OCU Workshop 19/09/19

Hanany-Witten Transitions & Quantum Curves

Sanefumi Moriyama (Osaka City Univ/NITEP)

Based on: N.Kubo, S.M., 2019.

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Duality in Brane Physics

Hanany-Witten Transitions & Quantum Curves

Quantum Toroidal Algebra

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Gauge Theories & Integrability

- Many Physical Quantities of Gauge Theories with Large SUSY Enjoy Integrability
- The Most Famous Example

Anomalous Dimensions of Trace Operators

in *D*=4 $\mathcal{N}$=4 U(*N*) Super Yang-Mills Theory

N x D3-branes

$\Updownarrow$

Integrable Spin Chain

[Minahan-Zarembo, ..., Beisert, ..., many talks in this workshop]

- Many More ...

Today, – ABJM Theory –

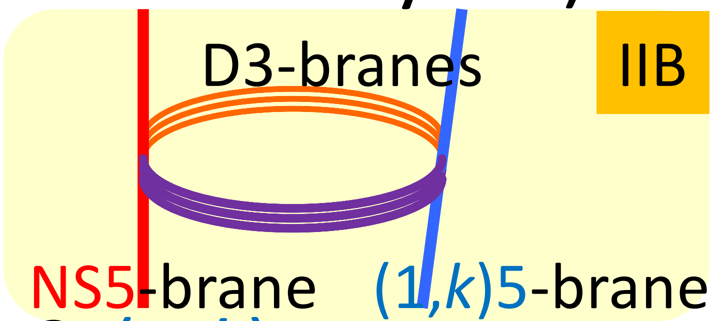
- ABJM Theory

D=3 $\mathcal{N}$=6 U(N)$_k$ x U(N)$_{-k}$

Super Chern-Simons Theory

$\leftrightarrow$ N x M2-branes on C^4/Z$_k$ (In Brane Physics)

- Why?



D3-branes on S^1

with perpendicular NS5-brane & (1,k)5-brane

(After T-Duality & M-Lift)

Integrability for ABJM

- Aspects of Integrability

Partition Function or One-Point Function of Half-BPS Wilson Loop in Grand Canonical Ensemble

$\Updownarrow$

Giambelli Identity, Jacobi-Trudi Identity,

Modified KP Hierarchy, 2D Toda Lattice Hierarchy

[Matsumoto-M 2013, Matsuno-M, Furukawa-M, Kubo-M, Furukawa-M 2019]

Another Aspect: More 5-branes

- Generalizations with More NS5's & $(1,k)5$'s

(M2-branes on More Complicated Background)

D3 with 2 x NS5 & 2 x $(1,k)5$ on S^1

$\Updownarrow$

Quantum Curve of Del Pezzo D5

Branes

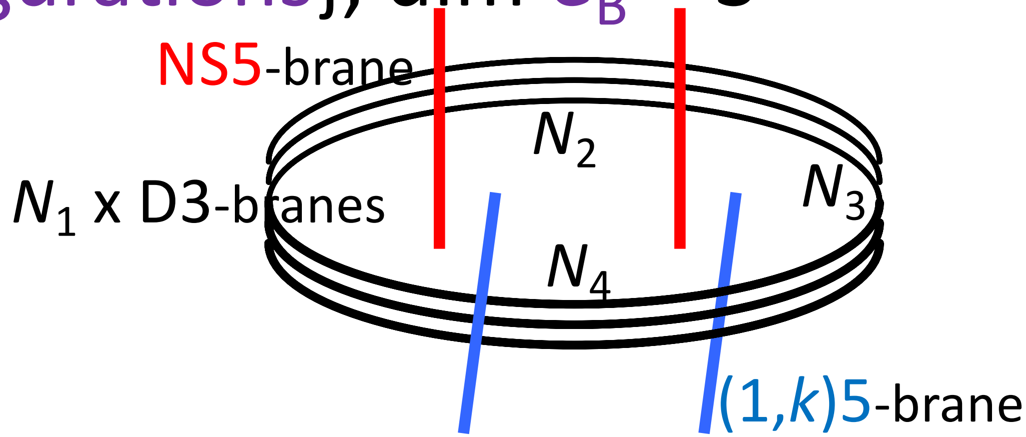
Dynkin's
Classification
D5 = so(10)

Deformations

- D3 with 2 x NS5 & 2 x (1,*k*)5 on S^1

  Four Intervals with 1 Overall Rank & 3 Relative Ranks

  C_B = {Brane Configurations}, dim C_B = 3



- Quantum D5 Curve

  Characterized by 5 Parameters [Kajiwara-Noumi-Yamada 2015]

