On the Generalised Hydrodynamics of Integrable QFT with non-diagonal scattering

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ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Experimental results and theoretical progresses

Advent of *cold atoms*, *optical lattices* and *molecular electronic devices*

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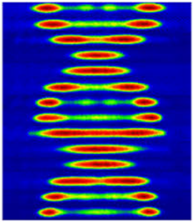
new experimental insight in phenomena of *quantum non-equilibrium statistical mechanics*



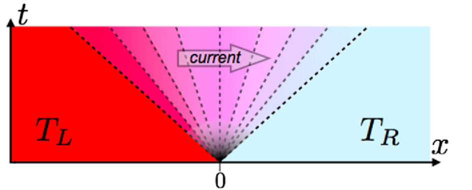
Kinoshita, Wenger, Weiss: ***A quantum Newton's cradle***, Nature 440 (2006) 900

There has been much theoretical progress in the last decade. Studies focused on:

- **quantum quench**: responses to excitations or pulses
- **emergent hydrodynamics**: steady properties (do not vary in time)
  Paradigms:
  - effective reservoirs ⟶ open, non-unitary systems
  - hamiltonian reservoirs ⟶ close, unitary systems

Partition protocol

1. Prepare two semi-infinite halves of a homogeneous 1D quantum system thermalized independently at temperatures T_L and T_R
2. At time $t = 0$ connect the two halves so that they can exchange energy and particles
3. The initial state $|\text{ini}\rangle$ evolves for $t > 0$ with Hamiltonian $H = H_L + H_R + \delta H$. At large times it reaches a steady regime



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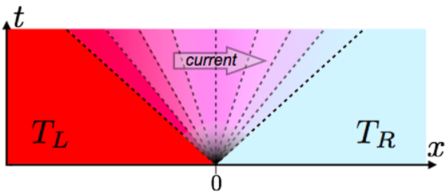

For an observable $\mathcal{O}$ the **steady state limit** means

$$\mathcal{O}_{\text{st}} := \lim_{t\to\infty} \lim_{L\to\infty} \langle e^{iHt} \mathcal{O} e^{-iHt} \rangle_{\text{ini}}$$

Local thermodynamic equilibrium

Assumption: **local thermodynamic equilibrium**

- We observe the system on the scale of clouds of particles (10^{-6}m) rather than at the scale of particles (10^{-10}m).
- After some *local relaxation time*, physical properties vary only on space-time scales much larger than microscopic ones.
- The system decomposes in *fluid cells*, each one in thermal equilibrium. Potentials $\beta(x,t)$ vary slowly in adjacent cells.

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As a consequence:

- Averages of local observables tend at large times, to averages evaluated in local Gibbs ensembles with space-time dependent potentials

$$\langle \mathcal{O}(x,t) \rangle = \mathrm{Tr}[\rho(x,t)\mathcal{O}] \rightsquigarrow \langle \mathcal{O} \rangle_{\beta(x,t)}$$

Conservation laws and normal modes

Conserved quantities $Q_i = \int dx q_i(x,t)$ in involution $\Longrightarrow$ conservation laws

$$\partial_t q_i(x,t) + \partial_x j_i(x,t) = 0 \quad , \quad i = 1, ..., N$$

Average of densities $\mathfrak{q}_i(x,t) = \langle q_i \rangle$ and currents $\mathfrak{j}_i(x,t) = \langle j_i \rangle$ also satisfy

$$\boxed{\partial_t \mathfrak{q}_i(x,t) + \partial_x \mathfrak{j}_i(x,t) = 0}$$

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Equation of State links $\mathfrak{j}_i = \mathcal{F}_i(\mathfrak{q})$
Jacobian

$$J_{ij} = \frac{\partial \mathcal{F}_i(\mathfrak{q})}{\partial \mathfrak{q}_j} \qquad \Longrightarrow \qquad \partial_t \mathfrak{q}_i(x,t) + \sum_j J_{ij} \partial_x \mathfrak{q}_j(x,t)) = 0$$

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J can be diagonalized with a change of coordinates $\mathfrak{q} \to \mathfrak{n}$ (*normal modes*)

$$\partial_t \mathfrak{n}_i(x,t) + v_i^{\text{eff}} \partial_x \mathfrak{n}_i(x,t)) = 0$$

v_i^{eff} can be interpreted as velocity of propagation of the i-th normal mode $\mathfrak{n}_i$.
Invariance of this equation under rescaling $(x,t) \mapsto (ax, at)$ shows that the solutions should depend only on $\xi = x/t$.

