Field-Theoretical Formulation of the Thermodynamical Bethe Ansatz

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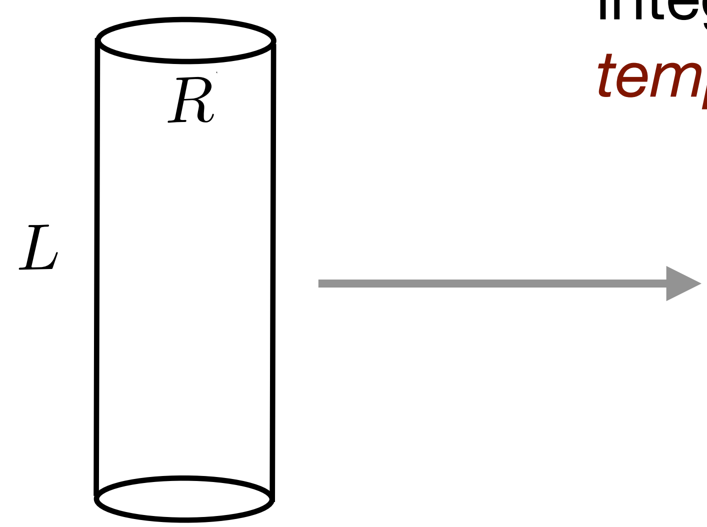
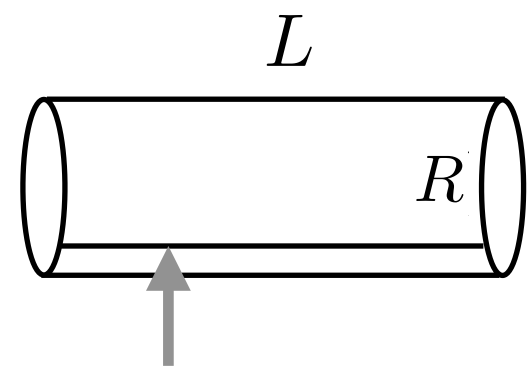
I.K., arXiv[hep-th]1909xxxx

I.K., Didina Serban, D. L. Vu, arXiv[hep-th]1805.02591, 1809.05705, 1906.01909

Thermodynamic Bethe Ansatz (TBA) [Yang&Yang, 1969]
— thermodynamics of 1-dim. integrable systems at *finite temperature*.

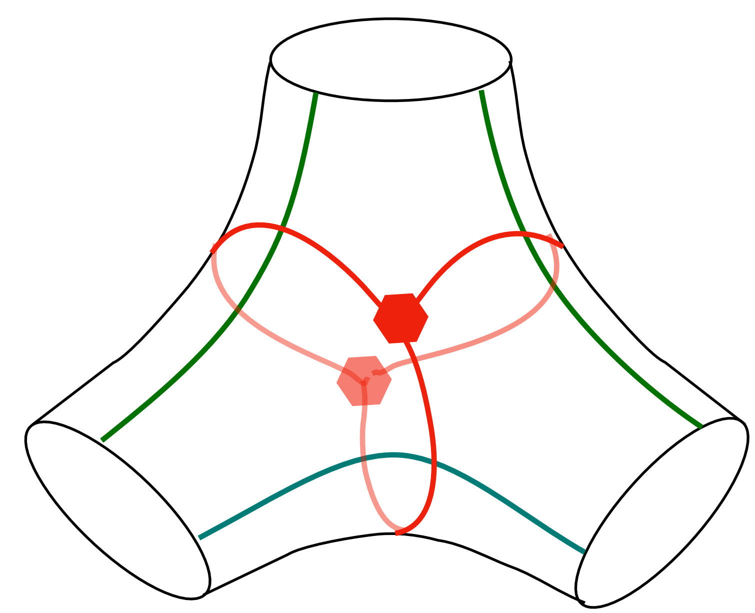
Since [Al. Zamolodchikov (1990) TBA became the main tool to compute *finite size* effects in 1+1 dim. relativistic field theories





Cut the cylinder and glue it back by inserting a complete set of virtual states (wrapping particles)

More recently TBA related methods are used in computation of correlation functions, e.g. hexagonalization method in N=4 SYM (see Didina's talk).



Need to learn how to evaluate efficiently the sum over the virtual particles in different problems.

Is it possible to replace the original TBA arguments by a more refined QFT/statistical formulation?

The question was posed decades ago and the answer is in principle yes although the effective QFT has not been yet formulated

Balog'94, Saleur 1999

Woynarovich, 2004: gaussian fluctuations around the saddle point of the Y-Y potential.

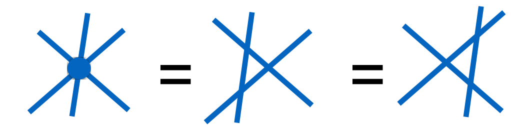
Pozsgay, 2010: showed that there is another O(1) contribution from the measure.

Kato&Wadati, 2004: exact cluster expansion.
I.K., Serban, Vu 2018 graph expansion for the free energy with periodic and open b.c.

In this talk we construct from scratch an effective QFT generating the exact cluster expansion for TBA

— For simplicity we take an integrable theory with one single neutral particle.
— In view of applications to N=4 SYM the scattering matrix is not supposed to be of difference type and no relativistic symmetry is assumed, only a mirror transformation.

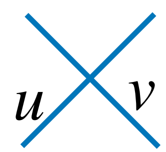
Euclidean 1+1 dimensional integrable field theory with factorized scattering



(c.f. Roberto's talk)

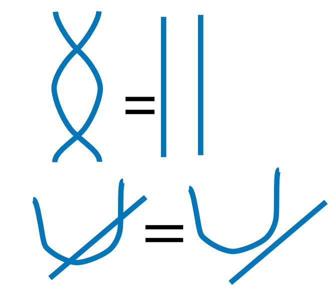
Rapidity variable: $p = p(u),\ E = E(u)$ $\qquad (\theta \equiv \pi u)$

Two-particle S-matrix:



unitarity: $S(u, v)\ S(v, u) = 1$

crossing $S(u, v)S(u, v^{2\gamma}) = 1$

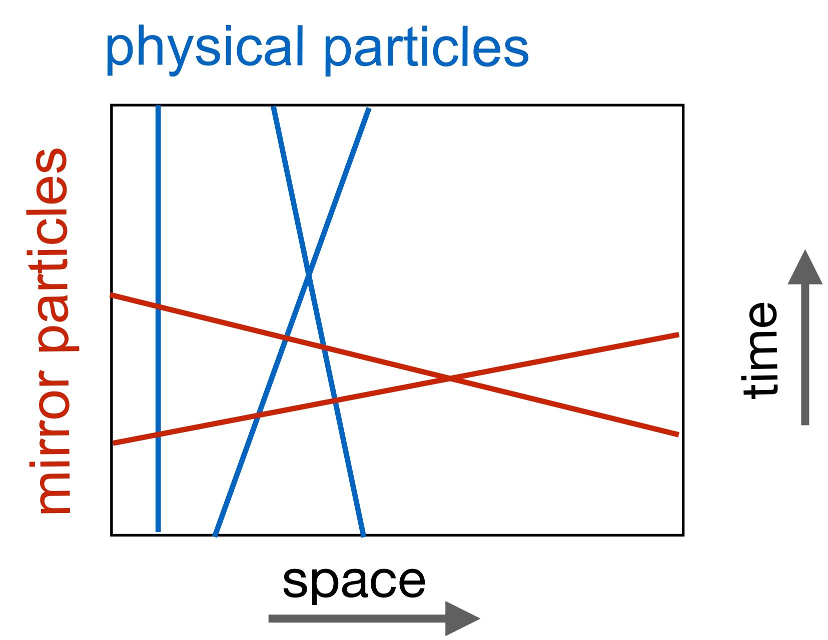


$S(u, u) = -1$

— no Lorentz invariance assumed, only a **mirror** transformation exchanging space and time