  C_P = {Point Configurations}, dim C_P = 5

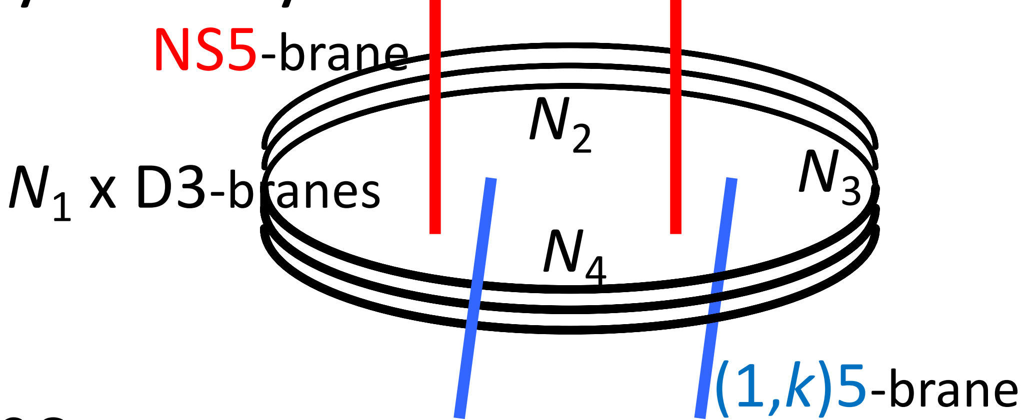
Embedding of C_B in C_P ?

Clues

- In Brane Configurations

Exchange of 5-branes

→ #{D3-branes} Change by Hanany-Witten Transitions



- In Point Configurations

Parameters of Quantum Curve Change

by Similarity Transformations

Embedding of C_{B} in C_{P} ?

Difficulties

- Disastrous in Continuing HW Transitions Uncritically and Without Strategies

$$\langle N_1 \textcolor{red}{\bullet} N_2 \textcolor{red}{\bullet} N_3 \textcolor{blue}{\bullet} N_4 \textcolor{blue}{\bullet} \rangle$$
$$= \langle N_1 \textcolor{red}{\bullet} N_2 \textcolor{blue}{\bullet} N_2+N_4-N_3+k \textcolor{red}{\bullet} N_4 \textcolor{blue}{\bullet} \rangle$$
$$= \langle N_1 \textcolor{red}{\bullet} N_2 \textcolor{blue}{\bullet} N_2+N_4-N_3+k \textcolor{blue}{\bullet} N_1+N_2-N_3+2k \textcolor{red}{\bullet} \rangle$$
$$= \langle N_1+N_2-N_3+2k \textcolor{red}{\bullet} N_1 \textcolor{red}{\bullet} N_2 \textcolor{blue}{\bullet} N_2+N_4-N_3+k \textcolor{blue}{\bullet} \rangle$$

First Two Eq: HW Transit., $K \textcolor{red}{\bullet} L \textcolor{blue}{\bullet} M = K \textcolor{blue}{\bullet} K+M-L+k \textcolor{red}{\bullet} M$

NS5 $(1,k)5$ [Hanany-Witten 1997]

Last Eq: Cyclicity

- Corresponding Curve ??

Embedding of C_B in C_P ?

- Key Ideas

To Avoid Uncritical Use of Cyclicity.

To Fix Reference Frame in Mechanics or Local Chart in Geometry.

To Distinguish the 5-branes.

- Messages

M2 without IIB Brane Configurations

(Non Lagrangian Theories, Non Matrix Models)

Brane Transitions, Unknown Previously

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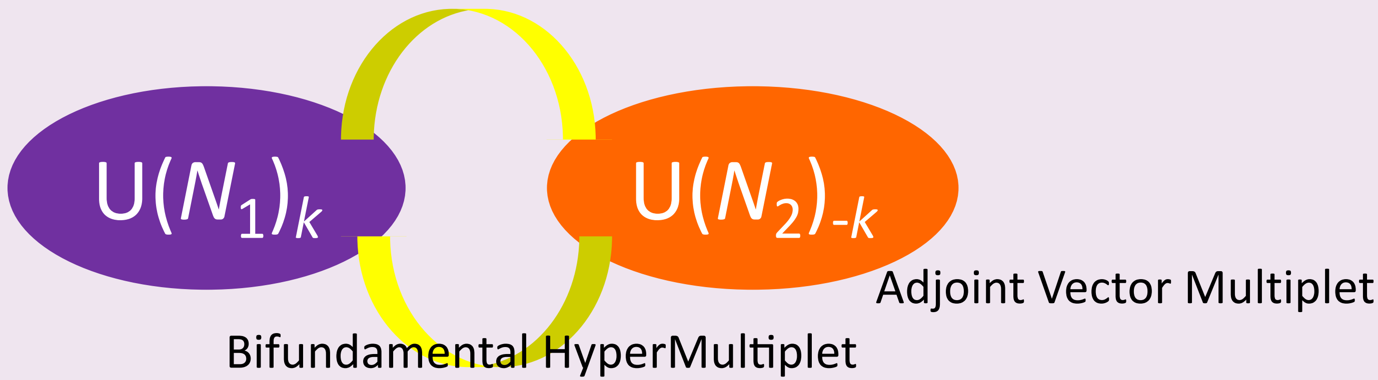
0. Introduction
1. ABJM Theory
2. Super Chern-Simons Theories

