 0$.

The **eigenstates** are written as Bethe states of the general form ($a \in \{sp, d, g\}$, $\epsilon = \pm$) :

$$\bar{\pi}\left(\mathsf{I}(\kappa, \kappa^*, \kappa_+, \kappa_-)\right) |\Psi^M_{a,\epsilon}(\bar{u})\rangle = \Lambda^M_{a,\epsilon} |\Psi^M_{a,\epsilon}(\bar{u})\rangle \ .$$

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- For each case, the system of **Bethe equations** satisfied by the Bethe roots $\bar u = \{u_1, ..., u_M\}$ is either of the '**homogeneous**' or '**inhomogeneous**' form. An **alternative presentation** of the Bethe equation is given in terms of the 'symmetrized' Bethe roots :

$$P_a(U_i) = 0\ \ i = 1, ..., dim(\bar V) \quad \text{where} \quad U_i = \frac{qu_i^2 + q^{-1}u_i^{-2}}{q + q^{-1}} \quad \text{with} \quad a \in \{sp, d, g\}.$$

Main results 2

q-difference HAW operator and TQ relations

Let (π, V) be an infinite dim. vector space such that $\pi(\mathrm{A}), \pi(\mathrm{A}^*)$ are second order q-difference operators that satisfy the Askey-Wilson relations.

- For each case $a \in \{sp, d, g\}$, define the Baxter Q-polynomial :

$$Q^a_M(U) = \prod_{j=1}^{M} (U - U_j) \quad \text{with} \quad U_i = \frac{q u_i^2 + q^{-1} u_i^{-2}}{q + q^{-1}} \ .$$

Action of I?

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$$\begin{aligned}
(Special) \qquad \pi(\mathsf{I}(0, \kappa^*, 0, 0))Q^{sp}_M(U) &= \Lambda^M_{sp,+} Q^{sp}_M(U) , \\
(Diagonal) \qquad \pi(\mathsf{I}(\kappa, \kappa^*, 0, 0))\tilde{Q}^d_{2s}(U) &= \Lambda^{2s}_{d,+} \tilde{Q}^d_{2s}(U) - \kappa\delta^{2s}_d H(U) , \\
(Generic) \qquad \pi(\mathsf{I}(\kappa, \kappa^*, \kappa_+, \kappa_-))\tilde{Q}^g_{2s}(U) &= \Lambda^{2s}_{g,+} \tilde{Q}^g_{2s}(U) - \kappa\delta^{2s}_g \underbrace{H(U)}_{\text{inhomogeneous term}}
\end{aligned}$$

with $H(U) = \prod_{k=0}^{2s} b(q^{1/2+k-s}vu)b(q^{1/2+k-s}v^{-1}u)$. N.B. : $b(x) = x - x^{-1}$.

$\Rightarrow$ **All TQ relations from ABA are recovered**.

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$\Rightarrow$ **All TQ relations from ABA are recovered.**

- **Baxter Q-polynomial** : transition coefficient $Q^a_M(Z) = \mathcal{N}_M(\bar{u})^{-1}\langle z|\Psi^M_{a,+}(\bar{u})\rangle$.

Preliminaries (AW algebra, Leonard pairs and triple)

• The **Askey-Wilson algebra** is generated by A, A^* subject to the relations [Zhedanov,'91]

$$\left[\mathsf{A}, \left[\mathsf{A}, \mathsf{A}^*\right]_q\right]_{q^{-1}} = \rho\, \mathsf{A}^* + \omega\, \mathsf{A} + \eta \mathcal{I} \ ,$$
$$\left[\mathsf{A}^*, \left[\mathsf{A}^*, \mathsf{A}\right]_q\right]_{q^{-1}} = \rho\, \mathsf{A} + \omega\, \mathsf{A}^* + \eta^* \mathcal{I} \ .$$

• **Irreducible finite dimensional representations** are classified [Terwilliger et al.,'03] for q not a root of unity. Let $(\bar{\pi}, \bar{V})$ be a finite dim. rep. such that $\bar{\pi}(\mathsf{A}), \bar{\pi}(\mathsf{A}^*)$ are (i) diagonalizable and spectra non-degenerate ; (ii) $\bar{V}$ is irreducible. Then $\bar{\pi}(\mathsf{A}), \bar{\pi}(\mathsf{A}^*)$ is called a **Leonard pair** [Terwilliger et al.,'03].

(i) in the eigenbasis of $\bar{\pi}(\mathsf{A})$, then $\bar{\pi}(\mathsf{A}^*)$ acts as a tridiagonal matrix ;
(ii) in the eigenbasis of $\bar{\pi}(\mathsf{A}^*)$, then $\bar{\pi}(\mathsf{A})$ acts as a tridiagonal matrix.

$$\begin{aligned}
\bar{\pi}(\mathsf{A})|\theta_M\rangle &= \theta_M|\theta_M\rangle \ , \quad \bar{\pi}(\mathsf{A}^*)|\theta_M\rangle = a_{M,M+1}|\theta_{M+1}\rangle + a_{M,M}|\theta_M\rangle + a_{M,M-1}|\theta_{M-1}\rangle \ ,\\
\bar{\pi}(\mathsf{A}^*)|\theta^*_M\rangle &= \theta^*_M|\theta^*_M\rangle \ , \quad \bar{\pi}(\mathsf{A})|\theta^*_M\rangle = a^*_{M,M+1}|\theta^*_{M+1}\rangle + a^*_{M,M}|\theta^*_M\rangle + a^*_{M,M-1}|\theta^*_{M-1}\rangle \ ,
\end{aligned}$$

where $a_{0,-1} = a_{2s,2s+1} = a^*_{0,-1} = a^*_{2s,2s+1} = 0$ and

$$\theta_M = bq^{2M} + cq^{-2M} \ , \quad \theta^*_M = b^*q^{2M} + c^*q^{-2M} \ ,$$

Preliminaries (Examples of A, A*)

Let $\{q^{\pm s_3}, S_\pm\}$ be the generators of $U_q(sl_2)$ with relations :

$$[s_3, S_\pm] = \pm S_\pm \ , \quad [S_+, S_-] = \frac{q^{2s_3} - q^{-2s_3}}{q - q^{-1}} \ .$$

Casimir element : $\quad C = (q - q^{-1})^2 S_- S_+ + q^{2s_3+1} + q^{-2s_3-1}$

Coproduct : $\Delta : U_q(sl_2) \to U_q(sl_2) \otimes U_q(sl_2)$:

$$\Delta(S_+) = S_+ \otimes I\!I + q^{2s_3} \otimes S_+ \quad \Delta(S_-) = S_- \otimes q^{-2s_3} + I\!I \otimes S_- \ , \quad \Delta(q^{s_3}) = q^{s_3} \otimes q^{s_3} \ .$$

Preliminaries (Examples of A, A*)

Let $\{q^{\pm s_3}, S_\pm\}$ be the generators of $U_q(sl_2)$ with relations :

$$[s_3, S_\pm] = \pm S_\pm \ , \quad [S_+, S_-] = \frac{q^{2s_3} - q^{-2s_3}}{q - q^{-1}} \ .$$

Casimir element : $C = (q - q^{-1})^2 S_- S_+ + q^{2s_3+1} + q^{-2s_3-1}$

Coproduct : $\Delta : U_q(sl_2) \to U_q(sl_2) \otimes U_q(sl_2)$:

