Extended T-Systems, Q Matrices and T-Q Relations for $s\ell(2)$ Models at Roots of Unity

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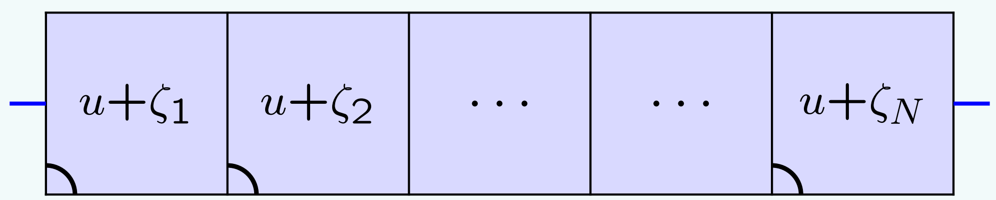
$\boldsymbol{T}(u) =$ 

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Critical Six-Vertex Model

- The R matrices and $\check{R}$ face operators of the critical six-vertex model are

$$\boxed{R:\mathbb{C}\to\mathrm{End}(\mathbb{C}^2\otimes\mathbb{C}^2):\quad R(u)=P_{12}\check{R}(u),\quad \check{R}(u)=\begin{pmatrix}\frac{\sin(\lambda-u)}{\sin\lambda} & 0 & 0 & 0\\ 0 & g & \frac{\sin u}{\sin\lambda} & 0\\ 0 & \frac{\sin u}{\sin\lambda} & g^{-1} & 0\\ 0 & 0 & 0 & \frac{\sin(\lambda-u)}{\sin\lambda}\end{pmatrix}}$$

where P_{12} permutes the two copies of $\mathbb{C}^2$ and $\lambda\in(0,\pi)$ is the crossing parameter.

- Choosing the gauge $g=z=e^{iu}$ gives

$$\boxed{\check{R}_{j,j+1}(u)=\frac{\sin(\lambda-u)}{\sin\lambda}\,I+\frac{\sin u}{\sin\lambda}\,e_j}$$

in terms of the generators of the Temperley-Lieb algebra with generators e_j

$$\boxed{e_j^2=\beta e_j,\quad e_je_{j\pm1}e_j=e_j,\quad \beta=2\cos\lambda}$$

- The commuting single row transfer matrices, for diagonal twisted boundary conditions, are

$$\boxed{\boldsymbol{T}(u)=\mathrm{tr}_a\,\boldsymbol{\Theta}_{a,a}\overset{a}{\mathcal{T}}(u),\qquad \boldsymbol{\Theta}=\begin{pmatrix}\omega & 0\\ 0 & \omega^{-1}\end{pmatrix},\quad \omega\in\mathbb{C},\qquad [\boldsymbol{T}(u),\boldsymbol{T}(v)]=0}$$

where the monodromy matrix is

$$\boxed{\overset{a}{\mathcal{T}}(u)=i^N\,R_{a1}(u+\zeta_1)\,R_{a2}(u+\zeta_2)\cdots R_{aN}(u+\zeta_N)}$$

For periodic boundaries, $\omega=1$. In this talk, the column inhomogeneities are $\zeta_j=0$.

Baxter's T-Q Relation

T-Q Relation: Baxter's famous T-Q relation (Baxter 1972/73) for the six-vertex model is

$$\boxed{\boldsymbol{T}(u)\boldsymbol{Q}(u) = f(u)\boldsymbol{Q}(u-\lambda) + f(u-\lambda)\boldsymbol{Q}(u+\lambda), \qquad f(u) = \left(\frac{\sin u}{\sin\lambda}\right)^N, \qquad [\boldsymbol{T}(u),\boldsymbol{Q}(v)] = 0}$$

- $\boldsymbol{T}(u)$ is the commuting family of six-vertex transfer matrices.
- $\boldsymbol{Q}(u)$ is an auxiliary matrix family with eigenvalues $Q(u)$ of the form

$$\boxed{Q(u) = Q(u+2\pi) = \prod_{m=1}^{M} \sin(u-u_m), \qquad M \in \mathbb{N}}$$

- The Bethe roots u_m are determined by solving the Bethe ansatz equations

$$\boxed{\frac{\sin^N(\lambda-u)}{\sin^N u} = -\prod_{m=1}^{M} \frac{\sin(u-\lambda-u_m)}{\sin(u+\lambda-u_m)}}$$

Q Matrices: The matrix $\boldsymbol{Q}(u)$ (Baxter's mysterious Q) is not unique:

- On the cylinder, there are two "Bloch wave" solutions $\boldsymbol{Q}^{\pm}(u)$.
- Other Q matrices satisfying the T-Q relation are given by

$$\boxed{\boldsymbol{Q}'(u) = \boldsymbol{Q}(u)\boldsymbol{Q}_0(u), \quad [\boldsymbol{Q}_0(u),\boldsymbol{T}(u)] = 0, \quad [\boldsymbol{Q}_0(u),\boldsymbol{Q}(u)] = 0, \quad \boldsymbol{Q}_0(u+\lambda) = \boldsymbol{Q}_0(u)}$$

At roots of unity, these relate to the freedom of adding/removing complete p'-strings in $Q(u)$.