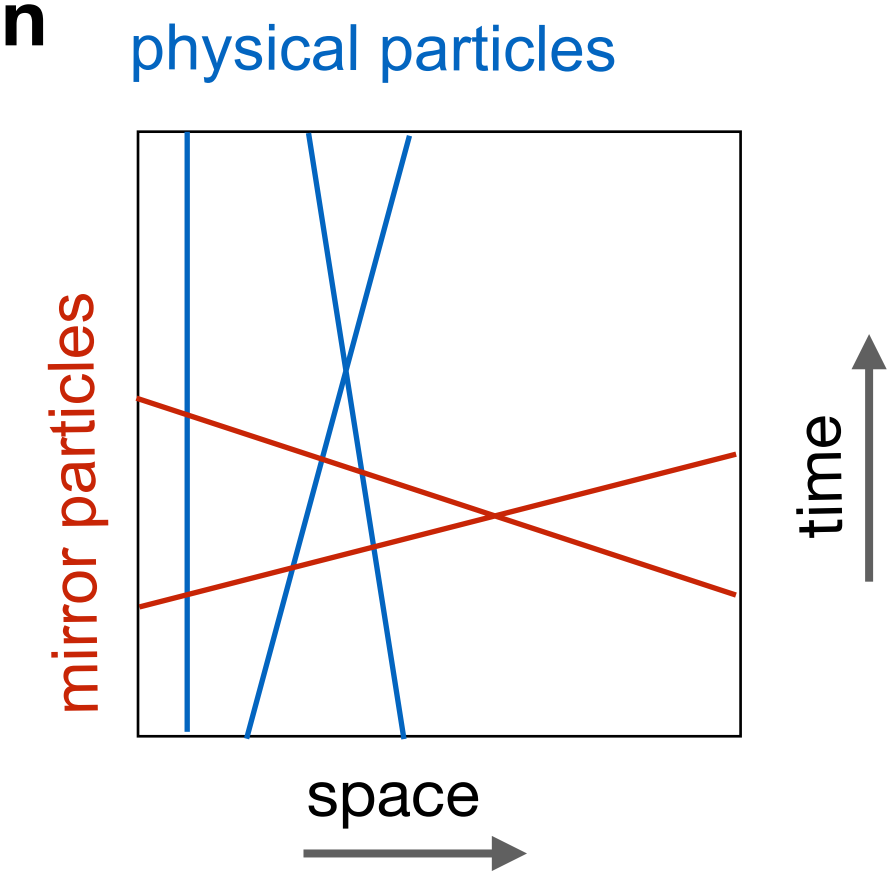
$$E \to i\tilde{p},\ p \to i\tilde{E}$$

Physical theory → Mirror theory



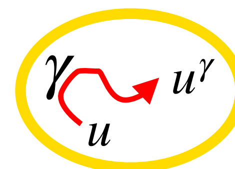
Mirror transformation as analytical continuation

For an observer living in the physical space,
a mirror particle looks as physical particle
with complex rapidity



Physical theory $\rightarrow$ Mirror theory

$$S(u,v) \;\rightarrow\; \tilde{S}(u,v) \equiv S(u^\gamma, v^\gamma)$$
$$p(u) \;\rightarrow\; \tilde{p}(u) = -iE(u^\gamma)$$
$$E(u) \;\rightarrow\; \tilde{E}(u) = -ip(u^\gamma)$$



$$\gamma : u \rightarrow u^\gamma$$

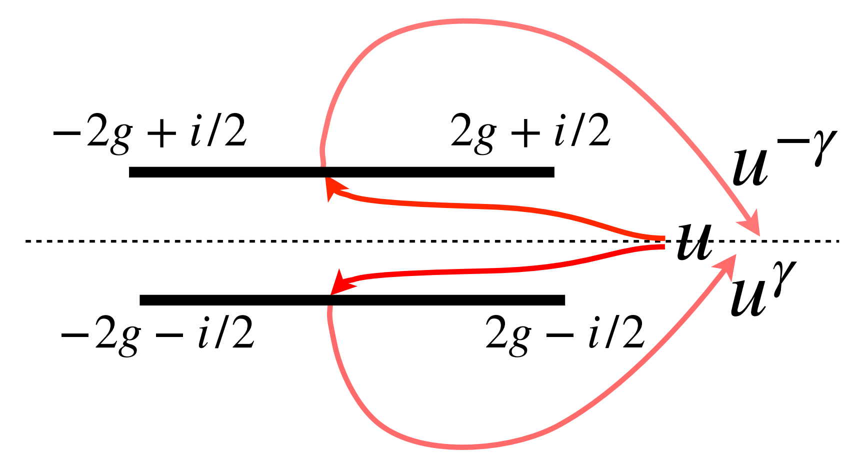
Example 1: Lorentz-invariant
massive integrable QFT

$$E^2 - p^2 = m^2$$
$$E = m\cosh(\pi u)$$
$$p = m\sinh(\pi u)$$

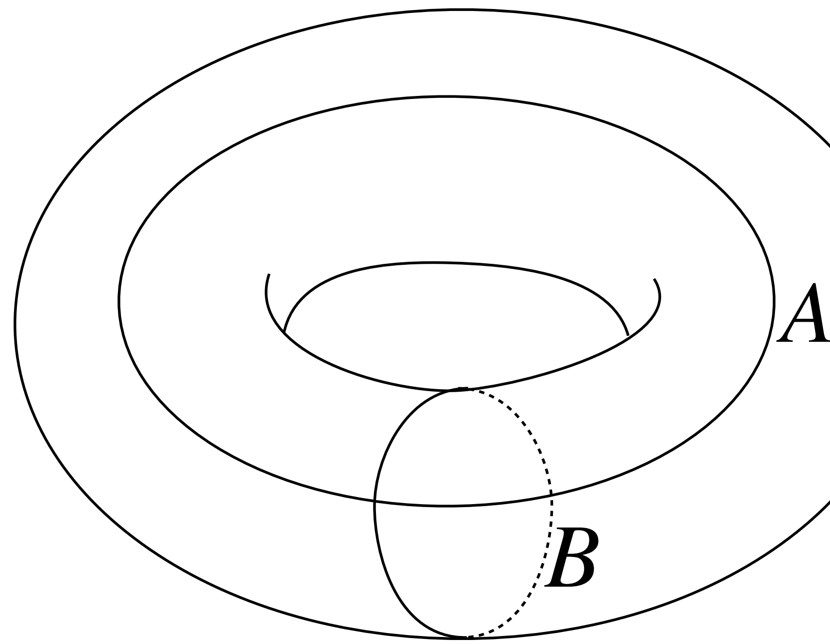
$$u^\gamma \equiv u + i/2$$
$$u$$
$$u^{-\gamma} \equiv u - i/2$$

Phys=Mirror

Example 2: N=4 SYM



I. PERIODIC BOUNDARY CONDITIONS

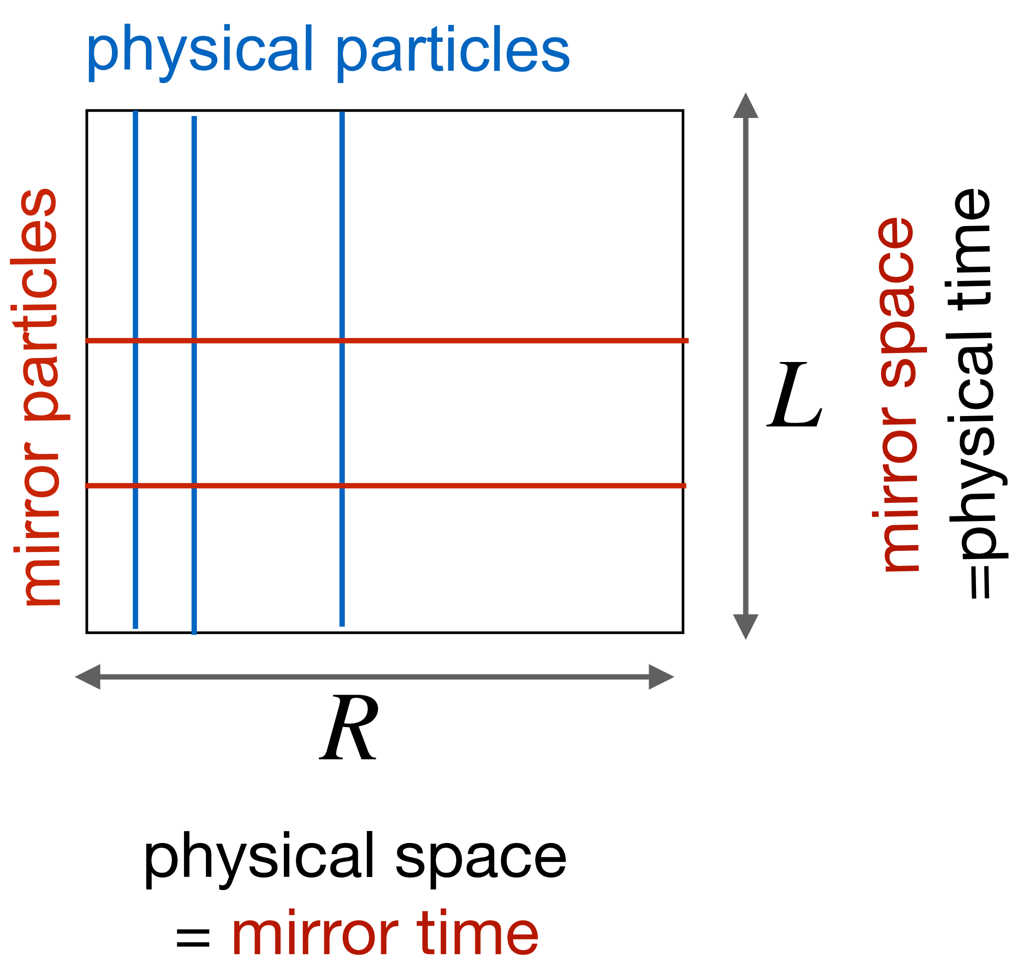


Physical theory defined on the A-cycle of length L,
Mirror theory defined on the B-cycle of length R