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• **Example 1**. [Granovskii-Zhedanov,'93] One has the map $AW \to U_q(sl_2)$.

$$\begin{aligned} \mathsf{A} \quad &\to \quad k_+ v q^{1/2} S_+ q^{s_3} + k_- v^{-1} q^{-1/2} S_- q^{s_3} + \epsilon_+ q^{2s_3} \ , \\ \mathsf{A}^* \quad &\to \quad k_+ v^{-1} q^{-1/2} S_+ q^{-s_3} + k_- v q^{1/2} S_- q^{-s_3} + \epsilon_- q^{-2s_3} \quad . \end{aligned}$$

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• **Example 2**. [Granovskii-Zhedanov,'93] One has the map $AW \to U_q(sl_2) \otimes U_q(sl_2) \otimes U_q(sl_2)$.

$$\begin{aligned} \mathsf{A} &\to \Delta(C) \otimes I\!I \;, \\ \mathsf{A}^* &\to I\!I \otimes \Delta(C) \;. \end{aligned}$$

Strategy :

AW algebra $\leftrightarrow$ Reflection algebra $\rightarrow$ (Modified) algebraic BA

• **Step 1 :** express the HAW element I in terms of a transfer matrix $t(u)$
$\rightarrow$ The double-row monodromy matrix satisfies a reflection algebra.

• **Step 2 :** the double-row monodromy matrix has **no** reference state in general $\rightarrow$ apply a gauge transformation parametrized by $\{m, \alpha, \beta, \epsilon\}$ to derive the reference states. The new transfer matrix is a combination of dynamical operators $\mathscr{A}^\epsilon(u,m)$, $\mathscr{B}^\epsilon(u,m)$, $\mathscr{C}^\epsilon(u,m)$, $\mathscr{D}^\epsilon(u,m)$ in terms of A, A^*

• **Step 3 :** Let π be such that $\pi(\mathsf{A}), \pi(\mathsf{A}^*)$ is a Leonard pair. Identify the parameters $\alpha, \beta, \epsilon = \pm$ of the gauge transformation such that there exists a reference state $\rightarrow \Omega^+ \equiv \theta_0^*$, $\Omega^- \equiv \theta_0$ of Leonard pair.

$\Rightarrow$ Diagonalization of $\bar{\pi}(\mathsf{I})$ using the (modified) Bethe ansatz !

Step 1 : Transfer matrix for the HAW operator

• The **Askey-Wilson algebra** with generators A, A^* admits a presentation in terms of a reflection algebra. Consider the R-matrix

$$R(u) = \begin{pmatrix} uq - u^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & u - u^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & u - u^{-1} & 0 \\ 0 & 0 & 0 & uq - u^{-1}q^{-1} \end{pmatrix} ,$$

then the **K-matrix** with $K_{11}(u) \equiv \mathcal{A}(u)$, $K_{12}(u) \equiv \mathcal{B}(u)$, $K_{21}(u) \equiv \mathcal{C}(u)$, $K_{22}(u) \equiv \mathcal{D}(u)$ such that : [PB,'04] (first example in [Zabrodin,'95])

$$\mathcal{A}(u) = (u^2 - u^{-2})\left(qu\,\mathsf{A} - q^{-1}u^{-1}\mathsf{A}^*\right) - (q + q^{-1})\rho^{-1}\left(\eta u + \eta^* u^{-1}\right) ,$$

$$\mathcal{D}(u) = (u^2 - u^{-2})\left(qu\,\mathsf{A}^* - q^{-1}u^{-1}\mathsf{A}\right) - (q + q^{-1})\rho^{-1}\left(\eta^* u + \eta u^{-1}\right) ,$$

$$\mathcal{B}(u) = \chi(u^2 - u^{-2})\left(\rho^{-1}\left(\left[\mathsf{A}^*, \mathsf{A}\right]_q + \frac{\omega}{q - q^{-1}}\right) + \frac{qu^2 + q^{-1}u^{-2}}{q^2 - q^{-2}}\right) ,$$

$$\mathcal{C}(u) = \rho\chi^{-1}(u^2 - u^{-2})\left(\rho^{-1}\left(\left[\mathsf{A}, \mathsf{A}^*\right]_q + \frac{\omega}{q - q^{-1}}\right) + \frac{qu^2 + q^{-1}u^{-2}}{q^2 - q^{-2}}\right) .$$

solves the **reflection algebra** [Sklyanin,'88]

$$R(u/v)\,(K(u) \otimes I\!I)\,R(uv)\,(I\!I \otimes K(v)) \;=\; (I\!I \otimes K(v))\,R(uv)\,(K(u) \otimes I\!I)\,R(u/v) .$$

Step 1 : Transfer matrix for the HAW operator

• Given a reflection algebra 'RKRK', a **generating function** for mutually commuting quantities is provided by the so-called transfer matrix [Sklyanin,'88]

$t(u) = \text{tr}\left(K^+(u)K(u)\right)$ where

$$K^+(u) = \begin{pmatrix} qu\kappa + q^{-1}u^{-1}\kappa^* & \kappa_+(q^2u^2 - q^{-2}u^{-2}) \\ \kappa_-\rho(q^2u^2 - q^{-2}u^{-2}) & qu\kappa^* + q^{-1}u^{-1}\kappa \end{pmatrix} ,$$

satisfies the "dual" reflection equation given by [DeVega-Gonzales-Ruiz,'93]. Using the K-matrix in terms of A, A^* one gets :

$$t(u) = (q^2u^2 - q^{-2}u^{-2})(u^2 - u^{-2}) \underbrace{\left(\kappa\,\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+\left[\mathsf{A}, \mathsf{A}^*\right]_q + \kappa_-\left[\mathsf{A}^*, \mathsf{A}\right]_q\right)}_{= \textbf{HAW element } \mathsf{I}(\kappa,\kappa^*,\kappa_+,\kappa_-)} + \underbrace{\mathcal{F}_0(u)}_{\text{scalar function}} ,$$

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$$t(u) = (q^2u^2 - q^{-2}u^{-2})(u^2 - u^{-2}) \underbrace{\left(\kappa\,\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+\left[\mathsf{A}, \mathsf{A}^*\right]_q + \kappa_-\left[\mathsf{A}^*, \mathsf{A}\right]_q\right)}_{= \textbf{HAW element } \mathsf{I}(\kappa, \kappa^*, \kappa_+, \kappa_-)} + \underbrace{\mathcal{F}_0(u)}_{\text{scalar function}} ,$$

Diagonalization of $t(u)$ for finite irreps. $\Rightarrow$ Diagonalization of I

<u>Technical tools :</u>
→ Theory of Leonard pairs [Terwilliger et al.,03],
→ Gauge transformations for open systems [Cao et al.,'03],
→ Modified algebraic Bethe ansatz [Belliard et al., '13-'15].