Generalized Gibbs Ensemble (GGE)

Goal of this approach:

Compute the profile of functions $\mathfrak{n}_i(\xi)$.

Integrable systems $\longrightarrow$ infinity of conserved charges = constraints on the ensemble
Density matrix has to take them into account to describe the dynamics of the system

$$\rho_{GGE} = \frac{e^{-\sum_i \beta_i Q_i}}{\mathrm{Tr}[e^{-\sum_i \beta_i Q_i}]}$$

Complete set of charges $Q_i = \{I_i, X_{s,i}\}$, local and quasi-local

Local charges I_j can be expressed in terms of densities $I_j = \sum_\ell i_j(\ell)$ which have support on a finite number of sites, *e.g.* in XXZ

$$I_j = -i \, \frac{d^j}{d\theta^j} \log T_1 \left(\theta + \frac{i\pi}{2} \right) \Big|_{\theta=0}$$

Quasi-local charges $X_{s,j}$ also have densities $X_{s,j} = \sum_\ell x_{s,j}(\ell)$ but their support is on an extended region with exponentially decaying norm [Ilievski et al. 2015]

$$X_{s,j} = -i \frac{d^j}{d\theta^j} \log T_s \left(\theta + \frac{i\pi}{2} \right) \Big|_{\theta=0}$$

Factorised S-matrix and TBA

- In QFT_2 integrability implies factorised S-matrix, no particle production and conservation of the set of momenta. [Parke, 1979]
- Study the steady states after local relaxation time by use of Thermodynamic Bethe Ansatz [Yang, Yang 1966 - Al. Zamolodchikov, 1990]
- For $IQFT_2$ with diagonal factorized S-matrix $\Longrightarrow$ Doyon, Castro-Alvaredo, Yoshimura, Phys. Rev. X6, 041065 (2016) (DCY)

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- For $IQFT_2$ with diagonal factorized S-matrix $\Longrightarrow$ Doyon, Castro-Alvaredo, Yoshimura, Phys. Rev. X6, 041065 (2016) (DCY)
- However, many theories have internal degrees of freedom and symmetries organising particles in multiplets. The S-matrix is non diagonal

$$(S_{ab})^{kl}_{mn}(\theta_{ij}) = U_{ab}(\theta_{ij})(R^{(a,b)})^{kl}_{mn}(\theta_{ij})$$

- State with N-particles $|\theta_1, a_1; \ldots; \theta_N, a_N\rangle$ on a periodic box of length L. Add to this state a probe particle with rapidity θ and impose periodic boundary conditions on the wave function.

Diagonalisation of color transfer matrix

- This leads to Bethe-Yang condition of quantization of momenta

$$e^{ip_a(\theta)L} = \prod_{j=1}^{N} S_{ab_j}(\theta-\theta_j) = \prod_{j=1}^{N} U_{ab}(\theta-\theta_j) \underbrace{\mathrm{Tr}_a \prod_{j=1}^{N} (R^{(a,b)})^{k_j m_{j+1}}_{m_j n_j}(\theta-\theta_j)}_{\mathcal{T}(\theta|\{\theta_j\})=\text{color transfer matrix}}$$

$R^{(a,b)}$-matrix acts on the a, b multiplets of particles, indices k, l, m, n run in a multiplet. Between different multiplets a, b the S-matrix is block-diagonal.

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- Diagonalisation of $\mathcal{T}$ by Bethe ansatz. The eigenvalues of $\mathcal{T}$ enter the Bethe-Yang equation (BYE).
- They depend on the rapidities θ_n of the particles but also on some parameters u_n characterising the states (the Bethe roots), that are determined by a set of Bethe Ansatz Equations (BAE).

Thermodynamic limit

- Thermodynamic limit $N \to \infty$. The set of rapidities $\theta_1, ..., \theta_N$ and of Bethe roots $u_1, ..., u_M$ tend to continuum.
  - solutions u_k organize in *n*-strings

$$u_{k,\alpha}^{(n)} = r_k^{(n)} + \frac{i\pi}{2}(n+1-2\alpha) \qquad , \qquad \alpha = 1, ..., n$$