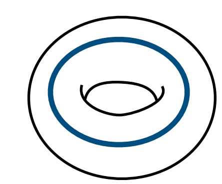
Degrees of freedom of the effective QFT:
particles winding around the A and B cycles

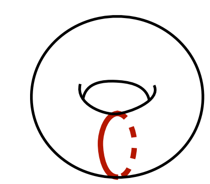
1. Wrapping operators

The Hilbert space of the effective QFT is spanned on the elementary excitations on a torus with asymptotically large space and time circles. Two types of them: time-wrapping particles in the physical theory and and space-wrapping particles in the mirror theory

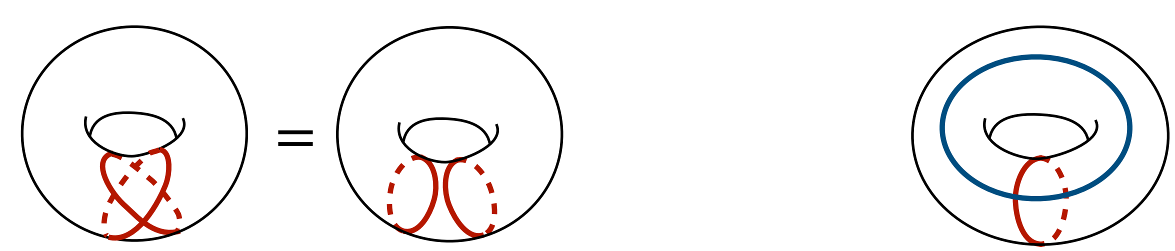


Operators creating wrapping particles:

$\mathbf{A}(u)$ = time-wrapping operator 

$\mathbf{B}(u)$ = space-wrapping operator 

Particles wrapping the same cycle do not scatter but particles wrapping different cycles do:



$\mathbf{A}(u)$ creates real particle in the physical theory
/virtual particle in the mirror theory

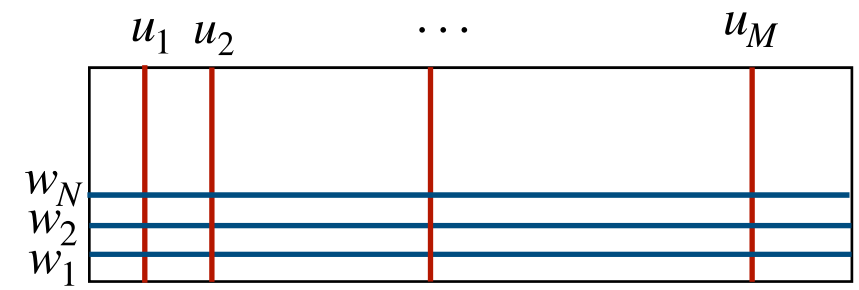
$\mathbf{B}(u)$ creates real particle in the mirror theory
/virtual particle in the physical theory

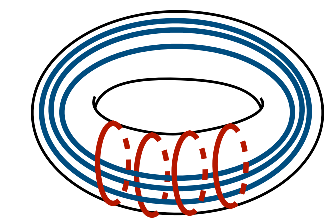
$\mathbf{A}(u^{\gamma})$ creates space-wrapping particle in the mirror theory

$\mathbf{B}(u^{-\gamma})$ creates space-wrapping particle in the physical theory

2. Algebra of the wrapping operators

Expectation value of N time-wrapping and M space-wrapping operators:





$$\langle \prod_{j=1}^{M} \mathbf{B}(v_j) \prod_{k=1}^{N} \mathbf{A}(w_k) \rangle = \prod_{j=1}^{M} \prod_{k=1}^{N} S(v_j^\gamma, w_k) \prod_{j=1}^{M} e^{-R\tilde{E}(v_j)} \prod_{k=1}^{N} e^{-LE(w_k)}$$

Fock-space realisation:

$$\mathbf{B}(v)\mathbf{A}(u) = S(v^\gamma, u)\, \mathbf{A}(u)\mathbf{B}(v), \quad [\mathbf{B}(u), \mathbf{B}(v)] = [\mathbf{A}(u), \mathbf{A}(v)] = 0$$

$$\langle L \,|\, \mathbf{A}(u) = e^{-LE(u)} \langle L\,|, \quad \mathbf{B}(u)\,|\,R\rangle = e^{-R\tilde{E}(u)}\,|\,R\rangle \qquad \langle L\,|\,R\rangle = 1$$

Fock-space expectation value:

$$\langle \prod_{j=1}^{M} \mathbf{B}(v_j) \prod_{k=1}^{N} \mathbf{A}(w_k) \rangle \equiv \langle L\,| \prod_{j=1}^{M} \mathbf{B}(v_j) \prod_{k=1}^{N} \mathbf{A}(w_k)\,|\,R\rangle$$

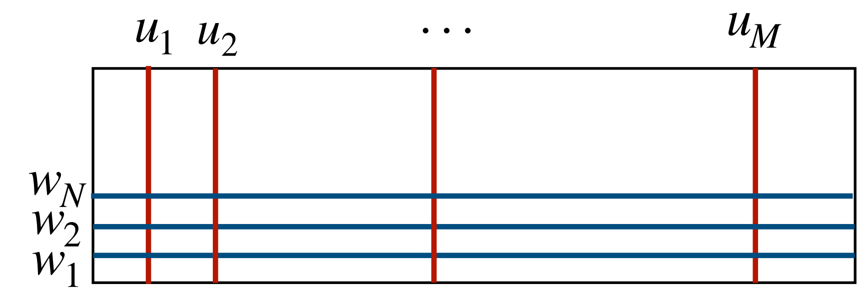
For any operator define

$$\langle \mathcal{O} \rangle = \langle L\,|\!:\mathcal{O}:\!|\,R\rangle$$

where $:\ :$ is the anti-normal product: all **B**'s are on the left of all **A**'s

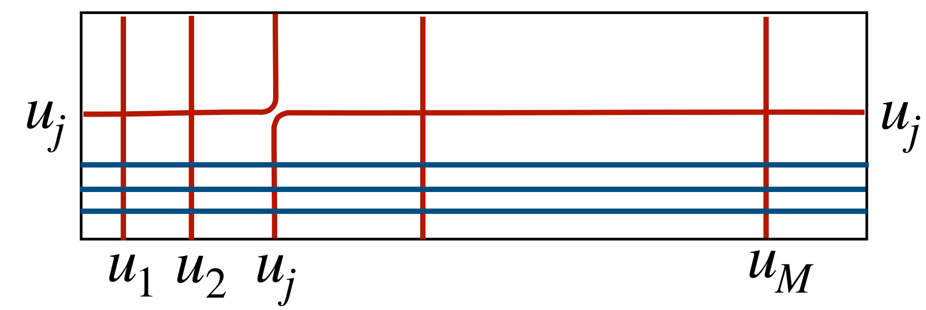
3. Operator form of Bethe-Yang equations