Step 2 : Gauge transformations and dynamical operators

• **Gauge transformations :** There is no reference state such that $\mathcal{C}(u)|\Omega\rangle = 0$!
Let $\epsilon = \pm 1$, α, β be generic complex parameters and m be an integer. Introduce the covariant (resp. contravariant) vectors [Cao et al., '03]

$$|X^\epsilon(u,m)\rangle = \begin{pmatrix} \alpha q^{\epsilon m} u^\epsilon \\ 1 \end{pmatrix}, \quad |Y^\epsilon(u,m)\rangle = \begin{pmatrix} \beta q^{-\epsilon m} u^\epsilon \\ 1 \end{pmatrix}$$

$$\langle \tilde{X}^\epsilon(u,m)| = -\epsilon \frac{q^{-\epsilon} u^{-\epsilon}}{\gamma_{m-1}} \begin{pmatrix} -1 & \alpha q^{\epsilon m} u^\epsilon \end{pmatrix}, \quad \langle \tilde{Y}^\epsilon(u,m)| = -\epsilon \frac{q^{-\epsilon} u^{-\epsilon}}{\gamma_{m+1}} \begin{pmatrix} 1 & -\beta q^{-\epsilon m} u^\epsilon \end{pmatrix}$$

where $\gamma^\epsilon(u,m) = \alpha^{\frac{1-\epsilon}{2}} \beta^{\frac{\epsilon+1}{2}} q^{-m} u - \alpha^{\frac{\epsilon+1}{2}} \beta^{\frac{1-\epsilon}{2}} q^m u^{-1}$.
Applying the gauge transformation to $K(u)$, the entries of $K(u|m)$ are given by :

$$\mathscr{A}^\epsilon(u,m) = \langle \tilde{Y}^\epsilon(u,m-2)|K(u)|X^\epsilon(u^{-1},m)\rangle, \quad \mathscr{B}^\epsilon(u,m) = \langle \tilde{Y}^\epsilon(u,m)|K(u)|Y^\epsilon(u^{-1},m)\rangle,$$

$$\mathscr{C}^\epsilon(u,m) = \langle \tilde{X}^\epsilon(u,m)|K(u)|X^\epsilon(u^{-1},m)\rangle,$$

$$\mathscr{D}^\epsilon(u,m) = \frac{\gamma^\epsilon(1,m+1)}{\gamma^\epsilon(1,m)} \langle \tilde{X}^\epsilon(u,m+2)|K(u)|Y^\epsilon(u^{-1},m)\rangle - \frac{(q-q^{-1})\gamma^\epsilon(u^{-2},m+1)}{(qu^2-q^{-1}u^{-2})\gamma^\epsilon(1,m)} \mathscr{A}^\epsilon(u,m).$$

The transfer matrix reads : $t(u) = \text{tr}\left(K^+(u)K(u)\right) = \text{tr}\left(\tilde{K}^+(u|m)K(u|m)\right)$.

$\Rightarrow$ **Given $\epsilon = \pm$ fixed, $\exists$ a reference state !**

Step 2 : Gauge transformations and dynamical operators

- **Commutation relations between the dynamical operators**

$$\begin{aligned}
\mathscr{B}^{\epsilon}(u,m+2)\mathscr{B}^{\epsilon}(v,m) &= \mathscr{B}^{\epsilon}(v,m+2)\mathscr{B}^{\epsilon}(u,m),\\
\mathscr{A}^{\epsilon}(u,m+2)\mathscr{B}^{\epsilon}(v,m) &= f(u,v)\mathscr{B}^{\epsilon}(v,m)\mathscr{A}^{\epsilon}(u,m)\\
&\quad +g(u,v,m)\mathscr{B}^{\epsilon}(u,m)\mathscr{A}^{\epsilon}(v,m)+w(u,v,m)\mathscr{B}^{\epsilon}(u,m)\mathscr{D}^{\epsilon}(v,m),\\
\mathscr{D}^{\epsilon}(u,m+2)\mathscr{B}^{\epsilon}(v,m) &= h(u,v)\mathscr{B}^{\epsilon}(v,m)\mathscr{D}^{\epsilon}(u,m),\\
&\quad +k(u,v,m)\mathscr{B}^{\epsilon}(u,m)\mathscr{D}^{\epsilon}(v,m)+n(u,v,m)\mathscr{B}^{\epsilon}(u,m)\mathscr{A}^{\epsilon}(v,m),\\
\mathscr{C}(u,m+2)\mathscr{B}(v,m) &= \mathscr{B}(v,m-2)\mathscr{C}(u,m)\\
&\quad +q(u,v,m)\mathscr{A}(v,m)\mathscr{D}(u,m)+r(u,v,m)\mathscr{A}(u,m)\mathscr{D}(v,m)\\
&\quad +s(u,v,m)\mathscr{A}(u,m)\mathscr{A}(v,m)+x(u,v,m)\mathscr{A}(v,m)\mathscr{A}(u,m)\\
&\quad +y(u,v,m)\mathscr{D}(u,m)\mathscr{A}(v,m)+z(u,v,m)\mathscr{D}(u,m)\mathscr{D}(v,m),
\end{aligned}$$

- **Transfer matrix in the dynamical operators**

$$t(u)=a(u,m)\mathscr{A}^{\epsilon}(u,m)+d(u,m)\mathscr{D}^{\epsilon}(u,m)+b(u,m)\mathscr{B}^{\epsilon}(u,m)+c(u,m)\mathscr{C}^{\epsilon}(u,m)$$

Step 3 : Reference states

The **gauge transformation** depends on the parameters α, β and ϵ. $\Rightarrow$ We fix α, β such that $|\Omega^{\pm}\rangle$ is **a reference state**.

Lemma 1. Let m_0 be an integer. If the parameters α, β are such that :

$$(q^2 - q^{-2})\alpha c^* q^{m_0} = 1 \qquad (\text{resp. } (q^2 - q^{-2})\beta c^* q^{-m_0} = 1)$$

then $|\Omega^+\rangle \equiv |\theta_0^*\rangle$ satisfies

$$\bar{\pi}(\mathscr{C}^+(u, m_0))|\Omega^+\rangle = 0 \qquad (\text{resp. } \bar{\pi}(\mathscr{B}^+(u, m_0))|\Omega^+\rangle = 0\,).$$

Lemma 2. Let m_0 be an integer. If the parameters α, β are such that :

$$(q^2 - q^{-2})\chi^{-1}\alpha b q^{-m_0} = -1 \qquad (\text{resp. } (q^2 - q^{-2})\chi^{-1}\beta b q^{m_0} = -1)$$

then $|\Omega^-\rangle \equiv |\theta_0\rangle$ satisfies

$$\bar{\pi}(\mathscr{C}^-(u, m_0))|\Omega^-\rangle = 0 \qquad (\text{resp. } \bar{\pi}(\mathscr{B}^-(u, m_0))|\Omega^-\rangle = 0\,).$$

$\Rightarrow$ **The HAW operator will be diagonalized starting either from $|\Omega^+\rangle \equiv |\theta_0^*\rangle$ or $|\Omega^-\rangle \equiv |\theta_0\rangle$**

Special case : $\kappa^* = \kappa_\pm = 0$ or $\kappa = \kappa_\pm = 0$

- For this choice of parameters, the HAW element reduces to :

$$\mathsf{I}(\kappa, 0, 0, 0) = \kappa\,\mathsf{A} \qquad \text{or} \qquad \mathsf{I}(0, \kappa^*, 0, 0) = \kappa^*\,\mathsf{A}^* \ .$$