  - introduce density of possible string centres $\sigma_n(\theta)$
  - density of occupied string centres $\rho_n(\theta)$
  - density of holes (unoccupied string centres) $\bar{\rho}_n(\theta) = \sigma_n(\theta) - \rho_n(\theta)$
  - density of occupied quasi-particle states $\rho_p(\theta)$ such that $\sum_{j=1}^{N} ... \longmapsto \int d\theta \rho_p(\theta)...$
- Split the product on all Bethe roots as

$$\prod_{k=1}^{M} \cdots = \prod_{n \in \mathfrak{U}} \prod_{k=1}^{M_n} \prod_{\alpha=1}^{n} \cdots$$

$\mathfrak{U}$ = set of all possible types of strings (depends on the model)

Perturbed coset CFT's and Dynkin TBA

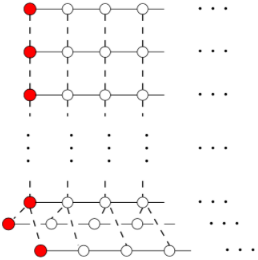
Here we study the case where $R(\theta)$ is the R-matrix of $\mathcal{U}_q(\mathcal{G})$, restricted at $q =$ root of 1, for some algebra $\mathcal{G} = A, D, E$. In the RSOS basis, these S-matrices describe the perturbed CFT coset models

$$\frac{\mathcal{G}_k \times \mathcal{G}_\ell}{\mathcal{G}_{k+\ell}} + \phi^{\mathrm{id,id}}_{\mathrm{adj}}$$

$$S(\theta) = X(\theta) S_k(\theta) \otimes S_\ell(\theta)$$

Spectrum of kinks of mass $m^{(a)}$ (Perron-Frobenius eigenvector of $\mathcal{G}$) separating colored vacua with RSOS(k) $\times$ RSOS(ℓ) strucutres.

TBA is encoded on product of Dynkin diagrams $\mathcal{G} \diamond A_{k+\ell-1}$
FR, Tateo, Valleriani 1992 — Quattrini, FR, Tateo 1993
see also: Kuniba, Nakanishi, Suzuki, 1994 and 2011



Thermodynamic Bethe Ansatz (TBA)

Log-derivatives of BYE+BAE lead to

$$\sigma_n^{(a)}(\theta) = \delta_{n\ell} p'^{(a)}(\theta) + \sum_{b=1}^{\mathrm{rank}\mathcal{G}} \varphi^{(a,b)} * \rho_n^{(b)}(\theta) + \sum_{m=1}^{k+\ell-1} \mathcal{I}_{n,m} \varphi * \rho_m^{(a)}(\theta)$$

Kernels ($g = \mathrm{dual\,Coxeter}\,\mathcal{G}$)

$$\tilde{\varphi}^{(a,b)}(\kappa) = 2\pi \left[\left(\delta^{ab} - \frac{1}{2\cosh\frac{\pi\kappa}{g}} \mathcal{G}^{ab} \right)^{-1} - \delta^{ab} \right] \quad , \quad \varphi(\theta) = \frac{g}{2\cosh\frac{g\theta}{2}}$$

$\mathcal{G}^{ab}$ is the incidence matrix of the $\mathcal{G}$ Dynkin diagram, $\mathcal{I}_{n,m}$ is the incidence matrix of $A_{k+\ell-1}$ Dynkin diagram

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Minimizaton of free energy with the constraints of BYE and BAE leads to the $\mathcal{G} \diamond A_{k+\ell-1}$ **TBA equations**

$$\boxed{\log y_n^{(a)}(\theta) = \delta_{n\ell} p'^{(a)} - \varphi^{(a,b)} * \log Y_n^{(b)}(\theta) + \mathcal{I}_{nm}\varphi * \log Y_m^{(a)}(\theta)}$$

Universal form and Y-system

Functions $y_n^{(a)}$ defined as

$$y_n^{(a)}(\theta) = \frac{\rho_n^{(a)}(\theta)}{\bar{\rho}_n^{(a)}(\theta)} = \frac{\text{density of "string centres"}}{\text{density of holes}}$$

$$Y_n^{(a)}(\theta) = 1 + y_n^{(a)}(\theta) = \frac{\sigma_n^{(a)}(\theta)}{\bar{\rho}_n^{(a)}(\theta)}$$

$$\hat{Y}_n^{(a)}(\theta) = (1 + y_n^{(a)}(\theta)^{-1})^{-1} = \frac{\rho_n^{(a)}(\theta)}{\sigma_n^{(a)}(\theta)} = n_n^{(a)}(\theta) = \text{occupation numbers}$$