M+N equations for M+N rapidities



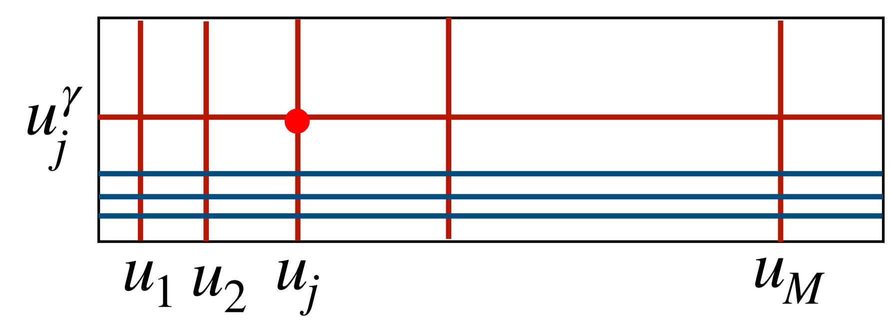
B-Y equations in the mirror channel

$$1 + e^{-iL\tilde{p}(u_j)} \prod_{k=1}^{M} \tilde{S}(u_k, u_j) = 0, \quad j = 1,\ldots, M$$



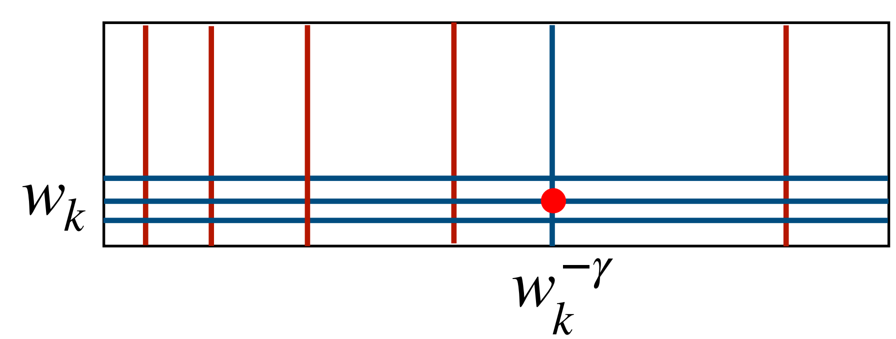
Operator form:

$$\left\langle \prod_{j=1}^{M} \mathbf{B}(u_j) \prod_{k=1}^{N} \mathbf{A}(w_k)\ (1 + \mathbf{A}(u_j^{\gamma})) \right\rangle = 0, \quad j = 1,\ldots,M$$



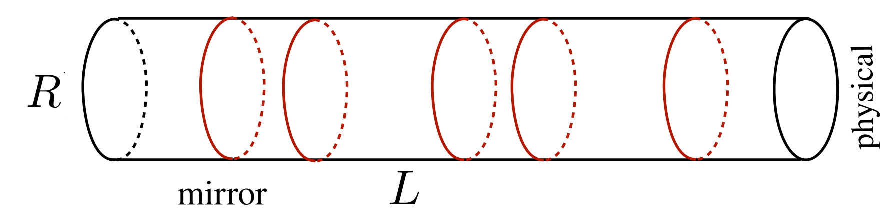
Similarly, in the physical channel

$$\left\langle \left(1 + \mathbf{B}(w_k^{-\gamma})\right) \prod_{j=1}^{M} \mathbf{B}(u_j) \prod_{k=1}^{N} \mathbf{A}(w_k) \right\rangle = 0, \quad k = 1,\ldots,N$$



4. Partition function at finite volume *R*

Finite volume partition function in the physical theory
= thermal partition function in the mirror theory
— can be computed by the techniques of the Thermodynamical Bethe Ansatz (justified in the thermodynamical limit when the number of the wrapping particles is large).



Instead, we will evaluate **exactly** the sum over particles without using TBA:

$$\mathcal{Z}(L,R) = \mathrm{Tr}[e^{-R\tilde{\mathbf{H}}}] = \sum_{\psi} \frac{\langle \psi | e^{-R\tilde{\mathbf{H}}} | \psi \rangle}{\langle \psi | \psi \rangle} = \sum_{M=0}^{\infty} \sum_{n_1 < n_2 < \ldots < n_M} e^{-R(\tilde{E}(u_1)+\ldots+\tilde{E}(u_M))}$$

The sum goes over the Bethe quantum numbers which appear in the logarithmic form of the Bethe-Yang equations:

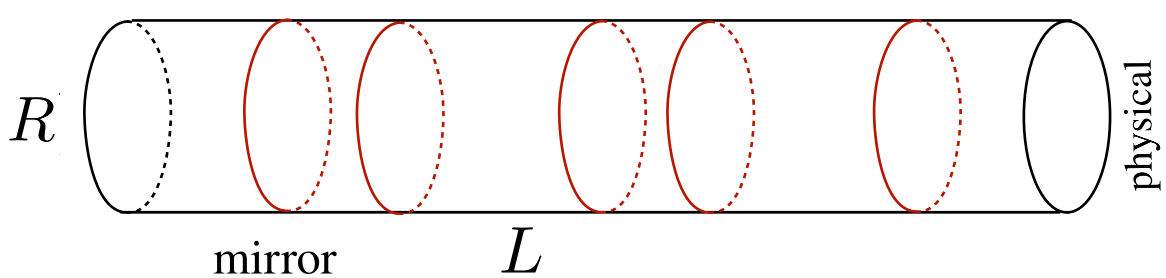
$$-L\tilde{p}(u_j) + i \sum_{k=1}^{M} \log \tilde{S}(u_k, u_j) = 2\pi n_j, \quad j = 1,\ldots,M$$

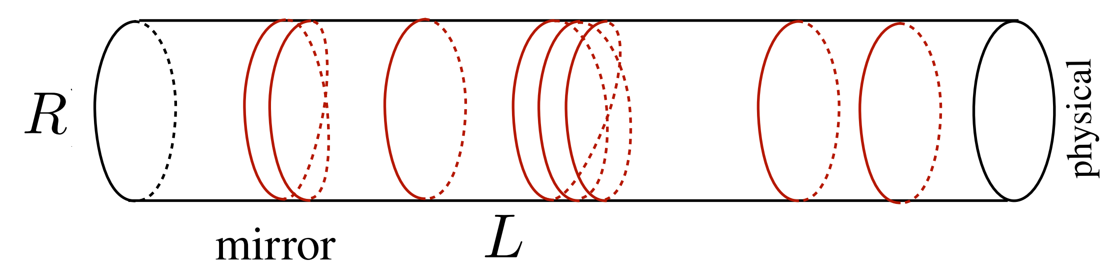
Our aim is to sum over the solutions of the Bethe-Yang equations (with no approximation) without solving them.

4. Fock-space representation of the partition function

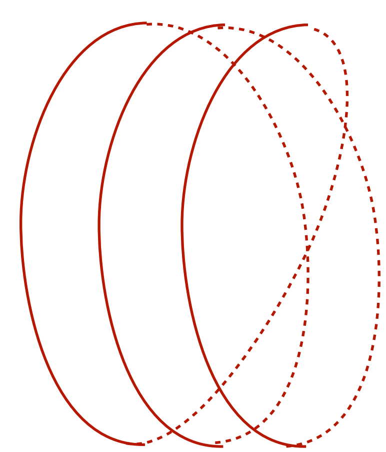
- Relax the constraint $n_1 < n_2 < \ldots < n_M$ by introducing multi-wrapping particles as explained yesterday by Dinh-Long

$$\mathscr{Z}(L,R) = \sum_{M=0}^{\infty} \sum_{n_1<n_2<\ldots<n_M} \prod_{j=1}^{M} e^{-R\tilde{E}(u_j)} \quad \rightarrow \quad \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{r_1,\ldots,r_m \geq 1} \sum_{n_1,\ldots,n_m} \prod_{j=1}^{m} \frac{(-1)^{r_j-1}}{r_j} e^{-r_j R\tilde{E}(u_j)}$$





r-wrapping particle:



scatters (r-1) times with itself, S(u,u)=-1

$$\frac{(-1)^{r-1}}{r} e^{-rR\tilde{E}(u)}$$

wraps length r R

cyclic symmetry of order r

- Instead of solving the B-Y equations we can impose them by the operator constraint

$$\prod_j (\mathbf{A}(u_j)+1) = 0$$ - selects both the physical and the unphysical solutions