- In terms of the dynamical operators of ABA, according to the choice of reference state $|\Omega^\pm\rangle$ one has $(\bar{\eta}(u) = (q+q^{-1})\rho^{-1}\left(\eta u + \eta^* u^{-1}\right)$, $b(x) = x - x^{-1})$:

$$\mathsf{A} = \frac{u^{-1}}{b(u^2)}\left(\frac{1}{b(qu^2)}\mathscr{A}^-(u,m) + \frac{1}{b(q^2u^2)}\mathscr{D}^-(u,m)\right) + \frac{(q\,u\,\bar{\eta}(u) + q^{-1}u^{-1}\bar{\eta}(u^{-1}))}{b(u^2)b(q^2u^2)} ,$$

$$\mathsf{A}^* = \frac{u}{b(u^2)}\left(\frac{1}{b(qu^2)}\mathscr{A}^+(u,m) + \frac{1}{b(q^2u^2)}\mathscr{D}^+(u,m)\right) + \frac{(q\,u\,\bar{\eta}(u^{-1}) + q^{-1}u^{-1}\bar{\eta}(u))}{b(u^2)b(q^2u^2)} .$$

- We define the string of dynamical operators of ABA $(\bar{u} = \{u_1, u_2, \ldots, u_M\})$:

$$B^\epsilon(\bar{u}, m, M) = \mathscr{B}^\epsilon(u_1, m + 2(M-1)) \cdots \mathscr{B}^\epsilon(u_M, m) .$$

- According to the choice of reference state $|\Omega^\pm\rangle$, we introduce the Bethe states :

$$|\Psi^M_{sp,-}(\bar{u}, m_0)\rangle = \bar{\pi}(B^-(\bar{u}, m_0, M))|\Omega^-\rangle ,$$
$$|\Psi^M_{sp,+}(\bar{u}, m_0)\rangle = \bar{\pi}(B^+(\bar{u}, m_0, M))|\Omega^+\rangle .$$

Proposition 1 :

$$\bar{\pi}\left(\mathsf{I}(\kappa,0,0,0)\right)|\Psi^M_{sp,-}(\bar{u},m_0)\rangle = \frac{\kappa}{2}q^{\frac{1}{2}(\nu+\nu')}\left(e^{-\mu}q^{-2s+2M}+e^{\mu}q^{2s-2M}\right)|\Psi^M_{sp,-}(\bar{u},m_0)\rangle$$

where the set $\bar{u}=\{u_1,u_2,\dots,u_M\}$ satisfies the Bethe equations :

$$\prod_{j=1,j\neq i}^{M}\left(\frac{b(u_i/(qu_j))b(u_iu_j)}{b(qu_i/u_j)b(q^2u_iu_j)}\right)=$$

$$\frac{\left(qe^{\mu'}u_i+q^{-1}e^{\mu}u_i^{-1}\right)\left(qe^{-\mu}u_i+q^{-1}e^{\mu'}u_i^{-1}\right)b\left(q^{\frac{1}{2}-s}vu_i\right)b\left(q^{\frac{1}{2}-s}v^{-1}u_i\right)}{\left(e^{\mu'}u_i+e^{-\mu}u_i^{-1}\right)\left(e^{\mu}u_i+e^{\mu'}u_i^{-1}\right)b\left(q^{s+\frac{1}{2}}vu_i\right)b\left(q^{s+\frac{1}{2}}v^{-1}u_i\right)}$$

for $i=1,\dots,M$.

Remark :

$\longrightarrow$ Spectrum has the typical form for Leonard pairs ($\theta_M = bq^{2M}+cq^{-2M}$)

$\longrightarrow$ Similar result for the special case $\kappa=0$ (**Proposition 1*** **:**) :

$$\bar{\pi}\left(\mathsf{I}(0,\kappa^*,0,0)\right)|\Psi^M_{sp,+}(\bar{u},m_0)\rangle = \frac{\kappa^*}{2}q^{\frac{1}{2}(\nu+\nu')}\left(e^{-\mu'}q^{2s-2M}+e^{\mu'}q^{-2s+2M}\right)|\Psi^M_{sp,+}(\bar{u},m_0)\rangle$$

Diagonal case : $\kappa_\pm = 0,\ \kappa, \kappa^* \neq 0$

Proposition 2 : For $\epsilon = \pm 1$, one has :

$$\bar{\pi}\left(\mathsf{I}(\kappa,\kappa^*,0,0)\right)|\Psi^{2s}_{d,\epsilon}(\bar{u},m_0)\rangle = \Lambda^{2s}_{d,\epsilon}|\Psi^{2s}_{d,\epsilon}(\bar{u},m_0)\rangle$$

with

$$\Lambda^{2s}_{d,+} = \kappa^*\theta^*_{2s} + \kappa e^{\mu-\mu'} b\Big((v^2+v^{-2})[2s]_q + 2e^{\mu'}\cosh(\mu) - q\sum_{j=1}^{2s}(qu_j^2 + q^{-1}u_j^{-2})\Big)\ ,$$

$$\Lambda^{2s}_{d,-} = \kappa\theta_{2s} + \kappa^* e^{\mu'-\mu} c^*\Big((v^2+v^{-2})[2s]_q + 2e^{\mu}\cosh(\mu') - q^{-1}\sum_{j=1}^{2s}(qu_j^2 + q^{-1}u_j^{-2})\Big)\ ,$$

where the set $\bar{u}$ satisfies the (inhomogeneous) Bethe equations for $i = 1, \dots, 2s$:

$$\frac{b(u_i^2)}{b(qu_i^2)}(\kappa\, u_i + \kappa^* u_i^{-1})\prod_{j=1,j\neq i}^{2s} f(u_i,u_j)\Lambda_1^\epsilon(u_i) - q^{-\epsilon}u_i^{-2\epsilon}(q\kappa^* u_i + q^{-1}\kappa u_i^{-1})\prod_{j=1,j\neq i}^{2s} h(u_i,u_j)\Lambda_2^\epsilon(u_i)$$

$$\underbrace{+(-1)^{2s}\,\epsilon(q-q^{-1})^{-1}q^\epsilon\kappa^{(1+\epsilon)/2}\kappa^{*(1-\epsilon)/2}\delta_d\,\frac{u_i^{-2\epsilon}b(u_i^2)\prod_{k=0}^{2s} b(q^{1/2+k-s}vu_i)b(q^{1/2+k-s}v^{-1}u_i)}{\prod_{j=1,j\neq i}^{2s} b(u_iu_j^{-1})b(qu_iu_j)}}_{\text{inhomogeneous term}} = 0.$$

Generic case : $\kappa_\pm \neq 0$, $\kappa, \kappa^* \neq 0$

Proposition 3 : For $\epsilon = \pm 1$, one has :

$$\bar{\pi}\left(\mathsf{I}(\kappa,\kappa^*,\kappa_+,\kappa_-)\right)|\Psi^{2s}_{g,\epsilon}(\bar{u},m_0)\rangle = \Lambda^{2s}_{g,\epsilon}|\Psi^{2s}_{g,\epsilon}(\bar{u},m_0)\rangle$$

with

$$\Lambda^{2s}_{g,+} = \kappa^*\theta^*_{2s} + \kappa^*\theta^*_{2s}|_{\mu'\to\xi}\frac{\cosh(\mu)}{\cosh(\xi')} + \left(-\frac{\kappa^*}{2\cosh(\xi')}\theta^*_{3s+1/2}|_{\mu'\to\mu'+\xi} + (-1)^{2s+1}\delta_g[2s+1]_q\right)(v^2+v^{-2})$$
$$-\omega\frac{(\chi^{-1}\kappa_+ + \chi\kappa_-)}{(q-q^{-1})} + \left(\frac{\kappa^* q^{\nu+\nu'}}{4\cosh(\xi')}(q-q^{-1})(q^{2s}e^{\mu'+\xi} - q^{-2s}e^{-\mu'-\xi}) - (-1)^{2s+1}\delta_g\right)\sum_{j=1}^{2s}(qu_j^2 + q^{-1}u_j^{-2}$$