- Baxter constructed two Q matrices: $\boldsymbol{Q}_{p,p'}(u)$ in 1972 for roots of unity and arbitrary N and $\boldsymbol{Q}_\lambda(u)$ in 1973 for generic λ with N even. But these do not agree

$$\boxed{\lim_{\lambda\to\lambda_{p,p}} \boldsymbol{Q}_\lambda(u) \neq \boldsymbol{Q}_{p,p'}(u), \qquad N \text{ even}}$$

Fusion Hierarchy

- The fusion hierarchy equations are

$$\boxed{\begin{aligned} \boldsymbol{T}_0^n \boldsymbol{T}_n^1 &= f_n \boldsymbol{T}_0^{n-1} + f_{n-1} \boldsymbol{T}_0^{n+1} \\ \boldsymbol{T}_0^1 \boldsymbol{T}_1^n &= f_{-1} \boldsymbol{T}_2^{n-1} + f_0 \boldsymbol{T}_0^{n+1} \end{aligned} \qquad n \in \mathbb{Z}}$$

$$\boldsymbol{T}_k^n = \boldsymbol{T}^n(u+k\lambda), \qquad f_k = f(u+k\lambda), \qquad \boldsymbol{T}^{-1}(u) = 0, \qquad \boldsymbol{T}^0(u) = f(u-\lambda)\boldsymbol{I}$$

$$\boldsymbol{T}_0^n := -\boldsymbol{T}_{n+1}^{-n-2}, \quad n < 0, \qquad \boldsymbol{T}^n(u+\pi) = (-)^N \boldsymbol{T}^n(u)$$

- For $n \geq 0$, these recursively define the fused transfer matrices $\boldsymbol{T}^n(u)$ as polynomials in the fundamental transfer matrix $\boldsymbol{T}(u) = \boldsymbol{T}^1(u)$.

- Pronko (2000) applied a shift of $-n\lambda$ in the fusion hierarchy to obtain

$$\boxed{\begin{aligned} \boldsymbol{T}^1(u)\boldsymbol{T}^n(u-n\lambda) &= f(u)\,\boldsymbol{T}^{n-1}(u-n\lambda) + f(u-\lambda)\,\boldsymbol{T}^{n+1}(u-n\lambda) \\ &= f(u)\,\boldsymbol{T}^{n-1}(u-\lambda-(n-1)\lambda) + f(u-\lambda)\,\boldsymbol{T}^{n+1}(u+\lambda-(n+1)\lambda) \end{aligned}}$$

which takes the form of a *"generalized T-Q relation"*

$$\boxed{\boldsymbol{T}(u)\boldsymbol{Q}^n(u) = f(u)\boldsymbol{Q}^{n-1}(u-\lambda) + f(u-\lambda)\boldsymbol{Q}^{n+1}(u+\lambda), \qquad \boldsymbol{Q}^n(u) = \boldsymbol{T}^n(u-n\lambda)}$$

- Taking an infinite fusion limit and assuming existence, Yang-Nepomechie-Zhang (2006) "formally regained" a T-Q relation for generic λ from Pronko's generalized T-Q relation

$$\boxed{\boldsymbol{T}(u)\boldsymbol{Q}(u) = f(u)\boldsymbol{Q}(u-\lambda) + f(u-\lambda)\boldsymbol{Q}(u+\lambda), \qquad \boldsymbol{Q}(u) = \lim_{n\to\infty} \boldsymbol{T}^n(u-n\lambda)}$$

- To make sense of an infinite fusion limit, we restrict to roots of unity $\lambda = \frac{(p'-p)\pi}{p'}$.

Extended Q Matrices

- At roots of unity $\lambda = \frac{(p'-p)\pi}{p'}$, the fused transfer matrices $\boldsymbol{T}^n(u)$ admit the closure

$$\boxed{\boldsymbol{T}^{p'}(u) = \boldsymbol{T}^{p'-2}(u+\lambda) + 2\sigma \boldsymbol{J}\,\boldsymbol{T}^0(u), \qquad \sigma = i^{-N(p'-p)}}$$

(BLZ94,KSS98)

where the diagonal matrix $\boldsymbol{J}$ is given in terms of the twist $\omega \in \mathbb{C}$ by

$$\boxed{\boldsymbol{J} = \frac{1}{2}(\omega^{p'} i^{-2p\boldsymbol{S}^z} + \omega^{-p'} i^{2p\boldsymbol{S}^z}), \qquad [\boldsymbol{T}^n(u), \boldsymbol{J}] = 0}$$

The diagonal matrix $\boldsymbol{S}^z$ is the spin-$\frac{1}{2}$ magnetization with eigenvalues $S^z \in \frac{1}{2}\mathbb{Z}$.

- The fused transfer matrices with $n = yp' + j > p'$ satisfy the generalized closure relations

$$\boxed{\boldsymbol{T}^{yp'+j}(u) = \sigma^y U_y(\boldsymbol{J})\,\boldsymbol{T}^j(u) + \sigma^{y-1} U_{y-1}(\boldsymbol{J})\,\boldsymbol{T}^{p'-2-j}(u+(j+1)\lambda), \quad y \in \mathbb{Z}, \quad j = 0, 1, \ldots, p'-1}$$

where $U_k(x)$ is the k-th Chebyshev polynomial of the second kind. We write

$$\boxed{\boldsymbol{J} = \cos \boldsymbol{\Lambda} = \frac{1}{2}\big(e^{i\boldsymbol{\Lambda}} + e^{-i\boldsymbol{\Lambda}}\big), \qquad e^{i\boldsymbol{\Lambda}} = \omega^{p'} i^{-2p\boldsymbol{S}^z}}$$

- For $\omega = 1$ and $pS^z \in \mathbb{Z}$, the infinite fusion limits $n \to \pm\infty$ of the eigenvalues $T^n(u)$ exist provided the limit is taken through suitable subsequences. Explicitly, setting $n = yp' + j$

$$\boxed{\lim_{y \to \pm\infty} \frac{T^{yp'+j}(u)}{\sigma^{y-1} U_{y-1}(J)} = \begin{cases} \mathcal{Q}^{j,\pm}(u), & |e^{i\Lambda}| > 1 \text{ or } e^{i\Lambda} = \pm 1 \\ \mathcal{Q}^{j,\mp}(u), & |e^{i\Lambda}| < 1 \end{cases}}$$

where $\mathcal{Q}^{j,\pm}(u)$ are the eigenvalues of the fundamental objects we call extended Q matrices

$$\boxed{\boldsymbol{\mathcal{Q}}^{j,\pm}(u) = \sigma\, e^{\pm i\boldsymbol{\Lambda}}\, \boldsymbol{T}^j(u) + \boldsymbol{T}^{p'-j-2}(u + (j+1)\lambda), \qquad j = 0, 1, \ldots, p' - 1}$$