(Generalizations of ABJM)

3. Quantum Curves: C_{P}
4. Hanany-Witten Transitions: C_{B}
5. QC vs HW
6. Symmetries
7. Conclusions

ABJM Theory

D=3 $\mathcal{N}$=6 Super Chern-Simons Theory



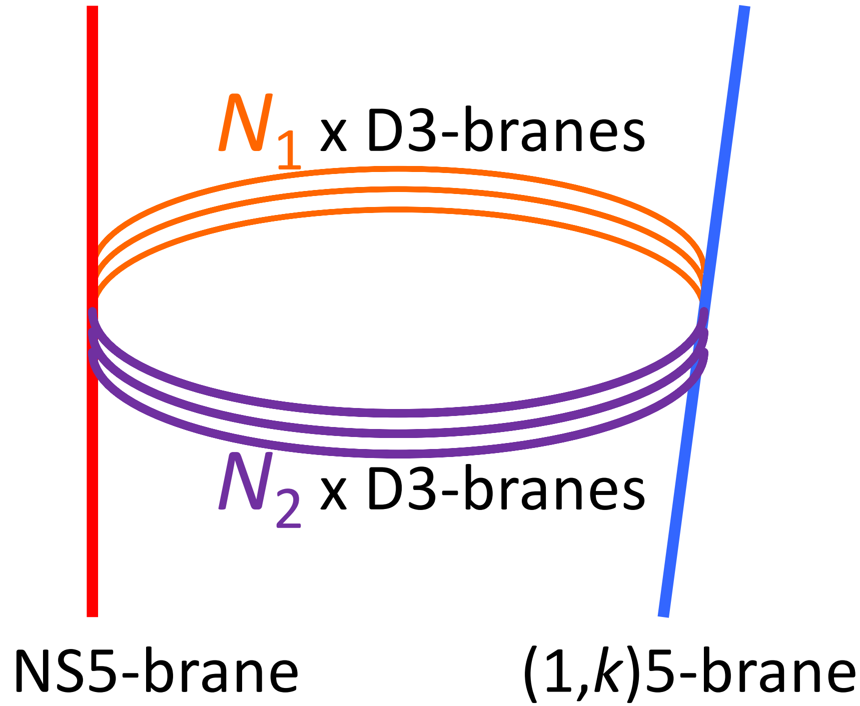
[Aharony-Bergman-Jafferis-Maldacena
Hosomichi-Lee-Lee-Lee-Park, ABJ 2008]

$\min(N_1,N_2)$ M2 + $|N_2-N_1|$ fractional M2 on C^4 / Z_k

Brane Configuration in IIB

From Large Supersymmetries

[Kitao-Ohta-Ohta 1998, ...]



(IIB String Theory)

→ T-duality to IIA

→ Lift to M-Theory

ABJM Matrix Model

Partition Function & VEV of ½-BPS Wilson Loop

- Defined by Infinite-Dim Path Integral
- Localized to Finite-Dim Matrix Integration

(Cancellations between Bosons & Fermions in SUSY Theories)

[Witten, Pestun, Kapustin-Willett-Yaakov 2009]

ABJM Matrix Model

Characters labeled by Young Diagram

$$\langle s_\lambda \rangle_k (N_1 | N_2) = \frac{i^{-\frac{1}{2}\left(N_1^2 - N_2^2\right)}}{N_1! N_2!} \int \prod_{m=1}^{N_1} \frac{d\mu_m}{2\pi} \prod_{n=1}^{N_2} \frac{d\nu_n}{2\pi} \, s_\lambda\left(e^\mu | e^\nu\right)$$