$$\Lambda^{2s}_{g,-} = \kappa\theta_{2s} + \kappa\theta_{2s}|_{\mu\to-\xi'}\frac{\cosh(\mu')}{\cosh(\xi)} + \left(-\frac{\kappa}{2\cosh(\xi)}\theta_{3s+1/2}|_{\mu\to\mu-\xi'} + (-1)^{2s+1}\delta_g[2s+1]_q\right)(v^2+v^{-2})$$
$$-\omega\frac{(\chi^{-1}\kappa_+ + \chi\kappa_-)}{(q-q^{-1})} + \left(\frac{\kappa}{2\cosh(\xi)}(q-q^{-1})\theta_{2s}|_{\mu\to\mu-\xi} - (-1)^{2s+1}\delta_g\right)\sum_{j=1}^{2s}(qu_j^2 + q^{-1}u_j^{-2}),$$

where the set $\bar{u}$ satisfies the (inhomogeneous) Bethe equations :

$$-\frac{b(u_i^2)}{b(qu_i^2)}\Delta_g(u_i)\prod_{j=1,j\neq i}^{2s} f(u_i,u_j)\Lambda_1^\epsilon(u_i) + \Delta_g(q^{-1}u_i^{-1})\prod_{j=1,j\neq i}^{2s} h(u_i,u_j)\Lambda_2^\epsilon(u_i)$$
$$-(-1)^{2s}\delta_g(q-q^{-1})^{-1}\frac{u_i^{-\epsilon}b(u_i^2)\prod_{k=0}^{2s} b(q^{1/2+k-s}vu_i)b(q^{1/2+k-s}v^{-1}u_i)}{\prod_{j=1,j\neq i}^{2s} b(u_iu_j^{-1})b(qu_iu_j)} = 0$$

for $i = 1, \ldots, 2s$.

Comment 1 : BA equations, an alternative presentation

From the **algebraic Bethe ansatz** calculations, for each case $(a \in \{sp, d, g\})$ the Bethe equations enjoy the symmetries :

$$u_i \longleftrightarrow \pm q^{-1} u_i^{-1} \;, \quad u_i \longleftrightarrow -u_i$$
$$u_j \longleftrightarrow \pm q^{-1} u_j^{-1} \;, \quad u_j \longleftrightarrow -u_j \quad \text{for} \quad j \neq i \;.$$

Example (special case). For $i = 1, \ldots, M$

$$\prod_{j=1, j\neq i}^{M} \left(\frac{b(u_i/(qu_j))b(u_iu_j)}{b(qu_i/u_j)b(q^2u_iu_j)} \right) =$$
$$\frac{\left(qe^{\mu'}u_i + q^{-1}e^{\mu}u_i^{-1}\right)\left(qe^{-\mu}u_i + q^{-1}e^{\mu'}u_i^{-1}\right) b\left(q^{\frac{1}{2}-s}vu_i\right) b\left(q^{\frac{1}{2}-s}v^{-1}u_i\right)}{\left(e^{\mu'}u_i + e^{-\mu}u_i^{-1}\right)\left(e^{\mu}u_i + e^{\mu'}u_i^{-1}\right) b\left(q^{s+\frac{1}{2}}vu_i\right) b\left(q^{s+\frac{1}{2}}v^{-1}u_i\right)} \,.$$

$\Rightarrow$ Each system of Bethe equations admits an **alternative presentation** in terms of the 'symmetrized' variables (Why interesting ? Numerics/Asymptotic $M \to \infty$)

$$U_i = \frac{qu_i^2 + q^{-1}u_i^{-2}}{q + q^{-1}} \quad \text{with} \quad i = 1, ..., M.$$

Define $\bar{U}_i = \{U_1, U_2, ..., U_{i-1}, U_{i+1}, ..., U_M\}$ with 'symmetrized' variables $U_i = \frac{qu_i^2+q^{-1}u_i^{-2}}{q+q^{-1}}$. Define the polynomial of maximal degree M for $a = sp$ and $M = 2s+1$ for $a = d, g$ and denote $p(\ell) \equiv odd/even$:

$$P_a^M = \sum_{k=0}^{M-1} \frac{(-1)^k (q+q^{-1})^{M-1}}{2^{M-1-k}} \underbrace{e_k(\bar{U}_i)}_{\text{elem. sym. poly.}} \left(\sum_{\ell=0}^{M-1-k} \binom{M-1-k}{\ell} \frac{(q-q^{-1})^l U_i^{M-1-k-\ell}}{(q^2+q^{-2})^{1+k+\ell-M}} \underbrace{g_0^{2[\frac{\ell}{2}]}(U_i)}_{\text{deg. } \ell} \underbrace{g_{a,\epsilon}^{(p[\ell])}(U_i)}_{\text{deg. } 1,2,3} \right)$$

$$+\bar{\Delta}_a H(U_i)$$

with

$$\bar{\Delta}_a = \begin{cases} 0 & \text{for } a = sp, \\ -2(-1)^{2s} \epsilon \frac{q^{-(\nu+\nu')/2+\epsilon}}{(q-q^{-1})} \kappa^{*\,\frac{1-\epsilon}{2}} \kappa^{\frac{1+\epsilon}{2}} \delta_d & \text{for } a = d, \\ 2\frac{(-1)^{2s} q^{-(\nu+\nu')/2}}{(q-q^{-1})} \delta_g & \text{for } a = g \end{cases}$$

and

$$H(U_i) = (q+q^{-1})^{2s+1} \sum_{k=0}^{2s+1} (-1)^k e_k(X_0, X_1, \ldots, X_{2s}) U_i^{2s+1-k} \quad \text{with} \quad X_k = \frac{q^{2k-2s} v^2 + q^{-2k+2s} v^{-2}}{q+q^{-1}} .$$

The **BA equations** given by $E_a(u_i, \bar{u}_i) = 0$ for all i. **Alternative presentation :**

$$E_a(u_i, \bar{u}_i) = \frac{u_i^{-\epsilon} b(u_i^2) q^{(\nu+\nu')/2}}{2\prod_{j\neq i}^M b(u_i/u_j) b(q u_i u_j)} P_a^M(U_i, \bar{U}_i) \;\Rightarrow\; P_a^M(U_i, \bar{U}_i) = 0$$

Strategy :

(Modified) ABA $\rightarrow$ (in)homogeneous TQ relations $\leftarrow$ Inf. dim. reps. of AW algebra

• **Step 1 :** extract the TQ relations from ABA calculations.

• **Step 2 :** Let (π, V) be an infinite dim. rep. of AW algebra on which $\pi(\mathsf{A}), \pi(\mathsf{A}^*)$ act. Compute the action of $\pi(\mathsf{I})$ on the Baxter Q-polynomial.