Extended T-Q Relations

- For twisted boundary conditions $\omega \neq 1$, let us forget about convergence and work directly with the extended Q matrices $\boldsymbol{\mathcal{Q}}^{j,\pm}(u)$. These satisfy the periodicity and conjugacy properties

$$\boldsymbol{\mathcal{Q}}^{p'+j,\pm}(u) = \sigma\, e^{\pm i\boldsymbol{\Lambda}} \boldsymbol{\mathcal{Q}}^{j,\pm}(u), \qquad \boldsymbol{\mathcal{Q}}^{j,\pm}(u+\pi) = (-1)^N \boldsymbol{\mathcal{Q}}^{j,\pm}(u)$$

$$\boldsymbol{\mathcal{Q}}^{p'-j-2,\pm}(u) = \sigma\, e^{\pm i\boldsymbol{\Lambda}} \boldsymbol{\mathcal{Q}}^{j,\mp}(u-(j+1)\lambda)$$

- Remarkably, the extended Q matrices satisfy the bilinear factorization identities

$$\boxed{\boldsymbol{\mathcal{Q}}^{j,+}(u)\boldsymbol{\mathcal{Q}}^{j,-}(u) = \boldsymbol{T}^{p'-1}(u)\boldsymbol{T}^{p'-1}(u+(j+1)\lambda), \qquad j \in \mathbb{Z}}$$

Indeed, after a shift of $-j\lambda$, the bilinear factorization identities take the explicit form

$$\boxed{\Big(\sigma e^{i\boldsymbol{\Lambda}}\boldsymbol{T}^j(u-j\lambda) + \boldsymbol{T}^{p'-j-2}(u+\lambda)\Big)\Big(\sigma e^{-i\boldsymbol{\Lambda}}\boldsymbol{T}^j(u-j\lambda) + \boldsymbol{T}^{p'-j-2}(u+\lambda)\Big) = \boldsymbol{T}^{p'-1}(u-j\lambda)\boldsymbol{T}^{p'-1}(u+\lambda)}$$

This identity is the case $k = p'-j-1$ of the two-index extended T-system of bilinear identities

$$\boxed{\boldsymbol{T}^{j+k}_{-j}\boldsymbol{T}^{p'-1}_{1} = \boldsymbol{T}^{j}_{-j}\boldsymbol{T}^{p'-1-k}_{k+1} + 2\sigma \boldsymbol{J}\,\boldsymbol{T}^{j}_{-j}\boldsymbol{T}^{k-1}_{1} + \boldsymbol{T}^{k-1}_{1}\boldsymbol{T}^{p'-2-j}_{1}, \quad j,k \in \mathbb{Z}}$$

- Using the fusion hierarchy, we separately obtain the set of *extended T-Q relations*

$$\boxed{\begin{aligned} \boldsymbol{T}(u)\boldsymbol{\mathcal{Q}}^{j,\pm}(u-j\lambda) &= f(u)\boldsymbol{\mathcal{Q}}^{j-1,\pm}(u-j\lambda) + f(u-\lambda)\boldsymbol{\mathcal{Q}}^{j+1,\pm}(u-j\lambda) \\ \boldsymbol{T}(u)\boldsymbol{\mathcal{Q}}^{j,\pm}(u+\lambda) &= f(u-\lambda)\boldsymbol{\mathcal{Q}}^{j-1,\pm}(u+2\lambda) + f(u)\boldsymbol{\mathcal{Q}}^{j+1,\pm}(u) \end{aligned}}$$

The extended Q matrices $\boldsymbol{Q}^{j,\pm}(u)$ are no longer auxiliary quantities! They are defined as polynomials in $\boldsymbol{T}(u)$ through the T-system.

Bethe Roots

- The Bethe roots appear as the zeros of $T^{p'-1}(u+\lambda)$ or equivalently $\mathcal{Q}^{p'-1,\pm}(u+\lambda)$! Here we used $\boldsymbol{\mathcal{Q}}^{p'-1,\pm}(u) = \sigma\, e^{\pm i\boldsymbol{\Lambda}}\boldsymbol{T}^{p'-1}(u)$.

- Let $\{v_k\}$ be the set of zeros of $T^{p'-1}(u+\lambda)$ in a periodicty strip. Likewise let $\{w_\ell^{j,\pm}\}$ be the sets of zeros of $\mathcal{Q}^{j,\pm}(u-j\lambda)$. The bilinear factorization identities then imply that

$$\boxed{\prod_k \sin(u-v_k)\sin\Big(u-(j+1)\lambda-v_k\Big) = \prod_\ell \sin\left(u-w_\ell^{j,+}\right)\sin\left(u-w_\ell^{j,-}\right)}$$

For generic ω with $e^{i\Lambda}\neq\pm1$, we observe that the zeros are all simple and that there are no complete p'-strings. In contrast, at $e^{i\Lambda}=\pm1$, complete p' strings can and do occur.

- In the generic case, each zero $u=u_m$ of the above product belongs to $\{v_k\}$ or $\{v_k+(j+1)\lambda\}$, and likewise belongs to $\{w_\ell^{j,+}\}$ or $\{w_\ell^{j,-}\}$. Let us partition the zeros $\{v_k\}$

$$\boxed{E^\pm = E^{j,\pm} = \{v_k\}\cap\{w_\ell^{j,\pm}\},\qquad j=0,\ldots,p'-2,\qquad E^+\cup E^- = \{v_k\}}$$

Crucially, we observe that *$E^{j,\pm}$ do not depend on j*. This implies that the division of the zeros of $T^{p'-1}(u+\lambda)$ between $\mathcal{Q}^{j,+}(u-j\lambda)$ and $\mathcal{Q}^{j,-}(u-j\lambda)$ is identical for all j.

- The cardinalities of the sets $E^\pm$ are $|E^\pm| = \frac{N}{2}\mp S^z$. Defining

$$\boxed{Q^+(u) = \prod_{u_m\in E^+}\sin(u-u_m),\qquad Q^-(u) = \prod_{u_m\in E^-}\sin(u-u_m)}$$

it is readily verified numerically that $u_m\in E^\pm$ are the Bethe roots.