Vector Multiplet

$$\boxed{\frac{\prod_{m<m'}^{N_1}\left(2\sinh\frac{\mu_m-\mu_{m'}}{2}\right)^2 \prod_{n<n'}^{N_2}\left(2\sinh\frac{\nu_n-\nu_{n'}}{2}\right)^2}{\prod_{m=1}^{N_1}\prod_{n=1}^{N_2}\left(2\cosh\frac{\mu_m-\nu_n}{2}\right)^2}} \quad \boxed{\exp\frac{ik}{4\pi}\left(\sum_{m=1}^{N_1}\mu_m^2 - \sum_{n=1}^{N_2}\nu_n^2\right)}$$

HyperMultiplet

1-loop

Classical

Similar to Correlation Functions of XX model ? [Goehmann's talk]

Especially, For Partition Function,

$$Z_k(N_1,N_2) = \langle 1 \rangle_k(N_1|N_2)$$

- No Schur Functions

$$s_\lambda = 1$$

- Introducing Fresnel Integrations

$$\mathrm{D}\mu = \mathrm{d}\mu \exp(+\mathrm{i}\ldots) \ldots, \ \mathrm{D}\nu = \mathrm{d}\nu \exp(-\mathrm{i}\ldots) \ldots$$

$$Z_k(N_1,N_2) = \int \mathrm{D}^{N_1}\mu\, \mathrm{D}^{N_2}\nu$$

$$\prod \mathrm{sh}^2(\mu_m-\mu_{m'}) \prod \mathrm{sh}^2(\nu_n-\nu_{n'}) / \prod \mathrm{ch}^2(\mu_m-\nu_n)$$

$$\mathrm{sh}\, x = 2 \sinh x/2, \ \mathrm{ch}\, x = 2 \cosh x/2$$

Grand Canonical Ensemble

- WOLOG, Assuming $N_1 = N \leqq N_2 = N + M$

Overall Rank: $N = \min(N_1, N_2)$

Relative Rank: $M = |N_2 - N_1|$

- Grand Partition Function

[Marino-Putrov 2011]

$$\Xi_{k,M}(z) = \sum_{N=0}^{\infty} z^N Z_k(N, N+M)$$

(N : Particle Number, z : Dual Fugacity)

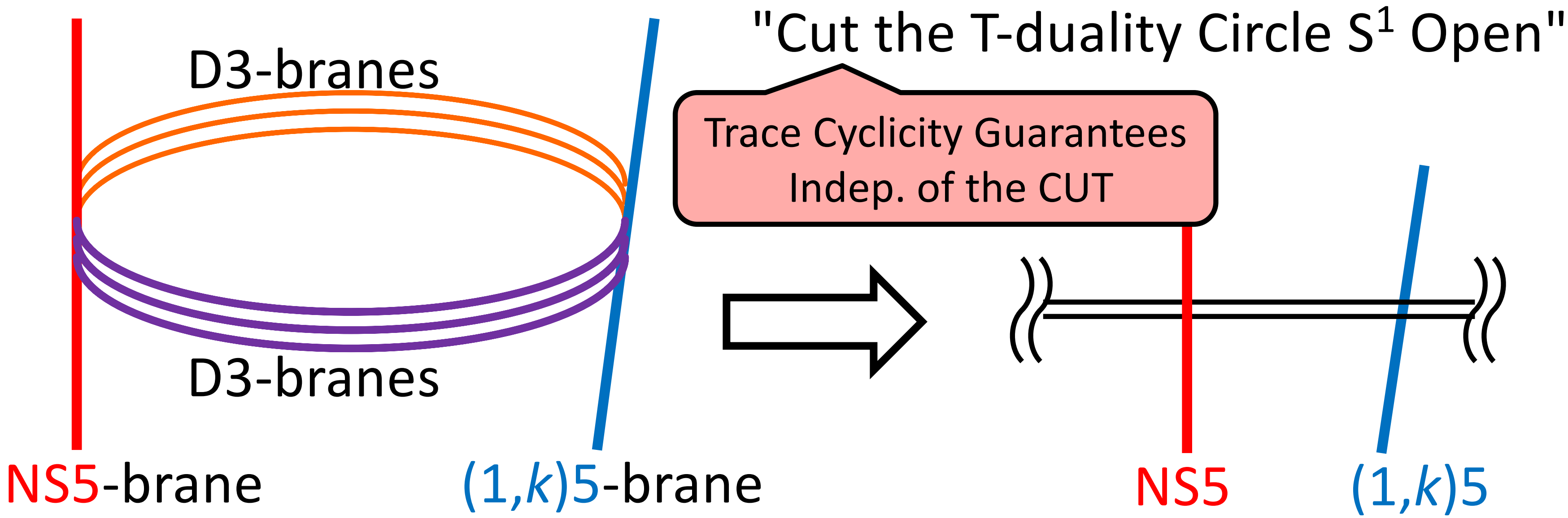
Spectral Determinant

$$\Xi_{k,M}(z) = \mathrm{Det}(\, 1 + z\, H^{-1}\,)$$

- At least, Without Rank Deform, $N_1 = N_2$, $M = 0$

$$H^{-1} = \mathcal{P}^{-1}\,\mathcal{Q}^{-1} \quad \text{or} \quad H = \mathcal{Q}\,\mathcal{P}$$