$\rightarrow$ Action of $\pi(\mathsf{I})$ on Q-polynomial $\Leftrightarrow$ (in)homogeneous TQ relations

Step 1 : T-Q relations of (in)homogeneous type

From the algebraic Bethe ansatz calculations, for each case $(a \in \{sp, d, g\})$ one has :

$$\underbrace{\Lambda^{2s}_{a,+}}_{\text{spectrum of I}} = \frac{1}{(u^2 - u^{-2})}\left(\frac{\Lambda_1^{a,+}(u)}{(qu^2 - q^{-1}u^{-2})}\prod_{j=1}^{2s} f(u,u_j) + \frac{\Lambda_2^{a,+}(u)}{(q^2u^2 - q^{-2}u^{-2})}\prod_{j=1}^{2s} h(u,u_j)\right)$$

$$+\bar{\delta}_a^{2s}\frac{\prod_{k=0}^{2s} b(q^{1/2+k-s}vu)b(q^{1/2+k-s}v^{-1}u)}{\prod_{i=1}^{2s} b(uu_i^{-1})b(quu_i)} \,.$$

Introduce the q-difference operators $T_\pm$ such that $T_\pm(f(u^2)) = f(q^{\pm 2}u^2)$. By elementary computations :

$$\prod_{j=1}^{M} h(u,u_j) = \frac{T_+ Q_M(U)}{Q_M(U)} \quad \text{and} \quad \prod_{j=1}^{M} f(u,u_j) = \frac{T_- Q_M(U)}{Q_M(U)} \quad \text{with} \quad Q_M(U) = \prod_{j=1}^{M}(U - U_j)$$

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$$\underbrace{\Lambda^{2s}_{a,+}}_{\text{spectrum of I}} = \frac{1}{(u^2 - u^{-2})}\left(\frac{\Lambda^{a,+}_1(u)}{(qu^2 - q^{-1}u^{-2})}\prod_{j=1}^{2s} f(u,u_j) + \frac{\Lambda^{a,+}_2(u)}{(q^2u^2 - q^{-2}u^{-2})}\prod_{j=1}^{2s} h(u,u_j)\right)$$

$$+\bar{\delta}^{2s}_a \frac{\prod_{k=0}^{2s} b(q^{1/2+k-s}vu)b(q^{1/2+k-s}v^{-1}u)}{\prod_{i=1}^{2s} b(uu_i^{-1})b(quu_i)} .$$

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Proposition 4 : The eigenvalues $\Lambda^{2s}_{a,+}$ of the **Heun-Askey-Wilson** operator $\bar{\pi}(\mathsf{I}(\kappa,\kappa^*,\kappa_+,\kappa_-))$ are given by an **(in)homogeneous Baxter T-Q relation** of the form

$$\Lambda^{2s}_{g,+} Q_{2s}(U) = u\Delta_g(q^{-1}u^{-1})\Lambda^{a,+}_2(u)T_+Q_{2s}(U) + u\Delta_g(u)\Lambda^{a,+}_1(u)\frac{(q^2u^2 - q^{-2}u^{-2})}{(qu^2 - q^{-1}u^{-2})}T_-Q_{2s}(U)$$

$$+\frac{(q+q^{-1})^2}{\rho}\frac{(\kappa\eta^* + \kappa^*\eta + (\kappa\eta + \kappa^*\eta^*)U)}{\left((u^2 - u^{-2})(q^2u^2 - q^{-2}u^{-2})\right)}Q_{2s}(U) + \bar{\delta}'^{2s}_d H(U)$$

Step 2 : a q-difference realization of HAW operator

Based on [Terwilliger,'03], one has for instance the following q-difference realization of the Askey-Wilson algebra :

$$\begin{aligned}\pi(\mathsf{A}) &= q^{-1}z^{-1}\phi(z)(T_+ - 1) + q^{-1}z\phi(z^{-1})(T_- - 1)\\ &\quad + \frac{1}{2}q^{\frac{\nu+\nu'}{2}}e^{-\mu'}q^{2s}\left(2e^{\mu'}\cosh(\mu) - (v^2+v^{-2})q^{-2s-1} + q^{-1}(z+z^{-1})\right) ,\\ \pi(\mathsf{A}^*) &= \phi(z)(T_+ - 1) + \phi(z^{-1})(T_- - 1) + \frac{1}{2}q^{(\nu+\nu')/2}(e^{\mu'}q^{-2s} + e^{-\mu'}q^{2s})\end{aligned}$$

where

$$\phi(z) = \frac{1}{2}q^{\frac{\nu+\nu'}{2}}e^{-\mu'}q^{2s}\frac{(1+qe^{-\mu+\mu'}z)(1+qe^{\mu+\mu'}z)(1-q^{-2s}v^2z)(1-q^{-2s}v^{-2}z)}{(1-z^2)(1-q^2z^2)} .$$

⇒ What is the action of the Heun-Askey-Wilson q-difference operator on the Baxter Q-polynomial ?

$$\pi(\mathsf{I}) \mapsto \pi(\kappa\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+[\mathsf{A},\mathsf{A}^*]_q + \kappa_-[\mathsf{A}^*,\mathsf{A}]_q)$$

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⇒ What is the action of the Heun-Askey-Wilson q-difference operator on the Baxter Q-polynomial ?

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- For the special and diagonal case : T-Q relations are recovered.
- For the generic case and a different realization of the HAW element, the T-Q relations are recovered.

TQ relations from the action of q-diff. HAW operator $\pi(\mathsf{I})$

Proposition 5 :
Let (π, V) be the infinite dim. vector space such that $\pi(\mathsf{A}), \pi(\mathsf{A}^*)$ are second order q-difference operators that satisfy the Askey-Wilson relations.

- For each case $a \in \{sp, d, g\}$, define the Baxter Q-polynomial :

$$Q^a_M(U) = \prod_{j=1}^{M} (U - U_j) \quad \text{with} \quad U_i = \frac{qu_i^2 + q^{-1}u_i^{-2}}{q + q^{-1}} \ .$$

The **HAW operator** $\pi(\mathsf{I})$ with $\mathsf{I} = \kappa\mathsf{A} + \kappa^*\mathsf{A}^* \ + \kappa_+ \left[\mathsf{A}, \mathsf{A}^*\right]_q + \kappa_- \left[\mathsf{A}^*, \mathsf{A}\right]_q$ acts as

$$\begin{array}{lrcl} (Special) & \pi(\mathsf{I}(0,\kappa^*,0,0))Q^{sp}_M(U) & = & \Lambda^M_{sp,+} Q^{sp}_M(U) \ , \\ (Diagonal) & \pi(\mathsf{I}(\kappa,\kappa^*,0,0))\tilde{Q}^d_{2s}(U) & = & \Lambda^{2s}_{d,+} \tilde{Q}^d_{2s}(U) - \kappa\delta^{2s}_d H(U) \ , \\ (Generic) & \pi(\mathsf{I}(\kappa,\kappa^*,\kappa_+,\kappa_-))\tilde{Q}^g_{2s}(U) & = & \Lambda^{2s}_{g,+} \tilde{Q}^g_{2s}(U) - \kappa\delta^{2s}_g \underbrace{H(U)}_{\text{inhomogeneous term}} \end{array}$$

with $H(U) = \prod_{k=0}^{2s} b(q^{1/2+k-s}vu)b(q^{1/2+k-s}v^{-1}u)$.

$\Rightarrow$ **Derive the inhomogeneous term solely using the theory of Leonard pairs ?**

Comment 1 : More about the inhomogeneous term...

In the **algebraic Bethe ansatz** calculations, a crucial ingredient is the following Lemma (diagonal case).