Baxter's T-Q Eigenvalue Relations

- The extended T-Q eigenvalue relations imply the usual Baxter T-Q eigenvalue relation.
- Using conjugacy to divide the factors, we find the BLZ1996 eigenvalue decompositions

$$\boxed{\begin{aligned} \mathcal{Q}^{j,+}(u-j\lambda) &= R^{j,+}(u)Q^{+}(u)Q^{-}(u-(j+1)\lambda) \\ \mathcal{Q}^{j,-}(u-j\lambda) &= R^{j,-}(u)Q^{-}(u)Q^{+}(u-(j+1)\lambda) \\ T^{p'-1}(u+\lambda) &= \phi(u)Q^{+}(u)Q^{-}(u) \\ T^{p'-1}(u-j\lambda) &= \phi(u)Q^{+}(u-(j+1)\lambda)Q^{-}(u-(j+1)\lambda) \end{aligned}}$$

- In the generic case, $R^{j,\pm}(u)$ and $\phi(u)$ are constants satisfying $R^{j,+}(u)R^{j,-}(u) = \phi(u)^2$. Their calculation yields

$$\boxed{\frac{R^{j+1,\pm}(u)}{R^{j,\pm}(u)} = \tilde{\omega}^{\pm 1}, \qquad \tilde{\omega} = \omega\, e^{-i\pi S^z}}$$

- The extended T-Q and Baxter T-Q relations now take the equivalent scalar forms

$$\boxed{\begin{aligned} T(u)\mathcal{Q}^{j,\pm}(u-j\lambda) &= f(u)\mathcal{Q}^{j-1,\pm}(u-j\lambda) + f(u-\lambda)\mathcal{Q}^{j+1,\pm}(u-j\lambda) \\ T(u)Q^{\pm}(u) &= \tilde{\omega}^{\mp 1} f(u)Q^{\pm}(u-\lambda) + \tilde{\omega}^{\pm 1} f(u-\lambda)Q^{\pm}(u+\lambda) \end{aligned}}$$

The second form follows using the BLZ decompositions and dividing by $R^{j,\pm}(u)Q^{\mp}(u-(j+1)\lambda)$. Strikingly, the dependence on j magically disappears!

- Substituting $u = u_m$, we recover the Bethe ansatz equations for periodic twisted boundary conditions on the cylinder. The presence of the factors $e^{\pm i\pi S^z}$ in the T-Q relation, which are absent in Baxter's T-Q equation are due to our choice of gauge in the R-matrix.

Non-Generic Case: Complete p' Strings

- Consider a sector with magnetization $S^z = m$ and twist such that $e^{i\Lambda} = \pm 1$, for example, $S^z = 0$ and $\omega = 1$. In such cases, the bilinear factorization identity for eigenvalues is

$$\boxed{\Big[\sigma\, T^j(u - j\lambda) \mp T^{p'-j-2}(u+\lambda)\Big]^2 = T^{p'-1}(u - j\lambda) T^{p'-1}(u+\lambda)}$$

- All the zeros of the left side are double, so the same must hold for the right side. Numerically, we see each of these zeros is twice degenerate and never more. The zeros of $T^{p'-1}(u+\lambda)$ can either be single or double.

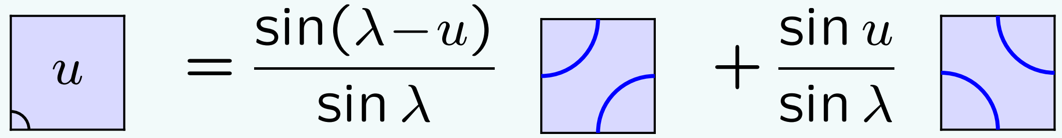
- If $u = u_0$ is a single zero of $T^{p'-1}(u+\lambda)$, the double zero of the left is evenly split between the two factors of the right side, implying that $T^{p'-1}(u_0 - j\lambda) = 0$. This holds true for $j \in \mathbb{Z}$, so that $T^{p'-1}(u+\lambda) = 0$ for $u = u_0, u_0 + \lambda, \ldots, u_0 + (p'-1)\lambda$. The zeros of $T^{p'-1}(u+\lambda)$ are therefore either double zeros or single zeros forming complete p'-strings.

- In the BLZ decompositions, the zeros of the complete p'-strings are encoded in the functions $R^{j,\pm}(u)$ and $\phi(u)$, which in this case are not constants. Instead, $\phi(u) = \phi(u+\lambda)$ and the sets E^+ and E^- are equal, implying that $Q^+(u) = Q^-(u) = Q(u)$, $Q^+(u)Q^-(u) = Q(u)^2$ and $R^{j,+}(u) = R^{j,-}(u) = R^j(u)$. This last function satisfies $R^j(u)^2 = \phi(u)^2$ and therefore $R^j(u) = R^j(u+\lambda)$.

- The occurrence of complete p' strings is thus properly accounted for and they occur at fixed not arbitrary positions.

Logarithmic Minimal Models on the Strip

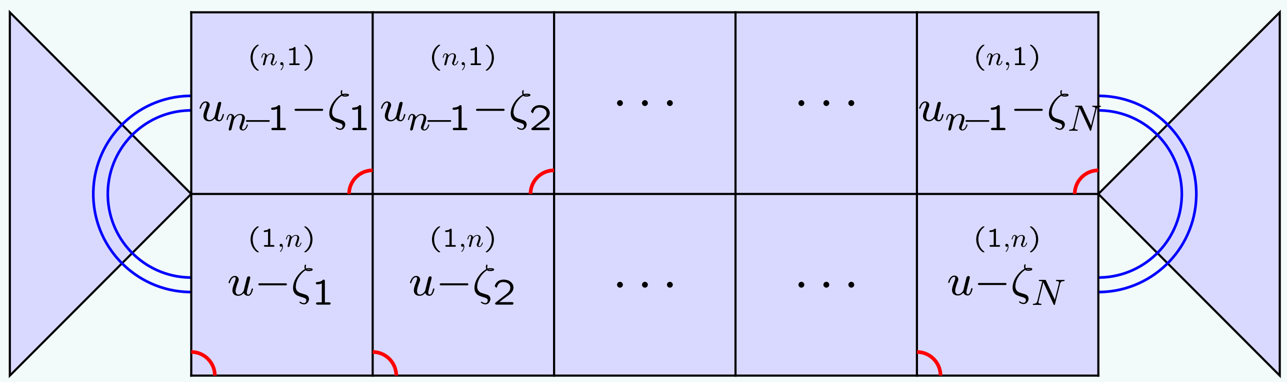
- The elementary face operator for the dense loop model is defined as



where u is the spectral parameter, λ is the crossing parameter and $\beta = 2\cos\lambda$ is the fugacity of closed loops in the bulk.