- Express the discrete sum over Bethe and wrapping numbers as a contour integral around the real axis

$$\sum_{r_1,\dots,r_m}\sum_{n_1,\dots,n_m}\prod_{j=1}^{M} e^{-r_j R\tilde{E}(u_j)} = \left\langle \oint_{\mathbb{R}}\cdots\oint_{\mathbb{R}}\prod_{j=1}^{m}\frac{du_j}{2\pi i}\frac{\partial \log\left(1+\ \mathbf{A}(u_j^{\gamma})\right)}{\partial u_j}\sum_{r_1,\dots,r_m}\frac{(-1)^{r_j-1}}{r_j}\prod_{j=1}^{m}\mathbf{B}(u_j)^{r_j}\right\rangle$$

- perform the sum over all wrapping numbers:

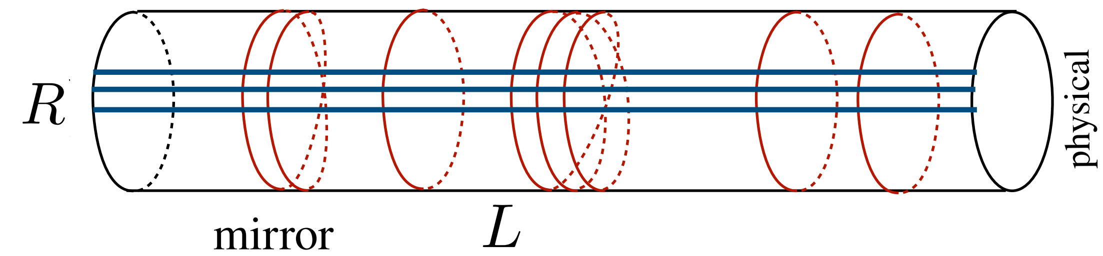
$$\mathscr{Z}(L,R) = \langle \mathbf{\Omega} \rangle$$

$$\mathbf{\Omega} \equiv \exp\left[\oint_{\mathbb{R}}\frac{du}{2\pi i}\log\left(1+\mathbf{B}(u)\right)\ \frac{\partial}{\partial u}\log\left(1+\mathbf{A}(u^{\gamma})\right)\right]$$

Ω — operator creating the physical vacuum (finite ***R***) out of the bare vacuum (asymptotically large ***R***)

5. Excited states in the physical channel

Partition function for an excited state with rapidities $\mathbf{w} = \{w_1, \ldots, w_N\}$



(1) $$\mathscr{Z}(L, R, \mathbf{w}) = \left\langle \prod_{k=1}^{N} \mathbf{A}(w_k) \ \ \mathbf{\Omega} \right\rangle$$

The rapidities of the excited state are determined by the exact Bethe equations

(2) $$\left\langle \left(1 + \mathbf{B}(w_j^{-\gamma})\right) \prod_{k=1}^{N} \mathbf{A}(w_k) \ \ \mathbf{\Omega} \right\rangle = 0, \quad j = 1,2,\ldots,N$$

Relation to Dorey-Tateo:

Write (1) as a contour integral:

$$\mathscr{Z}(L, R, \mathbf{w}) = (-1)^N \left\langle \exp\left[-\oint_{\mathbf{w}^{-\gamma}} \frac{du}{2\pi i} \log\left(1 + \mathbf{A}(u^\gamma)\right) \ \frac{\partial}{\partial u} \log\left(1 + \mathbf{B}(u)\right) \right] \mathbf{\Omega} \right\rangle = \langle \mathbf{\Omega}_{\mathbf{w}} \rangle$$

$$\mathbf{\Omega}_{\mathbf{w}} \equiv \exp\left[\oint_{\mathbb{R}+\mathbf{w}^{-\gamma}} \frac{du}{2\pi i} \log\left(1 + \mathbf{B}(u)\right) \ \frac{\partial}{\partial u} \log\left(1 + \mathbf{A}(u^\gamma)\right) \right]$$

5. Free-field representation

Wrapping operators as vertex operators: $\mathbf{B}(u) = e^{-\varphi(u)}, \quad \mathbf{A}(u^\gamma) = e^{-i\bar{\varphi}(u)}$

$$[\varphi(u), \bar{\varphi}(v)] = i \log \tilde{S}(u, v)$$

$$[\bar{\varphi}(u), \bar{\varphi}(v)] = [\varphi(u), \varphi(v)] = 0$$

$$\langle L | \bar{\varphi}(u) = \bar{\varphi}^\circ(u) \langle L |, \quad \varphi(u) | R \rangle = \varphi^\circ | R \rangle$$

$$\varphi^\circ(u) = R\tilde{E}(u), \qquad \bar{\varphi}^\circ(u) = L\tilde{p}(u)$$

Expectation value: $\langle \cdots \rangle = \langle L | \!:\! \cdots \!:\! | R \rangle$ with anti-normal product $: \, :$ (all φ's on the left of all φ's)

$$\langle \bar{\varphi}(u)\, \varphi(v) \rangle = i\tilde{S}(u, v) + \bar{\varphi}(u)^\circ \, \varphi^\circ(v)$$

$$\langle e^{-i\bar{\varphi}(u)} \, e^{-\varphi(v)} \rangle = \tilde{S}(v, u) \, e^{-i\bar{\varphi}^\circ(u) - \varphi^\circ(v)}$$

$$\boxed{\mathscr{Z}(L, R) = \langle \boldsymbol{\Omega} \rangle \qquad \boldsymbol{\Omega} = \exp \left[\oint_{\mathscr{C}} \frac{du}{2\pi i} \log\langle 1 + e^{-\varphi(u)} \rangle \, \partial_u \log \left(1 + e^{-i\bar{\varphi}(u)} \right) \right]}$$

5. Continuum spectrum approximation

Now expand the contour slightly away the real axis:

$$\oint_{\mathbb{R}} \frac{d \log\left(1 + e^{-i\bar{\varphi}(u)}\right)}{2\pi i} (\cdots) \quad \to \quad \int_{-\infty}^{\infty} \frac{d\bar{\varphi}(u)}{2\pi} \quad + \text{exponentially small in } L$$

$$d\bar{\varphi}(u) = \bar{\varphi}' du + \ldots (?)$$

by the Ward identity $\left\langle \left(\bar{\varphi}(u) - \int_{\mathbb{R}} \frac{dv}{2\pi} \partial_v \bar{\varphi}(v) \frac{i \log \tilde{S}(v,u)}{1 + e^{\varphi(v)}} \right) \boldsymbol{\Omega} \right\rangle = 0$

$$d\bar{\varphi}(u) \;\to\; \left[\bar{\varphi}'(u) - \int \frac{dv}{2\pi} \hat{K}(u,v)\ \bar{\varphi}'(v) \right] du \qquad \hat{K}(u,v) = -\frac{1}{i} \frac{\partial_u \log S(v,u)}{1 + e^{\varphi(v)}}$$

=> Jacobian = (functional) Gaudin determinant

Introduce by hand fermions which generate the non-diagonal terms in the Gaudin determinant:

$$\mathscr{Z}(L,R) = \left\langle \check{\boldsymbol{\Omega}} \right\rangle$$

$$\check{\boldsymbol{\Omega}} = \exp \int_{-\infty}^{\infty} \frac{du}{2\pi} \left[\log\left(1 + e^{-\varphi(u)}\right) \bar{\varphi}'(u) + \frac{\bar{\psi}(u)\psi'(u)}{1 + e^{\varphi(u)}} \right]$$

$$\langle \bar{\varphi}(u)\, \varphi(v) \rangle_c = i \log \tilde{S}(u,v), \quad \langle \psi(u) \bar{\psi}(v) \rangle = i \log \tilde{S}(u,v)$$

5. Path integral

Impose the correlators by introducing a pair of auxiliary fields $\rho, \bar{\rho}$ $\theta, \bar{\theta}$

$\varphi, \bar{\varphi}$ $\rho, \bar{\rho}$
commutative

$\bar{\psi}, \psi$ $\vartheta, \bar{\vartheta}$
grassmanian

$$\mathcal{Z}(L,R) = \int \mathcal{D}[\text{fields}]\; e^{-\mathcal{A}[\text{fields}]}$$

$$-\mathcal{A}[\text{fields}] = \int \frac{du}{2\pi} \left(\log(1+e^{-\varphi})\,\partial\bar{\varphi} + \frac{\bar{\psi}\,\partial\psi}{1+e^{\varphi}} + (\bar{\varphi}-\bar{\varphi}^{\circ})\rho + (\varphi-\varphi^{\circ})\bar{\rho} + \bar{\vartheta}\psi + \bar{\psi}\vartheta \right)$$

$$-i \int \frac{du}{2\pi}\frac{dv}{2\pi} \log \tilde{S}(u,v) \big(\bar{\rho}(u)\rho(v) - \vartheta(u)\bar{\vartheta}(v) \big)$$

the dependence on R and L through the classical fields:

$$\varphi^{\circ}(u) = R\tilde{E}(u), \quad \bar{\varphi}^{\circ}(u) = L\tilde{p}(u)$$