Lemma :

$$(*) \quad \bar{\pi}(\mathscr{B}^{\epsilon}(u, m_0 + 4s))|\Psi^{2s}_{d,\epsilon}(\bar{u}, m_0)\rangle - \delta_d \frac{u^{-\epsilon} b(u^2) H(U)}{\prod_{i=1}^{2s} b(uu_i^{-1}) b(q^{-1}u^{-1}u_i^{-1})} |\Psi^{2s}_{d,\epsilon}(\bar{u}, m_0)\rangle + \delta_d \sum_{i=1}^{2s} \frac{u_i^{-\epsilon} b(u_i^2) H(U_i)}{b(uu_i^{-1}) b(q^{-1}u^{-1}u_i^{-1}) \prod_{j=1, j\neq i}^{2s} b(u_i u_j^{-1}) b(q^{-1}u_i^{-1}u_j^{-1})} |\Psi^{2s}_{d,\epsilon}(\{u, \bar{u}_i\}, m_0)$$

In ABA calculations, the 1^{st} term produces the inhomogeneous term $H(U)$.

∃ examples of inhomogeneous terms conjectured by [Cao et al. '13] → Proof using SoV approach, for certain representations [Niccoli et al.,Belliard et al.].

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Lemma :

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In ABA calculations, the 1^{st} term produces the inhomogeneous term $H(U)$.

$\exists$ examples of inhomogeneous terms conjectured by [Cao et al. '13] $\rightarrow$ Proof using SoV approach, for certain representations [Niccoli et al.,Belliard et al.].

$\Rightarrow$ Proof using the representation theory of AW algebra only (Leonard pairs)

The inhomogeneous term is simply related with the characteristic polynomial of the Leonard triple $\mathsf{A}^\diamond$: $H(U) \sim \prod_{k=0}^{2s} (U - X_k)$

$$(*) \leftrightarrow \prod_{k=0}^{2s} \left(\frac{i(q - q^{-1})}{\sqrt{\rho}} \mathsf{A}^\diamond - X_k \right) = 0 \quad \text{with} \quad X_k = \frac{v^2 q^{2k-2s} + v^{-2} q^{-2k+2s}}{q + q^{-1}}$$

Comment 2 : The Q-polynomial as a transition coefficient

In the infinite dimensional representation $\mathcal{V}$, note that :

$$\pi([\mathsf{A}^*,\mathsf{A}]_q) = -\frac{q^{\nu+\nu'}(q-q^{-1})}{4}\Big((q+q^{-1})(z+z^{-1}) - (q^{2s+1}+q^{-2s-1})(v^2+v^{-2}) + 4\cosh(\mu)\cosh(\mu')\Big)$$

$\Rightarrow$ It is possible to **interpret** the **Baxter Q-polynomial** as a transition coefficient between Bethe states and an eigenbasis of the dynamical operators $\pi(\mathscr{B}^+(u,m))$. Consider the choice of gauge parameter $\beta = 0$. One has :

$$\pi(\mathscr{B}^+(u,m)) = \frac{\chi b(u^2)}{\alpha(q-q^{-1})q^{2+m}u}\left(U - \underbrace{\frac{z+z^{-1}}{q+q^{-1}}}_{\equiv Z}\right).$$

It follows :

$$Q_M(Z) = \mathcal{N}_M(\bar{u})^{-1}\langle z|\Psi^M_{a,+}(\bar{u})\rangle$$

$$\mathcal{N}_M(\bar{u}) = (-1)^M\frac{(q+q^{-1})^M}{2^M}\left(q^{\frac{\nu+\nu'}{2}}e^{-\mu'}q^{2s-M-1}\right)^M\prod_{i=1}^{M}\frac{b(u_i^2)}{u_i}$$

where $\langle z|\Psi^M_{a,+}(\bar{u})\rangle = \pi(\mathscr{B}^+(u_1,m_0+2(M-1))\cdots\mathscr{B}^+(u_M,m_0))|_{\beta=0}$

Examples

• **The quantum Euler top revisited.** Related works : [Wiegmann-Zabrodin,'95], [Turbiner,'16]

$$I(\kappa,\kappa^*,\kappa_+,\kappa_-) = t_{+-}S_+S_- + t_{00}q^{2s_3} + t'_{00}q^{-2s3} + t_{++}S_+^2 + t_{--}S_-^2$$
$$+t_{0+}S_+q^{s_3} + t_{0-}S_-q^{s_3} + t'_{0+}S_+q^{-s_3} + t'_{0-}S_-q^{-s_3} + I_0 \ ,$$

• Many examples of **three sites Heisenberg spin chain** (a toy model for generalizations to N sites, related with recent works of [Kuniba-Okada-Pasquier-Honeyama,'18,'19])

Example $s = 1/2$ **:**

$$\frac{I(\kappa,\kappa^*,\kappa_+,\kappa_-)}{2(q-q^{-1})^2} = \sum_a\sum_{i,j} J^a_{ij}\sigma^a_i\sigma^a_j + \sum_{a\neq b}\sum_{i,j} K^{ab}_{ij}\sigma^a_i\sigma^b_j + \sum_{\{a,b,c\}}\sum_{i,j,k} L^{abc}_{ijk}\sigma^a_i\sigma^b_j\sigma^c_k + \sum_i b^z_i\sigma^z_i + J_0 \ ,$$

→ **More general** than the well-known case $\kappa = \kappa^* \neq 0$ and $\kappa_\pm = 0$ [Sklyanin,'88]
→ **BA solution with inhomogeneous term** for $\kappa = \kappa^* \neq 0$ and $\kappa_\pm = 0 \neq$ BA sol. in [Sklyanin,'88]
→ **Three body terms** (atoms on a triangle, quantum optics...)

The quantum Euler top revisited

$$I(\kappa,\kappa^*,\kappa_+,\kappa_-) = t_{+-}S_+S_- + t_{00}q^{2s_3} + t'_{00}q^{-2s_3} + t_{++}S_+^2 + t_{--}S_-^2 + t_{0+}S_+q^{s_3} + t_{0-}S_-q^{s_3} + t'_{0+}S_+q^{-s_3} + t'_{0-}S_-q^{-s_3} + I_0 \ ,$$

Spin $s = 1$	Direct diagonalization	Diagonalization via ABA $\Lambda^2_{a,+}$	Bethe roots $\{U_1, \ldots, U_{2s}\}$
$\kappa = 1,\ \kappa^* = 0.25,$ $\kappa_\pm = 0$	17.8556	17.8556	{-7.53525, -2.25731}
	10.5068 + 9.82751 i	10.5068 + 9.82751 i	{-2.89915 - 7.58381 i, -0.941451 - 0.375642 i}
	10.5068 - 9.82751 i	10.5068 - 9.82751 i	{-2.89915 + 7.58381 i, -0.941451 + 0.375642 i}
$\kappa = \frac{10i(1+\sqrt{5})}{\sqrt{3}},\ \kappa^* = \frac{20i}{\sqrt{3}},$ $\kappa_+ = \frac{\sqrt{3}}{2},\ \kappa_- = -\frac{3}{2}$ $\chi = -\frac{40}{3\sqrt{3}}$	-2394.67 + 986.732 i	-2394.67 + 986.732 i	{2.98826 - 0.846233 i, -0.155658 + 1.20672 i}
	-6079.21 + 1505.54 i	-6079.21 + 1505.54 i	{4.06015 + 0.244047 i, 1.69724 - 0.997537 i}
	-1543.12 - 1249.58 i	-1543.12 - 1249.58 i	{2.43438 + 1.09148 i, 0.117738 + 1.26215 i}

TABLE – Numerical results for the parameters $q = 3$, $\nu = \nu' = 1$, $\mu = 0.2$, $\mu' = 0.3$, $v = 1.1$.