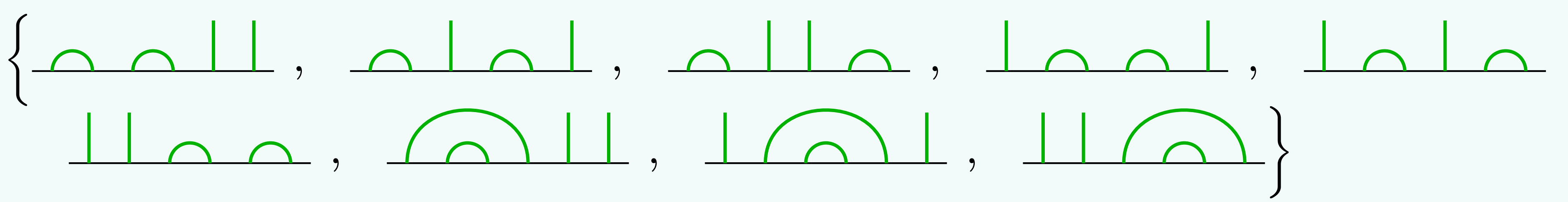
- For Kac vacuum boundary conditions, the $1 \times n$ fused transfer tangle is defined diagrammatically by

$$\boldsymbol{D}^n(u) =$$

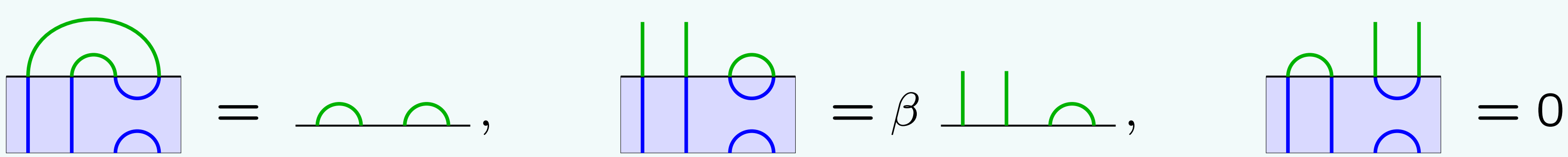


- The transfer matrices act on a vector space V_N^d of link states: $\dim V_N^d = \binom{N}{\frac{N-d}{2}} - \binom{N}{\frac{N-d-2}{2}}$

$$V_{N=6}^{d=2} =$$



The (defect preserving) action of the TL algebra on the strip is



Fusion Hierarchies on the Strip

- The double-row transfer matrices $\boldsymbol{D}^n(u)$ with Kac vacuum boundary conditions satisfy the fusion hierarchy relations

$$\boxed{\begin{array}{rcl} s_{n-2}s_{2n-1}\boldsymbol{D}_0^n\boldsymbol{D}_n^1 &=& s_{n-3}s_{2n}f_n\boldsymbol{D}_0^{n-1}+s_{n-1}s_{2n-2}f_{n-1}\boldsymbol{D}_0^{n+1} \\ s_ns_{-1}\boldsymbol{D}_0^1\boldsymbol{D}_1^n &=& s_{n+1}s_{-2}f_{-1}\boldsymbol{D}_2^{n-1}+s_{n-1}s_0f_0\boldsymbol{D}_0^{n+1} \end{array} \qquad n\in\mathbb{Z}}$$

where $\boldsymbol{D}_k^n=\boldsymbol{D}^n(u+k\lambda)$, $f_k=f(u+k\lambda)$, ζ_i are bulk inhomogeneities, $\boldsymbol{D}^n(u+\pi)=\boldsymbol{D}(u)$ and

$$\boldsymbol{D}_0^{-1}=0,\quad \boldsymbol{D}_0^0=f(u-\lambda)\boldsymbol{I},\quad f(u)=\prod_{i=1}^N\frac{\sin(u+\zeta_i)\sin(u-\zeta_i)}{\sin^2\lambda},\quad s_k=\frac{\sin(2u+k\lambda)}{\sin\lambda}$$

- The extra s_k functions appearing in the fusion hierarchy are order 1 *boundary terms*. The definition of $\boldsymbol{D}_0^n$ is extended to $n\le -2$ by the convention

$$\boxed{\boldsymbol{D}_0^n:=-\boldsymbol{D}_{n+1}^{-2-n}\qquad n\le -2}$$

- Applying a shift of $-n\lambda$ gives

$$\boxed{s_{-n-2}s_{-1}\boldsymbol{D}_0^1\boldsymbol{D}_{-n}^n=s_{-n-3}s_0f_0\boldsymbol{D}_{-n}^{n-1}+s_{-n-1}s_{-2}f_{-1}\boldsymbol{D}_{-n}^{n+1}}$$

Setting $\boldsymbol{Q}^n(u)=s_{-n-2}\boldsymbol{D}^n(u-n\lambda)$ gives the *generalized* T-Q relation

$$\boxed{s_{-1}\boldsymbol{D}^1(u)\boldsymbol{Q}^n(u)=s_0f(u)\boldsymbol{Q}^{n-1}(u-\lambda)+s_{-2}f(u-\lambda)\boldsymbol{Q}^{n+1}(u+\lambda),\qquad [\boldsymbol{D}(u),\boldsymbol{Q}^j(u)]=0}$$

- As observed in YNZ2006, this would lead to a T-Q equation if the limit $\lim_{n\to\infty}\boldsymbol{Q}^n(u)$ exists. But again, the existence of this simple limit is problematic.