This can be recast in a set of functional equations (Y-system of $\mathcal{G} \diamond A_{k+\ell-1}$ type)

$$y_n^{(a)}\left(\theta + \frac{i\pi}{g}\right) y_n^{(a)}\left(\theta - \frac{i\pi}{g}\right) = \prod_{b=1}^{\mathrm{rank}\mathcal{G}} \hat{Y}_n^{(b)}(\theta)^{\mathcal{G}_{ab}} \prod_{m=1}^{k+\ell-1} Y_m^{(a)}(\theta)^{\mathcal{I}_{nm}}$$

Compact notation

- Introduce the matrix $\mathbf{\Phi}$ of kernel functions with entries

$$\mathbf{\Phi}^{(a,b)}_{n,m} = \varphi^{(a,b)}(\theta)\delta_{n,m} - \mathcal{I}_{n,m}\delta^{a,b}\varphi(\theta)$$

- Also introduce a vector notation for the currents

$$\underline{q} = \{q_i, i = 1, 2, 3, ...\}$$

- TBA can be compactly written as

$$\log y^{(a)}_n = \nu^{(a)}_n - \mathbf{\Phi}^{(a,b)}_{n,m} * \log Y^{(b)}_m$$

where $\nu^{(a)}_n = m^{(a)}\delta_{n\ell} \cosh\theta$

- Averages of densities can be written as

$$\underline{q}^{(a)} = \sum_{n=1}^{k+\ell-1} \int d\theta \underline{h}^{(a)}(\theta) \log Y^{(a)}_\ell(\theta)$$

Formulation of TBA with GGE

Conserved charges act as

$$\underline{Q}|\theta_1, a_1; \ldots; \theta_N, a_N\rangle = \sum_{k=1}^{N} \underline{h}^{(a_k)}(\theta_k)|\theta_1, a_1; \ldots, \theta_N, a_N\rangle$$

$\underline{h}^{(a)}(\theta)$ one particle eigenvalue of $\underline{Q}$. If relativistic (θ = rapidity):

$$h_1^{(a)}(\theta) = m^{(a)} \cosh\theta \quad , \quad h_2^{(a)}(\theta) = m^{(a)} \sinh\theta$$

If Galilean (θ = velocity):

$$h_1^{(a)}(\theta) = e^{(a)}(\theta) = m^{(a)}\theta^2/2 \quad , \quad h_2^{(a)}(\theta) = p^{(a)}(\theta) = m^{(a)}\theta$$

$Q_0 = N$, $Q_1 = H$, $Q_2 = P$... and $[Q_i, Q_j] = 0$ (also quasi-local charges)

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In the formulation of TBA we assume the generalized hamiltonian:

$$H_{GGE} = \sum_{a,n} \int \rho_n^{(a)}(\theta) w_n^{(a)}(\theta) d\theta$$

New driving term $w_n^{(a)}(\theta, x, t) = \underline{\beta}(x, t) \cdot \delta_{n,\ell} \underline{h}^{(a)}(\theta, x, t)$

Formulation of TBA with GGE II

This allows to:

- Extend the phase space of the Hamiltonian considering the higher charges as interactions
- Calculate the averages of quantitites using GGE density matrix

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Minimization of generalised free energy F_{GGE} leads to generalised TBA

$$\boxed{\log y_n^{(a)} = w_n^{(a)} - \mathbf{\Phi}_{n,m}^{(a,b)} * \log Y_m^{(b)}}$$

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For any density $\underline{h}(\theta)$ define the dressing operation as

$$[\underline{h}_n^{(a)}]^{dr} = \delta_{n,\ell} \underline{h}^{(a)} - \mathbf{\Phi}_{n,m}^{(a,b)} * n_m^{(b)} [\underline{h}_m^{(b)}]^{dr}$$

Spectral current and effective velocity

$\underline{q}^{(a)} = \{q_i^{(a)}\}$ and $\underline{\rho}^{(a)} = \{\rho_n^{(a)}\}$ are alternative complete sets to characterize the GGE ensemble.

$$\underline{q}^{(a)} = \int d\theta \rho_\ell^{(a)}(\theta)\underline{h}^{(a)}(\theta) = \int d\theta n_\ell^{(a)}(\theta)[p_\ell^{(a)\prime}(\theta)]^{dr}\underline{h}^{(a)}(\theta)$$
$$= \int d\theta n_\ell^{(a)}(\theta)p^{(a)\prime}(\theta)[h_\ell^{(a)}(\theta)]^{dr}$$