- The bosonic part of the path integral was obtained using different arguments by Jiang, Komatsu and Veskovi [ARXIV:1906.07733]

5. Localisation

$$\mathscr{A} = \mathscr{A}^{\circ} + \int \frac{du}{2\pi}\, \mathrm{Q}(u)\ \mathscr{B}$$

Q-exact localisation term

$$\mathscr{A}^{\circ} = -\int \frac{du}{2\pi}\, \bar{\varphi}^{\circ} \rho$$

$$Q = \bar{\psi}\frac{\delta}{\delta\varphi} + \bar{\varphi}\frac{\delta}{\delta\psi} + \bar{\rho}\frac{\delta}{\delta\vartheta} + \bar{\vartheta}\frac{\delta}{\delta\rho} \qquad Q^2 = 0$$

$$\mathscr{B} = \int \frac{du}{2\pi}\Big(-\log(1+e^{-\varphi})\,\partial\psi + \psi\rho + \theta(\varphi - \varphi^{\circ})\Big) - i\int \frac{du}{2\pi}\frac{dv}{2\pi}\, \theta(u)\ \log \tilde{S}(u,v)\, \rho(v)$$

By standard localisation argument integral localises to the critical point:

$$\mathscr{Z} \ \to \ \mathscr{Z}_t = \int e^{-\mathscr{A}_t} \qquad \mathscr{A} \to \mathscr{A}_t = \mathscr{A}^{\circ} + t\mathrm{Q}\mathscr{B}, \quad \mathscr{A}^{\circ} \equiv -L\int \frac{du}{2\pi} p'(u)\rho(u)\,.$$

$$\frac{\partial \mathscr{Z}_t}{\partial t} = \int e^{-\mathscr{A}^{\circ} - t\mathrm{Q}\mathscr{B}}(-\mathrm{Q}\mathscr{B}) = \int \mathrm{Q}\left(e^{-\mathscr{A}^{\circ}-t\mathrm{Q}\mathscr{B}}\mathscr{B}\right) = 0$$

Take the limit of
infinite perturbation: $t \to \infty$:

$$\mathscr{Z} = e^{-\mathscr{A}^{\circ}}\Big|_{\mathrm{Q}\mathscr{B}=0}$$

The critical point: $$\varphi(u) = R\tilde{E}(u) - i\int \frac{dv}{2\pi} \log \tilde{S}(u,v)\, \rho(v)$$

$$\rho(u) = \partial_u \log(1 + e^{-\varphi(u)})$$

equation for the critical point identical to the TBA integral equation:

$$\epsilon(u) = R\tilde{E}(u) - \int \tilde{K}(v,u)\ \log(1 + e^{-\epsilon(v)})$$

$\epsilon(u) = \varphi^{\text{crit}}(u)$ - pseudo energy

The partition function: $$\mathscr{Z}(L,R) = \exp\left(L \int \frac{d\tilde{p}(u)}{2\pi}\ \log\left[1 + e^{-\epsilon(u)}\right]\right)$$

As a consequence of localisation : the theory is one-loop exact and the gaussian fluctuations of the bosons and the fermions cancel => no quantum corrections to the critical action at all

6. Feynman graphs = exact cluster expansion

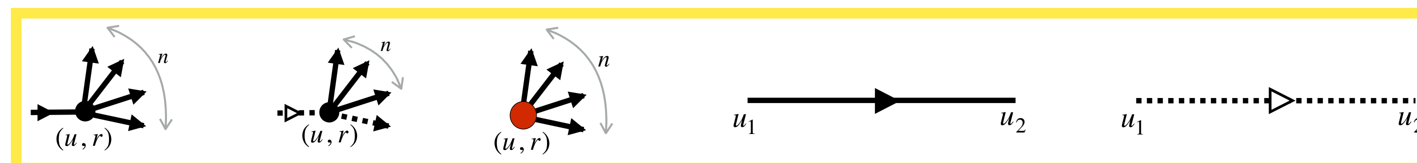
I.K., Didina Serban, D. L. Vu,
arXiv[hep-th]1805.02591,
1809.05705, 1906.01909

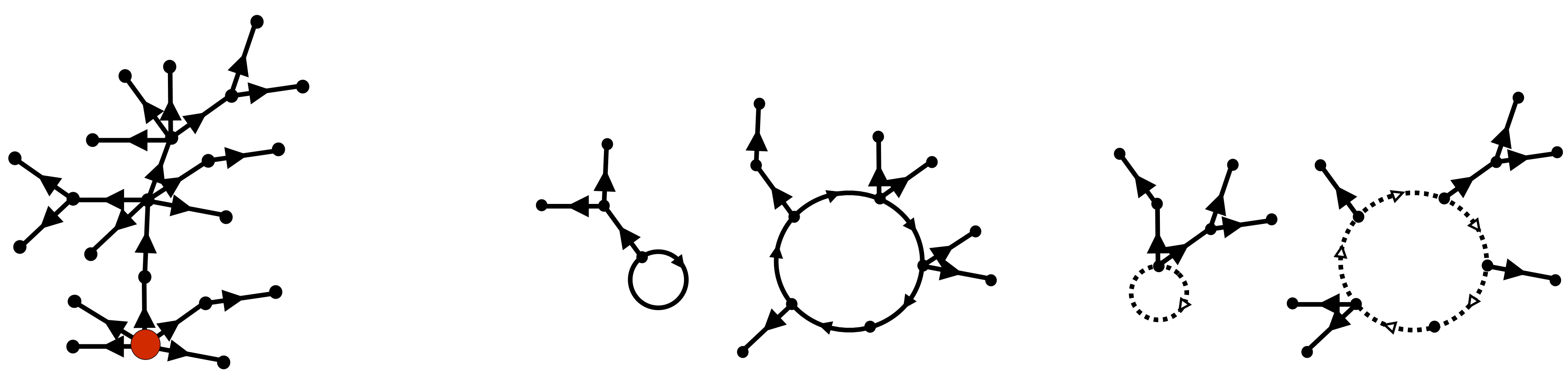
Vertices: $$\bar{\varphi}' \log(1+e^{-\varphi}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \bar{\varphi}' \varphi^n$$

Propagators: $$\langle \bar{\varphi}'(u)\varphi(v)\rangle = -\tilde{K}(u,v), \quad \langle \bar{\psi}'(u)\psi(v)\rangle = \tilde{K}(u,v)$$

$$\tilde{K}(u,v) = \frac{1}{i}\partial_u \log \tilde{S}(u,v)$$

Feynman rules (c.f. Dinh-Long's talk)





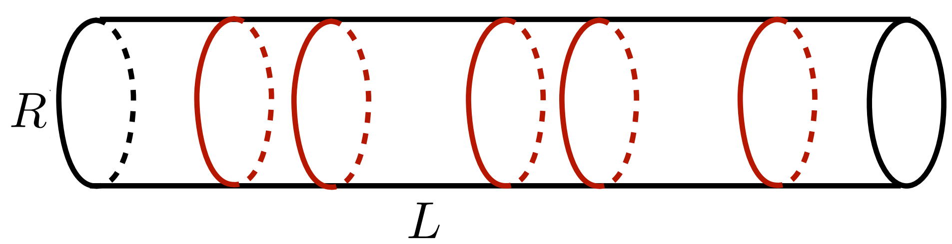
trees

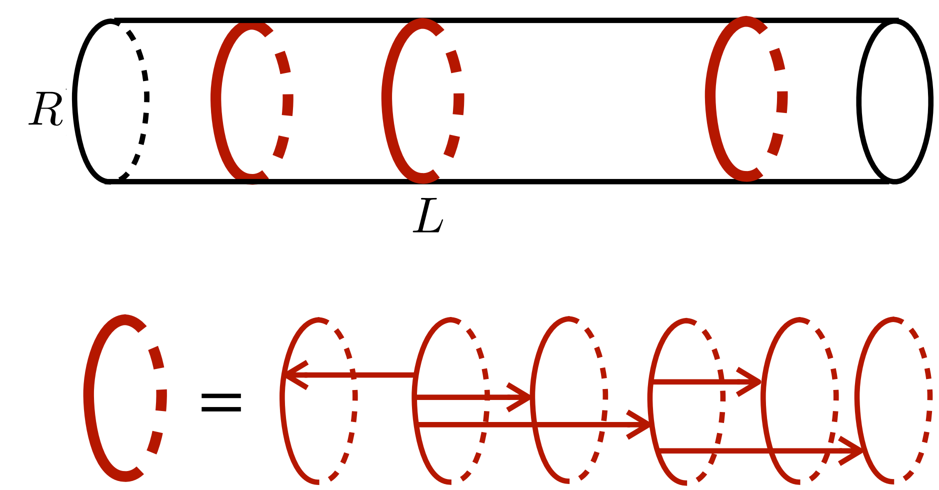
bosonic loops

fermionic loops

The bosonic loops and the fermionic loops cancel and the free energy is given by the sum over tree graphs.