Closure and Extended Q Matrices

- For $\lambda = \frac{(p'-p)\pi}{p'}$, the fused transfer matrices satisfy the simple closure relation

$$\boxed{\boldsymbol{D}_0^{p'} = 2\sigma \boldsymbol{D}_0^0 + \boldsymbol{D}_1^{p'-2}, \qquad \sigma = (-1)^{p'-p}}$$

Setting $n = yp' + j$, this closure relation generalizes to the higher fused transfer matrices

$$\boxed{\boldsymbol{D}_0^{yp'+j} = \sigma^y (y+1) \boldsymbol{D}_0^j + \sigma^{y-1} y \boldsymbol{D}_{j+1}^{p'-2-j}, \quad y \in \mathbb{Z}, \quad j = 0, 1, \ldots, p'-1}$$

- Taking the limit $n \to \infty$ of $\boldsymbol{D}^n(u)$ through subsequences with j finite gives

$$\boxed{\lim_{y\to\infty} \frac{\boldsymbol{D}^{yp'+j}(u)}{y\sigma^{y-1}} = \boldsymbol{\mathcal{Q}}^j(u)}$$

where the *extended Q matrices* are given as the linear combinations

$$\boxed{\boldsymbol{\mathcal{Q}}^j(u) = \sigma \boldsymbol{D}^j(u) + \boldsymbol{D}^{p'-j-2}(u+(j+1)\lambda), \qquad j \in \mathbb{Z}}$$

- In contrast to the cylinder case, the infinite fusion limit taken through subsequences always exists on the strip. Moreover, the same extended Q matrices are obtained by taking $y \to -\infty$.
- The extended Q matrices $\boldsymbol{\mathcal{Q}}^j(u)$ satisfy the periodicity and conjugacy properties

$$\boxed{\boldsymbol{\mathcal{Q}}^j(u+\pi) = \sigma\, \boldsymbol{\mathcal{Q}}^j(u), \qquad \boldsymbol{\mathcal{Q}}^{p'+j}(u) = \sigma\, \boldsymbol{\mathcal{Q}}^j(u), \qquad \boldsymbol{\mathcal{Q}}^{p'-j-2}(u) = \sigma\, \boldsymbol{\mathcal{Q}}^j(u-(j+1)\lambda)}$$

Extended T-Q Relations

- The extended Q matrices also satisfy the bilinear factorization identities

$$\boxed{\Big(s_{j-2}\boldsymbol{Q}^j(u)\Big)^2 = s_{-3}s_{2j-1}\boldsymbol{D}^{p'-1}(u)\boldsymbol{D}^{p'-1}(u+(j+1)\lambda), \qquad j\in\mathbb{Z}}$$

Indeed, after a shift of $-j\lambda$, this reads

$$\boxed{(s_{-j-2})^2\Big(\sigma\boldsymbol{D}^j_{-j}+\boldsymbol{D}^{p'-j-2}_1\Big)^2 = s_{-2j-3}s_{-1}\boldsymbol{D}^{p'-1}_{-j}\boldsymbol{D}^{p'-1}_1}$$

This is the special case $k=p'-1-j$ of the two-index extended T-system of bilinear identities

$$\boxed{\boldsymbol{D}^{j+k}_{-j}\boldsymbol{D}^{p'-1}_1 = \frac{s_{k-1}s_{-2-j}}{s_{k-j-2}s_{-1}}\Big(\boldsymbol{D}^j_{-j}\boldsymbol{D}^{p'-1-k}_{k+1}+2\sigma\boldsymbol{D}^j_{-j}\boldsymbol{D}^{k-1}_1+\boldsymbol{D}^{k-1}_1\boldsymbol{D}^{p'-2-j}_1\Big), \qquad j,k\in\mathbb{Z}}$$

These identities are proved using the fusion hierarchy and induction.

- Separately, from the fusion hierarchy, we obtain the *extended* T-Q relations

$$\boxed{\begin{aligned} s_{-j-2}s_{-1}\boldsymbol{D}(u)\boldsymbol{Q}^j(u-j\lambda) &= s_{-j-3}s_0f(u)\boldsymbol{Q}^{j-1}(u-j\lambda)+s_{-j-1}s_{-2}f(u-\lambda)\boldsymbol{Q}^{j+1}(u-j\lambda)\\ s_js_{-1}\boldsymbol{D}(u)\boldsymbol{Q}^j(u+\lambda) &= s_{j+1}s_{-2}f(u-\lambda)\boldsymbol{Q}^{j-1}(u+2\lambda)+s_{j-1}s_0f(u)\boldsymbol{Q}^{j+1}(u)\end{aligned}}$$

For example, setting $\boldsymbol{Q}^j_k=\boldsymbol{Q}^j(u+k\lambda)$, expanding the left side of the first relation and using the fusion hierarchy gives

$$\begin{aligned} s_{-j-2}s_{-1}\boldsymbol{D}^1_0\boldsymbol{Q}^j_{-j} &= s_{-j-2}s_{-1}\boldsymbol{D}^1_0\Big(\sigma\boldsymbol{D}^j_{-j}+\boldsymbol{D}^{p'-j-2}_1\Big)\\ &= s_{-j-3}s_0f_0\Big(\sigma\boldsymbol{D}^{j-1}_{-j}+\boldsymbol{D}^{p'-j-1}_0\Big)+s_{-j-1}s_{-2}f_{-1}\Big(\sigma\boldsymbol{D}^{j+1}_{-j}+\boldsymbol{D}^{p'-j-3}_2\Big)\\ &= s_{-j-3}s_0f_0\boldsymbol{Q}^{j-1}_{-j}+s_{-j-1}s_{-2}f_{-1}\boldsymbol{Q}^{j+1}_{-j}\end{aligned}$$