3-sites Heisenberg spin-$\frac{1}{2}$ chain (inhomogeneous, magn. field, 3-body terms)

Consider the realization of AW algebra :

$$\mathsf{I} = \kappa\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+\left[\mathsf{A},\mathsf{A}^*\right]_q + \kappa_-\left[\mathsf{A}^*,\mathsf{A}\right]_q \quad \text{with} \quad \mathsf{A} \to \Delta(C)\otimes I\!\!I\,, \quad \mathsf{A}^* \to I\!\!I\otimes\Delta(C)\,.$$

$$\frac{\mathsf{I}(\kappa,\kappa^*,\kappa_+,\kappa_-)}{2(q-q^{-1})^2} = \sum_a\sum_{i,j} J^a_{ij}\sigma^a_i\sigma^a_j + \sum_{a\neq b}\sum_{i,j} K^{ab}_{ij}\sigma^a_i\sigma^b_j + \sum_{\{a,b,c\}}\sum_{i,j,k} L^{abc}_{ijk}\sigma^a_i\sigma^b_j\sigma^c_k + \sum_i b^z_i\sigma^z_i + J_0\,,$$

Spin-$\frac{1}{2}$ chain	Direct diagonalization (degeneracy)	Diagonalization via ABA $\Lambda^{2s}_{a,+}(s)$	Bethe roots $\{U_1,\dots,U_{2s}\}$
$\kappa=1\,,\ \kappa^*=1/3\,,$ $\kappa_\pm=0$	32.5 (4) 14.4069 (2) 28.0931 (2)	32.5 (0) 14.4069 (1/2) 28.0931 (1/2)	- {-1.0344} {-1.4906}
$\kappa=-\frac{5}{4\sqrt{2}}\,,\ \kappa^*=-\frac{9}{4\sqrt{2}}\,,$ $\kappa_+=\frac{1}{8}\,,\ \kappa_-=-\frac{1}{16}$ $\chi=-\frac{15}{4}$	-0.200512 (4) -6.25895 + 3.32745 i (2) -6.25895 - 3.32745 i (2)	-0.200512 (0) -6.25895 + 3.32745 i (1/2) -6.25895 - 3.32745 i (1/2)	- {-0.793147 - 1.40509 i} {-0.793147 + 1.40509 i}

TABLE – Numerical results for the parameters $q=2$, $\nu=\nu'=1$.

$\longrightarrow$ Results for any s. Here for $s=1/2$ only.

$\longrightarrow$ Results for arbitrary alternating spin chain j_1, j_2, j_3.

$\longrightarrow$ Generalizes the special case studied by [Sklyanin,88].

Some perspectives

• **Integrable systems generated from q-Onsager algebra :**
Example : L-sites open XXZ chain with generic boundary conditions

$$H_{XXZ} = \sum_{k=0}^{L} F_k \underbrace{\mathsf{I}_{2k+1}}_{\text{Generalizations of HAW op. !}} + F_0$$

Diagonalization of $\mathsf{I}_{2k+1}, k = 0, 1, ..., N-1$ via modified ABA ?

Let A, A^* be the generators of the **q-Onsager algebra** with defining relations [Terwilliger,'99],[PB,'04] :

$$\big[\mathsf{A}, \big[\mathsf{A}, [\mathsf{A}, \mathsf{A}^*]_q\big]_{q^{-1}}\big] = \rho[\mathsf{A}, \mathsf{A}^*] \ ,$$
$$\big[\mathsf{A}, \big[\mathsf{A}^*, [\mathsf{A}^*, \mathsf{A}]_q\big]_{q^{-1}}\big] = \rho[\mathsf{A}^*, \mathsf{A}]$$

$$Example: \quad \mathsf{I}_1 \mapsto \kappa\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+ [\mathsf{A}, \mathsf{A}^*]_q + \kappa_- [\mathsf{A}^*, \mathsf{A}]_q \ .$$

Reflection algebra (known). Bethe ansatz ? Inhomogeneous TQ relations ?

Some perspectives

• **Integrable systems generated from higher rank Askey-Wilson algebras :**
For the **Askey-Wilson algebra**, one has the following realization :
[Granovskii-Zhedanov,'93] One has the map $AW \to U_q(sl_2) \otimes U_q(sl_2) \otimes U_q(sl_2)$.

$$\begin{aligned} \mathsf{A} &\to \Delta(C) \otimes \mathbb{I} \ , \\ \mathsf{A}^* &\to \mathbb{I} \otimes \Delta(C) \ . \end{aligned}$$

Recently, **higher-rank Askey-Wilson algebras** $AW(N)$ have been introduced [DeBie et al.,19], with generators $\mathsf{A}_{12}, \mathsf{A}_{23}, ..., \mathsf{A}_{N-1\ N}$. One has the map
$AW(N) \to U_q(sl_2) \otimes U_q(sl_2) \otimes \cdots \otimes U_q(sl_2)$

$$\begin{aligned} \mathsf{A}_{12} &\to \Delta(C) \otimes \mathbb{I} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \ , \\ \mathsf{A}_{23} &\to \mathbb{I} \otimes \Delta(C) \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \ , \\ \mathsf{A}_{N-1\ N} &\to \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \Delta(C) \end{aligned}$$

Consider the element : $\quad \mathsf{I}_N = \kappa_1 \mathsf{A}_{12} + \kappa_2 \mathsf{A}_{23} + \cdots \kappa_{N-1N} \mathsf{A}_{N-1N}$

$\Rightarrow$ **Diagonalization of I_N via modified ABA ?**

$\to \exists$ Reflection algebra using connection with generalized q-Onsager algebras !

Open problem related with recent works [Kuniba-Okada-Pasquier-Honeyama,'18,'19])

The Heun-Askey-Wilson algebra

- **HAW algebra**
The **Askey-Wilson algebra** (AW) [Zhedanov, '92] generated by $\{A, A^*\}$ gives an algebraic framework for all the **orthogonal polynomials of the Askey-scheme**.
The **Heun-Askey-Wilson algebra** (HAW) is a generalization of the AW algebra. It is generated by $\{X, I\}$ subject to the following two relations ($\{e_1, e_2, e_4\}$, $\{b_i\}$ are generic scalars) [B-Tsujimoto-Vinet-Zhedanov,19] :

$$[X,[X,I]_q]_{q^{-1}} = e_1X^3 + b_1X^2 + b_2\{X,I\} + b_3X + b_4I + b_5\mathcal{I} ,$$
$$[I,[I,X]_q]_{q^{-1}} = e_2X^3 + e_3XIX + e_4X^2 + b_1'\{X,I\} + b_2I^2 + b_3'I + b_6X + b_7\mathcal{I}$$

$$e_3 = e_1(q^2+q^{-2}+1), \quad b_1' = b_1 + e_1b_2, \quad b_3' = b_3 + e_1b_4.$$

Special case : for $e_i = 0\ \forall i$, the HAW algebra reduces to the the AW algebra.
$X \mapsto A\ ,\ I \mapsto A^*$
There exists $\phi : HAW \to AW$ where $\kappa, \kappa^*, \kappa_\pm$ arbitrary scalars :

$$X \mapsto A\ , \qquad I \mapsto \kappa A + \kappa^* A^* + \kappa_+ [A, A^*]_q + \kappa_- [A^*, A]_q\ .$$