The meaning of the tree diagrams in the cluster expansion:





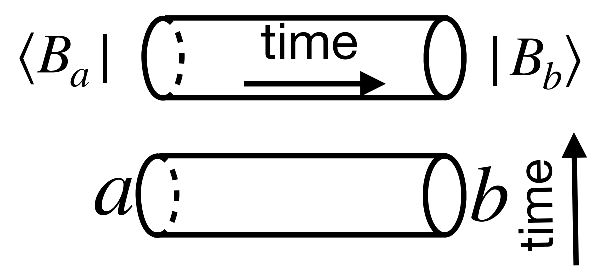
Wrapping particles weakly interacting after being put in a large box

Non-interacting clusters of wrapping particles: behave as free fermions with renormalized energy

II. OPEN B. C. IN THE MIRROR CHANNEL

Thermal partition function with open b.c.

$$\mathcal{Z}_{ab}(R,L) = \langle B_a | e^{-H(R)L} | B_b \rangle$$
$$= \mathrm{Tr}[e^{-\tilde{H}_{ab}(L)R}]$$



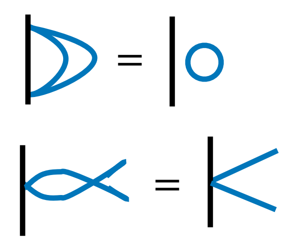
a,b - integrable boundary conditions

Parity conserving bulk scattering matrix $S(u,-v)S(-u,v) = 1$

boundary reflection matrix [Ghoshal-Zamolodchikov]

$$\tilde{R}_a(u)\tilde{R}_a(-u) = 1$$

$$\tilde{R}_a(-u) = \tilde{S}(u,-u)\tilde{R}_a(u)$$



$$\tilde{R}_a(u) \equiv R_a(u^\gamma)$$

Boundary entropy [Affleck-Ludvig' 91]

R-finite, L-asymptotically large

$$\mathcal{F}_{ab}(R,L) \equiv \log \mathcal{Z}_{ab}(R,L) - \log \mathcal{Z}(R,L) = \log g_a(R) + \log g_b(R) + O(e^{-mL})$$

"g-functions"

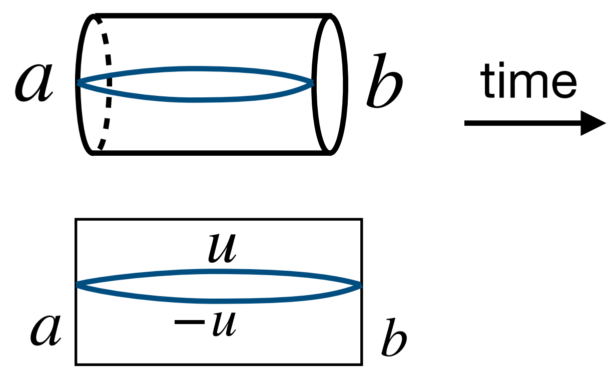
Asymptotically large cylinder= [circle of length R]x[interval of length L]

Excitations in the effective QFT:

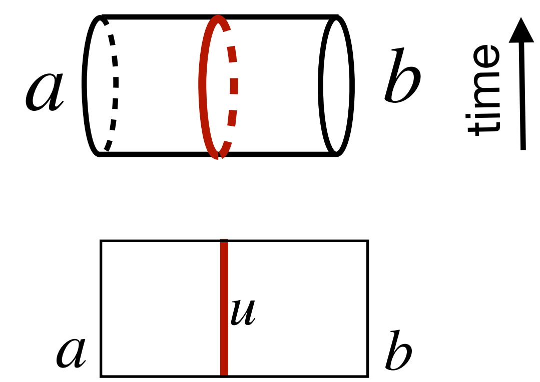
In the physical theory: parity invariant states propagating between the two boundaries

In the mirror theory: particles wrapping the cylinder

$\mathbf{A}_{ab}(u)$



$\mathbf{B}(u)$



Algebra of the wrapping operators:

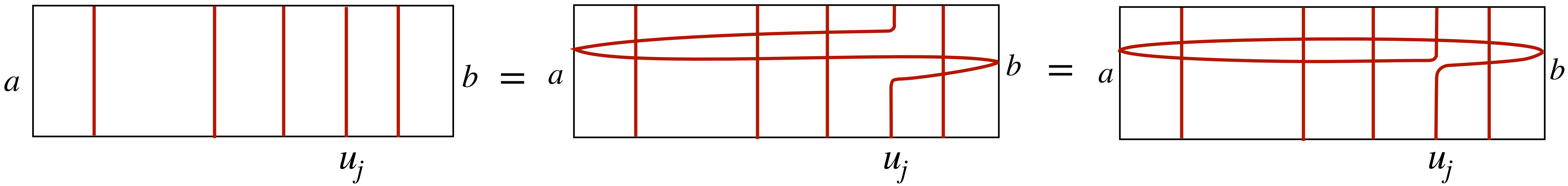
$$\mathbf{B}(v)\mathbf{A}_{ab}(u) = S(v^\gamma, u)S(-v^\gamma, u)\,\mathbf{A}_{ab}(u)\mathbf{B}(v)$$

$$[\mathbf{A}_{ab}(u), \mathbf{A}_{ab}(u)] = 0, \quad [\mathbf{B}(u), \mathbf{B}(v)] = 0$$

Fock space vacua: $\langle L \,|\, \mathbf{A}_{ab}(u) = \langle L \,|\, e^{-2LE(u)} R_a(u) R_b(u)$ $\qquad \mathbf{B}(u)\,|\,R\rangle = e^{-R\tilde{E}(u)}\,|\,R\rangle$

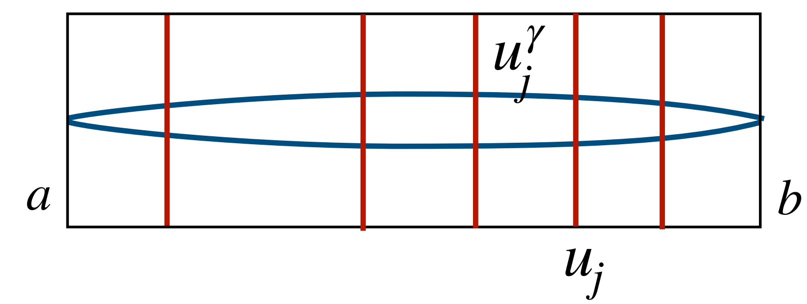
Expectation value: $\langle \mathcal{O} \rangle \equiv \langle L \,|\!: \mathcal{O} :\!|\, R \rangle$

Boundary Bethe-Yang equations



Operator representation:

$$\left\langle \prod_{k=1}^{M} \mathbf{B}(u_k)\,\left(1+\mathbf{A}_{ab}(u_j^{\gamma})\right) \right\rangle = 0, \qquad j = 1,\ldots,M\,.$$



Fock-space representation of the cylinder partition function

$$\mathscr{Z}_{ab}(L,R) = \langle \mathbf{\Omega}_{ab} \rangle$$

$$\mathbf{\Omega}_{ab} \equiv \exp \oint_{\mathbb{R}_+} \frac{du}{2\pi i} \log\left(1+\mathbf{B}(u)\right)\ \partial_u \log\left[1+\mathbf{A}_{ab}(u^{\gamma})\right]$$