Currents can be obtained by a Double Wick rotation (crossing operation $\mathcal{C}$)

$$(x,t) \mapsto (it, -ix) \quad , \quad \theta \mapsto \frac{i\pi}{2} - \theta \quad , \quad (e^{(a)}, p^{(a)}) \mapsto (ip^{(a)}, -ie^{(a)})$$

$$\underline{j}^{(a)} = \int d\theta n_\ell^{(a)}(\theta)[e_\ell^{(a)\prime}(\theta)]^{dr}\underline{h}^{(a)}(\theta)$$

Define the *spectral currents* $\hat{\rho}_n^{(a)}(\theta) = n_n^{(a)}(\theta)[e_n^{(a)\prime}(\theta)]^{dr}$

$$\underline{j}^{(a)} = \int d\theta \hat{\rho}_\ell^{(a)}(\theta)\underline{h}^{(a)}(\theta)$$

Effective velocity

$$\underline{j}^{(a)} = \int d\theta v_\ell^{(a)}(\theta)^{eff} \rho_\ell^{(a)}(\theta) \underline{h}^{(a)}(\theta)$$

Effective velocity (Group velocity is $v^{(a)}(\theta)^{gr} = e^{(a)\prime}(\theta)/p^{(a)\prime}(\theta)$)

$$v_n^{(a)}(\theta)^{eff} = \frac{[e_n^{(a)\prime}(\theta)]^{dr}}{[p_n^{(a)\prime}(\theta)]^{dr}} = \frac{\hat{\rho}_n^{(a)}(\theta)}{\rho_n^{(a)}(\theta)} = \frac{e^{(a)\prime}(\theta)\delta_{n\ell} - \sum_{b,m} \mathbf{\Phi}_{n,m}^{(a,b)} * \hat{\rho}_m^{(b)}(\theta)}{p^{(a)\prime}(\theta)\delta_{n\ell} - \sum_{b,m} \mathbf{\Phi}_{n,m}^{(a,b)} * \rho_m^{(b)}(\theta)}$$

$$\boxed{v_n^{(a)}(\theta)^{eff} = v^{(a)}(\theta)^{gr}\delta_{n\ell} + \frac{\rho_n^{(a)} * v_n^{(a)}(\theta)^{eff} - v_n^{(a)}(\theta)^{eff}}{p^{(a)\prime}(\theta)}}$$

This is the velocity of propagation of quasi-particles. It is determined by $\rho_n^{(a)}$ that completely characterize the steady state.

Bethe-Boltzmann equation

Euler equation $\partial_t \underline{h}^{(a)} + \partial_x \underline{j}^{(a)} = 0$ can be written in terms of $\rho_n^{(a)}$

$$\partial_t \rho_n^{(a)} + \partial_x \hat{\rho}_n^{(a)} = 0 = \partial_t \rho_n^{(a)} + v_n^{(a)\, eff} \partial_x \rho_n^{(a)}$$

or after some manipulation, in terms of the occupation number $\mathbf{n}$

$$\partial_t n_n^{(a)}(\theta) + v_n^{(a)\, eff}(\theta) \partial_x n_n^{(a)}(\theta) = 0$$

which is the diagonal form with normal modes propagating with velocity $\mathbf{v}^{eff}$.
Similar Euler equation holds for $\boldsymbol{\sigma}$ and $\bar{\boldsymbol{\rho}}$ and so entropy is conserved

$$\partial_t s + v^{eff} \partial_x s = 0$$

as it should be in a fluid without viscosity.

Partition protocol problem

Solving the two-reservoir system corresponds to solving the initial value problem

$$\begin{cases} \partial_t n(\theta, x, t) + v^{eff}(\theta)\partial_x n(\theta, x, t) = 0 \\ n(\theta, x, 0) = n^{in}(\theta, x) = n^L(\theta)\Theta(-x) + n^R(\theta)\Theta(x) \end{cases}$$

Ansatz

$$n(\theta, x, t) = n^{in}(\theta, x - v^{eff}(\theta)t)$$

giving

$$n(\theta, x, t) = n^L(\theta)\Theta(-x + v^{eff}(\theta)t) + n^R(\theta)\Theta(x - v^{eff}(\theta)t)$$

If $v^{eff}(\theta)$ is monotonic in θ, the equation $v^{eff}(\theta)t - x = 0$ has a unique solution $\theta^\star \longrightarrow$ Solution to initial value problem

$$\boxed{\begin{cases} n(\theta, x, t) = n^L(\theta)\Theta(\theta - \theta^\star(x, t)) + n^R(\theta)\Theta(\theta^\star(x, t) - \theta) \\ v^{eff}(\theta^\star(x, t)) = \frac{x}{t} \end{cases}}$$