Polynomial Reductions

- In the bilinear factorization identity, the transfer matrices $\boldsymbol{D}^{p'-1}(u)$ and extended Q matrices $\boldsymbol{\mathcal{Q}}^j(u)$ have order 1 factors in common. Setting $k=-j$ in the bilinear T-system gives

$$\boxed{f_{-j-1}\boldsymbol{D}_1^{p'-1}=\frac{s_{-j-1}s_{-j-2}}{s_{-2j-2}s_{-1}}\Big(\boldsymbol{D}_{-j}^{j}\boldsymbol{D}_{-j+1}^{p'-1+j}+2\sigma\boldsymbol{D}_{-j}^{j}\boldsymbol{D}_1^{-j-1}+\boldsymbol{D}_1^{-j-1}\boldsymbol{D}_1^{p'-2-j}\Big)}$$

For $j=-1$ and $j=0$, the trigonometric prefactor on the right side equals 1. For the other values of j, this prefactor is not 1. This implies that the transfer matrix $\boldsymbol{D}_1^{p'-1}$ vanishes if $\prod_{i=0}^{p'-2} s_i=0$. We therefore have the factorization

$$\boxed{\boldsymbol{D}_0^{p'-1}=\Big(\prod_{i=0}^{p'-2} s_{i-2}\Big)\hat{\boldsymbol{D}}_0^{p'-1}}$$

where the matrix entries of the reduced transfer matrix $\hat{\boldsymbol{D}}_0^{p'-1}$ are Laurent polynomials in e^{iu}.

- The bilinear factorization identities thus simplifies to

$$\boxed{\big(s_{j-2}\boldsymbol{\mathcal{Q}}_0^j\big)^2=\Big(\prod_{i=1}^{p'} s_i^2\Big)\hat{\boldsymbol{D}}_0^{p'-1}\hat{\boldsymbol{D}}_{j+1}^{p'-1},\qquad \boldsymbol{\mathcal{Q}}_0^j=\Big(\prod_{\substack{i=1\\ i\neq j-2}}^{p'} s_i\Big)\hat{\boldsymbol{\mathcal{Q}}}_0^j}$$

where $\boldsymbol{\mathcal{Q}}_0^j$ now factorizes and the reduced Q matrices $\hat{\boldsymbol{\mathcal{Q}}}_0^j$ are also Laurent polynomials in e^{iu}. After applying a shift of $-j\lambda$, the bilinear factorization identities take the reduced form

$$\boxed{(\hat{\boldsymbol{\mathcal{Q}}}_{-j}^j)^2=\hat{\boldsymbol{D}}_{-j}^{p'-1}\hat{\boldsymbol{D}}_1^{p'-1}}$$

Decomposition of Eigenvalues $\hat{\mathcal{Q}}^j(u-j\lambda)$

- The reduced bilinear factorization identity is satisfied eigenvalue by eigenvalue and the left side is a perfect square. Suppose the eigenvalues $D^{p'-1}(u)$ and $Q^{p'-1}(u)$ are both nonzero. Then it follows that $\hat{D}^{p'-1}(u+\lambda)$ can have double zeros as well as single zeros organized into complete p'-strings. For a given eigenstate, we denote by E^j the set of zeros common to $\hat{\mathcal{Q}}^j(u-j\lambda)$ and $\hat{D}^{p'-1}(u+\lambda)$, excluding the complete p' strings. As before, we find that $E^j = E^{j+1} = E$ and define

$$\boxed{Q(u) = \prod_{u_m \in E} \sin(u-u_m)}$$

- It follows that

$$\boxed{\hat{D}^{p'-1}(u+\lambda) = \phi(u)Q(u)^2, \qquad \hat{D}^{p'-1}(u-j\lambda) = \phi(u)Q(u-(j+1)\lambda)^2}$$

where $\phi(u) = \phi(u+\lambda)$ encodes the zeros of the complete p'-strings. The bilinear factorization identities then become

$$\boxed{\hat{\mathcal{Q}}^j(u-j\lambda)^2 = \Big(\phi(u)Q(u)Q(u-(j+1)\lambda)\Big)^2}$$

- Taking the square root yields the decomposition

$$\boxed{\hat{\mathcal{Q}}^j(u-j\lambda) = R^j\phi(u)Q(u)Q(u-(j+1)\lambda)}$$

where the R^j are constants satisfying $(R^j)^2 = 1$.

Baxter T-Q Relations

- The ratios R^{j+1}/R^j are obtained by comparing the coefficient of the maximal term in $z = e^{iu}$. In this case, because the coefficient of $z^{2(N-p'-1)}$ in each $\hat{\mathcal{Q}}^j(u-j\lambda)$ vanishes, one has to compare the coefficients of the next leading term. In all cases, we find

$$\boxed{\frac{R^{j+1}}{R^j} = 1}$$

- Finally, we rewrite the scalar extended T-Q relation in terms of $\hat{\mathcal{Q}}^j(u)$ and then $Q(u)$. Dividing this equation throughout by $R^j\phi(u)Q(u-(j+1)\lambda)\prod_{i=1}^{p'} s_i$ gives Baxter's T-Q relation

$$\boxed{s_{-1}D(u)Q(u) = s_0 f(u)Q(u-\lambda) + s_{-2}f(u-\lambda)Q(u+\lambda)}$$

- Assuming the zeros appear in pairs under crossing, we set

$$\boxed{Q(u) = \prod_{m=1}^{2M} \sin(u-u_m)}$$

Substituting this into Baxter's T-Q relation and setting $u = u_m$, so that the left side vanishes, then gives the Bethe ansatz equations. Using the crossing symmetry, these agree with AlcarazEtAl87, Sklyanin1988, Nepomechie2002.