Free-field representation: $\mathbf{A}_{ab}(u^\gamma) = e^{-i\bar{\varphi}_{ab}(u)}, \quad \mathbf{B}(u) = e^{-\varphi(u)}$

$$\left[\varphi(v), \bar{\varphi}_{ab}(u)\right] = -\, i \log[\tilde{S}(v,u)\tilde{S}(-v,u)]$$

$$\langle L \,|\, \bar{\varphi}_{ab}(u) = \bar{\varphi}^\circ_{ab}(u) \,\langle L\,|, \quad \varphi(u)\,|\,R\rangle = \varphi^\circ\,|\,R\rangle$$

$$\bar{\varphi}^\circ_{ab}(u) = 2L\tilde{p}(u) + i \log R_a(u) R_b(u) - 2\pi \operatorname{sign}(u)$$

$$\varphi^\circ(u) = R\tilde{E}(u)$$

Continuum spectrum approximation: add fermionic partners for the measure

$$[\psi_{ab}(v)\bar{\psi}(u)] = -\, i \log[\tilde{S}(v,u)\tilde{S}(-v,u)]$$

$$\boxed{\mathcal{Z}_{ab}(R,L) = \left\langle \check{\boldsymbol{\Omega}}_{ab} \right\rangle \qquad \check{\boldsymbol{\Omega}}_{ab} = \exp \int_0^\infty \frac{du}{2\pi} \frac{\bar{\varphi}_{ab}\partial_u \varphi - \bar{\psi}_{ab}\partial_u \psi}{1 + e^{\varphi}}}$$

Path integral for open boundary fonditions:

$$\mathcal{Z}_{ab}(L,R) = \int \mathcal{D}[\text{fields}]\ e^{-\mathcal{A}[\text{fields}]}$$

$$-\mathcal{A} = \int \frac{du}{2\pi}\left(\frac{\bar{\varphi}_{ab}\,\partial\varphi\ - \bar{\psi}_{ab}\,\partial\psi}{1+e^{\varphi}} + (\bar{\varphi}_{ab} - \bar{\varphi}^{\circ}_{ab})\rho + (\varphi - \varphi^{\circ})\bar{\rho} + \bar{\theta}\psi + \bar{\psi}_{ab}\theta\right)$$

$$+i\int \frac{du}{2\pi}\frac{dv}{2\pi}\log[\tilde{S}(u,v)\tilde{S}(u,-v)]\big(-\bar{\rho}(u)\rho(v) + \theta(u)\bar{\theta}(v)\big)$$

$$\bar{\varphi}^{\circ}_{ab}(u) = 2L\tilde{p}(u) + i\log\tilde{R}_a(u) + i\log\tilde{R}_b(u) - 2\pi\,\mathrm{sign}(u), \quad \varphi^{\circ}(u) = R\tilde{E}(u)$$

The theory is one-loop exact:

$$\mathcal{Z}_{ab} = \frac{\mathrm{Det}(1-\hat{K}^{-})}{\mathrm{Det}(1-\hat{K}^{+})}\exp\left(\int_0^{\infty}\frac{du}{2\pi}\partial_u\bar{\varphi}^{\circ}_{ab}\log[1+e^{-\varphi}]\right)$$

$$\hat{K}^{\pm}(u,v) = \frac{1}{i}\frac{1}{1+e^{\epsilon(u)}}\partial_u\left(\log\tilde{S}(u,v) \pm \log\tilde{S}(u,-v)\right)$$

Dorey, Fioravanti, Rim, Tateo’04; Pozsgay’2010

III. EXAMPLE: SINH-GORDON

$$\mathscr{A} = \int d^2x \left[\frac{1}{4\pi}(\partial_\mu \phi)^2 + \frac{2\mu^2}{\sin \pi b^2} \cosh(b\phi) \right]$$

$$\nu = 1 + \frac{1}{b^2}, \quad a = 1 - \frac{2}{\nu} \qquad (0 < b \leq 1)$$

One particle, no bound states;
relativistic theory: mirror=physical

$$p(u) = m \sinh \pi u, \quad E(u) = m \cosh \pi u$$

$$S(u,v) = \frac{\tanh\left(\frac{\pi u}{2} - \frac{i\pi}{2\nu}\right)}{\tanh\left(\frac{\pi u}{2} + \frac{i\pi}{2\nu}\right)}$$

$$\log S(u) = (\mathbb{D}^a + \mathbb{D}^{-a}) \log S_0(u)$$

$$S_0(u) = \tanh \frac{\pi(u - i/2)}{2}$$

Shift operator: $\mathbb{D} = \exp\left(\frac{i}{2}\frac{\partial}{\partial u}\right)$

$$K_0(u,v) = \frac{1}{i}\partial_u \log S_0(u,v)$$ - “universal kernel”

$$(\mathbb{D} + \mathbb{D}^{-1}) K_0(u) = 2\pi\delta(u)$$

$K \to \mathbb{K} = \frac{\mathbb{D}^a + \mathbb{D}^{-a}}{\mathbb{D} + \mathbb{D}^{-1}}$ $K(u) = K_0(u + ia/2) + K_0(u - ia/2) = (\mathbb{D}^a + \mathbb{D}^{-a}) K_0$ $K_0(u,v) = \frac{\pi}{\cosh \pi(u-v)}$ $K(u,v) = \frac{1}{i}\partial_u \log S(u,v)$ $\mathbb{D} \equiv e^{i\partial_u/2}$

(all expression below need to be rigorously defined)

$$\mathscr{A}[\text{fields}] = \int \frac{du}{2\pi} \left(\frac{\bar{\varphi}\,\varphi' - \bar{\psi}\,\psi'}{1+e^{\varphi}} + (\bar{\varphi} - \bar{\varphi}^{\circ})\rho + (\varphi - \varphi^{\circ})\bar{\rho} + \bar{\theta}\psi + \bar{\psi}\theta \right)$$

$$+i \int \frac{du}{2\pi}\frac{dv}{2\pi} \big(\bar{\rho}(u)\rho(v) + \theta(u)\bar{\theta}(v)\big)(\mathbb{D}^{a} + \mathbb{D}^{-a})\log S_0(u,v)$$

The action can be cast into a quasi-local form by a field redefinition

$$\bar{\varphi} \;\to\; Lp(u) + (\mathbb{D} + \mathbb{D}^{-1})\bar{\varphi} \qquad\qquad \bar{\psi} \;\to\; (\mathbb{D} + \mathbb{D}^{-1})\bar{\psi}$$

$$\mathscr{A} = \int \frac{du}{2\pi} \left[\varphi(\mathbb{D} + \mathbb{D}^{-1})\partial\bar{\varphi} - \frac{(\mathbb{D}^{a} + \mathbb{D}^{-a})\partial\bar{\varphi}}{1+e^{-\varphi}} \right]$$

$$+ \int \frac{du}{2\pi} \left[\psi(\mathbb{D} + \mathbb{D}^{-1})\bar{\partial}\psi - \frac{\bar{\psi}(\mathbb{D}^{a} + \mathbb{D}^{-a})\partial\psi}{1+e^{\varphi}} \right]$$

Critical point: $\qquad \left[\mathbb{D} + \mathbb{D}^{-1}\right] \varphi = -\,(\mathbb{D}^{a} + \mathbb{D}^{-a})\log(1+e^{-\varphi})$

"Discrete Liouville equation"

[Zamolodchikov, Lukyanov]

$\varphi \to \varphi + R\,E(u)$

Summary

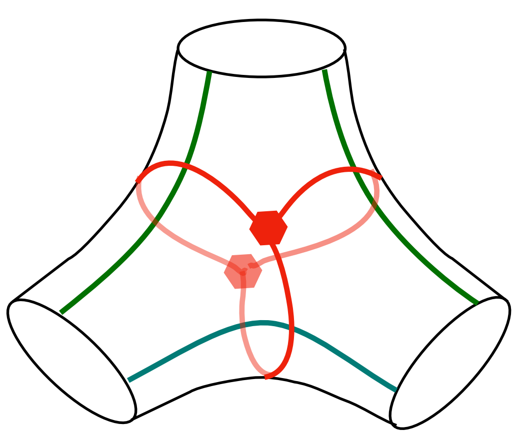
Path integral formulation of the Thermodynamic Bethe Ansatz

The theory is one-loop exact. Explains why there are only exponential corrections to the free energy

Works also for scattering matrices not of difference type, as in AdS/CFT

Can be generalised to the case of non-diagonal scattering (nested Bethe Ansatz) and bound states

Hopefully can be adapted to other geometries with application to AdS/CFT



Thank you!