Numerical implementation I

Functions $\theta^\star(\xi)$ are also solutions of

$$p(\theta^\star(\xi),\xi)^{dr} = 0$$

Numerical procedure

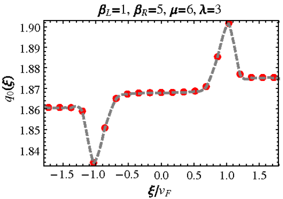
1. Solve TBA for right and left using $w(\theta) = \beta_{L,R}\cosh\theta \longrightarrow \epsilon_{L,R}(\theta)$
2. Compute
$$n_{L,R}(\theta) = \frac{1}{1+e^{\epsilon_{L,R}(\theta)}}$$
3. Fix ξ. Choose inital value $\theta^\star(\xi)_0 = 0$. Solve $p(\theta^\star(\xi),\xi)^{dr} = 0$ iteratively
$$n_n(\theta) = n_L(\theta)\Theta(\theta-\theta^\star_n) + n_R(\theta)\Theta(\theta^\star_n-\theta)$$
$$p(\theta^\star_{n+1})^{dr} = p(\theta^\star_n) + \int \frac{d\alpha}{2\pi}\varphi(\theta^\star_n-\alpha)n_n(\alpha)p(\theta^\star_n)^{dr}$$
$$p(\theta^\star_{n+1}) = 0 \quad \longrightarrow \quad \theta^\star_{n+1}$$

Numerical implementation II

1. Use the stable $\theta^\star(\xi)$ to compute the total occupation number

$$n(\theta,\xi) = n_L(\theta)\Theta(\theta-\theta^\star(\xi)) + n_R(\theta)\Theta(\theta^\star(\xi)-\theta)$$

2. We can now compute all dressed quantities $h(\theta,\xi)^{dr}$ as we need
3. Finally we can use $n(\theta,\xi)$ and $h_i(\theta,\xi)^{dr}$ to compute all average densities and currents $\mathbf{q}_i(\xi)$, $\mathbf{j}_i(\xi)$
4. The procedure can be repeated at different rays ξ

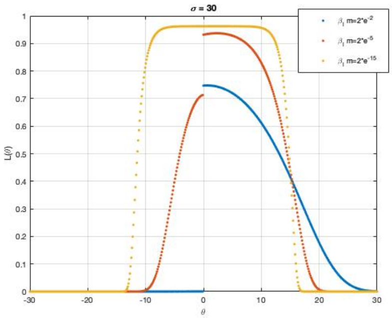


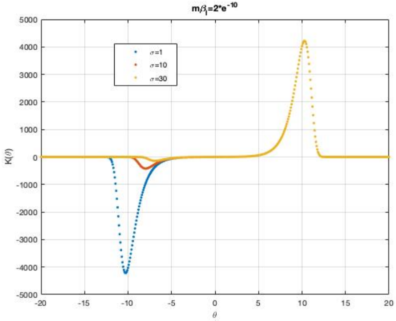
In the CFT limit one can compare this results with the expected prediction (Bernard, Doyon 2016)
$J_{CFT} = \frac{\pi c}{12}(T_L^2 - T_R^2)$

Preliminary numerical results for TIM

We have performed some numerical checks for the simplest non-diagonal $A_1 \diamond A_2$ model: the Tricritial Ising Model perturbed by its least relevant operator ϕ_{13}.
Initial solutions with discontinuity

$$\sigma = \frac{\beta_R}{\beta_L} = \frac{T_L}{T_R} \qquad , \qquad K(\theta) = n_\ell(\theta) p_\ell(\theta)$$





Conclusions

- TBA methods allow to access exact information about non-equilibrium features of integrable systems.
- In particular, in emergent hydrodynamic paradigma, one can describe exactly and non-perturbatively the stationary currents corresponding to steady states between two thermal reservoirs.
- The TBA has been generalized to cases with non-diagonal S-matrix and put in relation with Y-systems. Modifications in the Euler equations have been pointed out.

Many issues have still to be developed within this technique. For example:

- Where possible, an NLIE should be used instead. This would give access to studies in very important models, like e.g. sine-Gordon theory (next step)
- Numerical simulations are to be performed and checked against these theoretical results (TEBD method)

Thank you