T- and Y-Systems

- The periodic 6-vertex and loop transfer matrices satisfy the usual T- and Y-systems (KlümperP92,M-DuchesnePRasm2014)

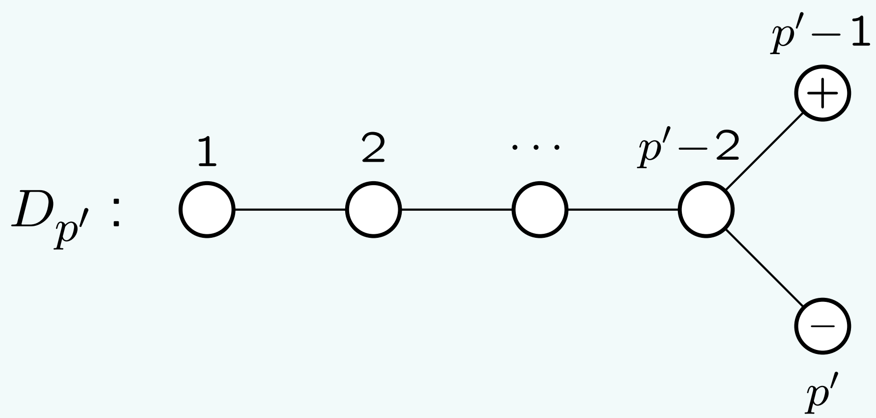
$$\boxed{\begin{array}{lll} \boldsymbol{T}_0^n\boldsymbol{T}_1^n = f_{-1}f_n\boldsymbol{I}+\boldsymbol{T}_0^{n+1}\boldsymbol{T}_1^{n-1}, & n\ge 0 & \\ & & \boldsymbol{t}_0^n = \dfrac{\boldsymbol{T}_1^{n-1}\boldsymbol{T}_0^{n+1}}{f_{-1}f_n} \\ \boldsymbol{t}_0^n\boldsymbol{t}_1^n = (\boldsymbol{I}+\boldsymbol{t}_1^{n-1})(\boldsymbol{I}+\boldsymbol{t}_0^{n+1}), & n=1,2,\ldots,p'-2 & \end{array}}$$

- At roots of unity, the closure relations are (BLZ97,KSS98,M-DuchesneKlümperP2017)

$$\boxed{\boldsymbol{I}+\boldsymbol{t}_0^{p'-1} = (\boldsymbol{I}+e^{i\boldsymbol{\Lambda}}\boldsymbol{K}_0)(\boldsymbol{I}+e^{-i\boldsymbol{\Lambda}}\boldsymbol{K}_0),\qquad \boldsymbol{K}_0\boldsymbol{K}_1 = 1+\boldsymbol{t}_1^{p'-2}}$$

$$\boldsymbol{K}_0 = \frac{i^{N(p'-p)}}{f_{-1}}\boldsymbol{T}_1^{p'-2},\qquad \boldsymbol{J} = \cos\boldsymbol{\Lambda} = T_{p'}(\tfrac{1}{2}\boldsymbol{T}(i\infty)) = \text{Chebyshev polynomial of first kind}$$

- The D-type TBA Dynkin diagram (with endpoint nodes distinguished by factors $e^{\pm i\boldsymbol{\Lambda}}$) is



- The same relations hold for boundary cases for double row transfer matrices with $\boldsymbol{\Lambda}=0$.
- In principle, the CFT spectra of all of these roots-of-unity models can be obtained by solving the D-type TBA non-linear integral equations.

Summary

- The rational points $\frac{\lambda}{\pi} \in \mathbb{Q}$ are dense on the six-vertex critical line $\frac{\lambda}{\pi} \in (0,1)$.

- At each rational point, the six-vertex model exhibits a higher loop algebra symmetry and is a distinct logarithmic theory. The higher symmetry is reflected in higher eigenvalue degeneracies and the extended T-systems of bilinear identities. The logarithmic nature is manifest in the appearance of nontrivial Jordan cells as studied by Gainutdinov-Nepomechie 2016. Indeed, $\lambda = \frac{\pi}{2}$ is dimers (PAP-VittoriniOrgeas 2017, PAP-VO-Rasmussen 2019).

- At roots of unity, the T-system closes and the six-vertex model satisfies extended T-Q relations. The extended Q matrices $\boldsymbol{\mathcal{Q}}^{j,\pm}(u)$ are not auxiliary. They are uniquely defined and unambiguously identified as explicit linear combinations of standard fused transfer matrices $\boldsymbol{T}^j(u)$ with locally defined Boltzmann face weights. The occurrence of complete p' strings in the eigenvalues is properly accounted for and there is no arbitrariness in their positions.

- The usual Baxter T-Q relations for eigenvalues are deduced from the extended T-Q relations. The analyticity/zeros of the Q matrices and Bethe roots are directly accessible numerically for modest finite sizes.

- The extended methods are systematic and general. On the strip, the methods apply to the six-vertex model with non-diagonal boundary conditions on the left and right (6 arbitrary boundary parameters) extending the results of Murgan-Nepomechie-Shi 2006 ($p = 1$, $p' \geq 2$) to all roots of unity. They also apply to the logarithmic minimal models $\mathcal{LM}(p,p')$ on the cylinder and on the strip with Robin vacuum boundary conditions.

- For critical bond percolation $\mathcal{LM}(2,3)$ on the strip with s-type boundary conditions, the new methods allow (M-DuchesneKlümperP2017) the patterns of zeros of $Q(u)$ to be completely classified in terms of q-binomials and skew q-binomials.

- Extended T-Q relations can be taken as the starting point to systematically derive NLIE.

Upcoming MathPhys/Integrability Meetings

- **Australia/New Zealand Assoc. Math. Physics (ANZAMP 2020)**
  5–7 February 2020
  Twin Towns, Tweed Heads, NSW, Australia
  `http://www.anzamp.austms.org.au/events/view/107`

- **Frontiers of Integrability (Baxter 2020)**
  10–14 February 2020
  Australian National University, Canberra, Australia
  `Webpage under construction`

- **Challenges in Integrability (Wuhan 2020)**
  25 May – 5 June 2020
  WIPM, CAS, Wuhan, China
  `http://integrable.csp.escience.cn/dct/page/1`