$$\mathcal{Q} = Q^{1/2}+Q^{-1/2},\ \mathcal{P} = P^{1/2}+P^{-1/2},\ Q = e^{q},\ P = e^{p},\ [q,p] = i\,\hbar,\ \hbar = 2\pi k$$



Spectral Determinant

$$\Xi_{k,M}(z) = \mathrm{Det}(\, 1 + z\, H^{-1}\,)$$

- With Rank Deformations

Still Same "Polynomial" ?

$$H = \#\, Q^{1/2}\, P^{1/2} + \#\, Q^{1/2}\, P^{-1/2} + \#\, Q^{-1/2}\, P^{1/2} + \#\, Q^{-1/2}\, P^{-1/2}$$

(Answer: YES, But we postpone the question.)

Why $\mathcal{P}^{-1} \leftrightarrow$ NS5, $\mathcal{Q}^{-1} \leftrightarrow (1,k)5$?

- ABJM Matrix Model

$$Z_k(N,N) = \int D^N\mu\, D^N\nu \prod \mathrm{sh}^2(\mu_m - \mu_{m'}) \prod \mathrm{sh}^2(\nu_n - \nu_{n'}) / \prod \mathrm{ch}^2(\mu_m - \nu_n)$$

- Cauchy Det

$$Z_k(N,N) = \int D^N\mu\, D^N\nu\, \det[1/\mathrm{ch}(\mu - \nu)]\, \det[1/\mathrm{ch}(\nu' - \mu')]$$

- (Continuous) Cauchy-Binet Formula

$$Z_k(N,N) = \int D^N\mu\, \det\Big[\int D\nu\, [\mathrm{ch}(\mu - \nu)]^{-1}\, [\mathrm{ch}(\nu - \mu')]^{-1}\Big]$$

- Fredholm Det

[Many talks in this workshop]

$$\Xi_{k,0} = \mathrm{Det}\,[\,1 + z\,H^{-1}\,]$$

$$H^{-1}(\mu,\mu') = [\mathrm{ch}(\mu - \nu)]^{-1} \bullet [\mathrm{ch}(\nu - \mu')]^{-1} \bullet$$

Why $\mathcal{P}^{-1} \leftrightarrow$ NS5, $\mathcal{Q}^{-1} \leftrightarrow (1,k)5$?

- Fredholm Det: $\Xi_{k,0} = \mathrm{Det}\,[\,1 + z\,H^{-1}\,]$

$$H^{-1}(\mu,\mu') = [\mathrm{ch}(\mu-\nu)]^{-1} \bullet [\mathrm{ch}(\nu-\mu')]^{-1} \bullet$$

- $[\mathrm{ch}]^{-1}$ Function is Fourier Self-Dual

$$H^{-1}(\mu,\mu') = \langle \mu | H^{-1} | \mu' \rangle$$

$$H^{-1} = (\mathrm{ch}\,p)^{-1} \exp(+\mathrm{i}q^2/(2\hbar))\,(\mathrm{ch}\,p)^{-1} \exp(-\mathrm{i}q^2/(2\hbar))$$

- Similarity Transformation

$\exp(+\mathrm{i}p^2/(2\hbar))\, H^{-1} \exp(-\mathrm{i}p^2/(2\hbar))$

$= (\mathrm{ch}\,p)^{-1} \exp(+\mathrm{i}p^2/(2\hbar)) \exp(+\mathrm{i}q^2/(2\hbar))\,(\mathrm{ch}\,p)^{-1} \exp(-\mathrm{i}q^2/(2\hbar)) \exp(-\mathrm{i}p^2/(2\hbar))$

$= (\mathrm{ch}\,p)^{-1} (\mathrm{ch}\,q)^{-1} = \mathcal{P}^{-1}\,\mathcal{Q}^{-1}$

($\mathcal{Q} = Q^{1/2}+Q^{-1/2}$, $\mathcal{P} = P^{1/2}+P^{-1/2}$, $Q = e^q$, $P = e^p$, $[q,p] = \mathrm{i}\,\hbar$, $\hbar = 2\pi k$)

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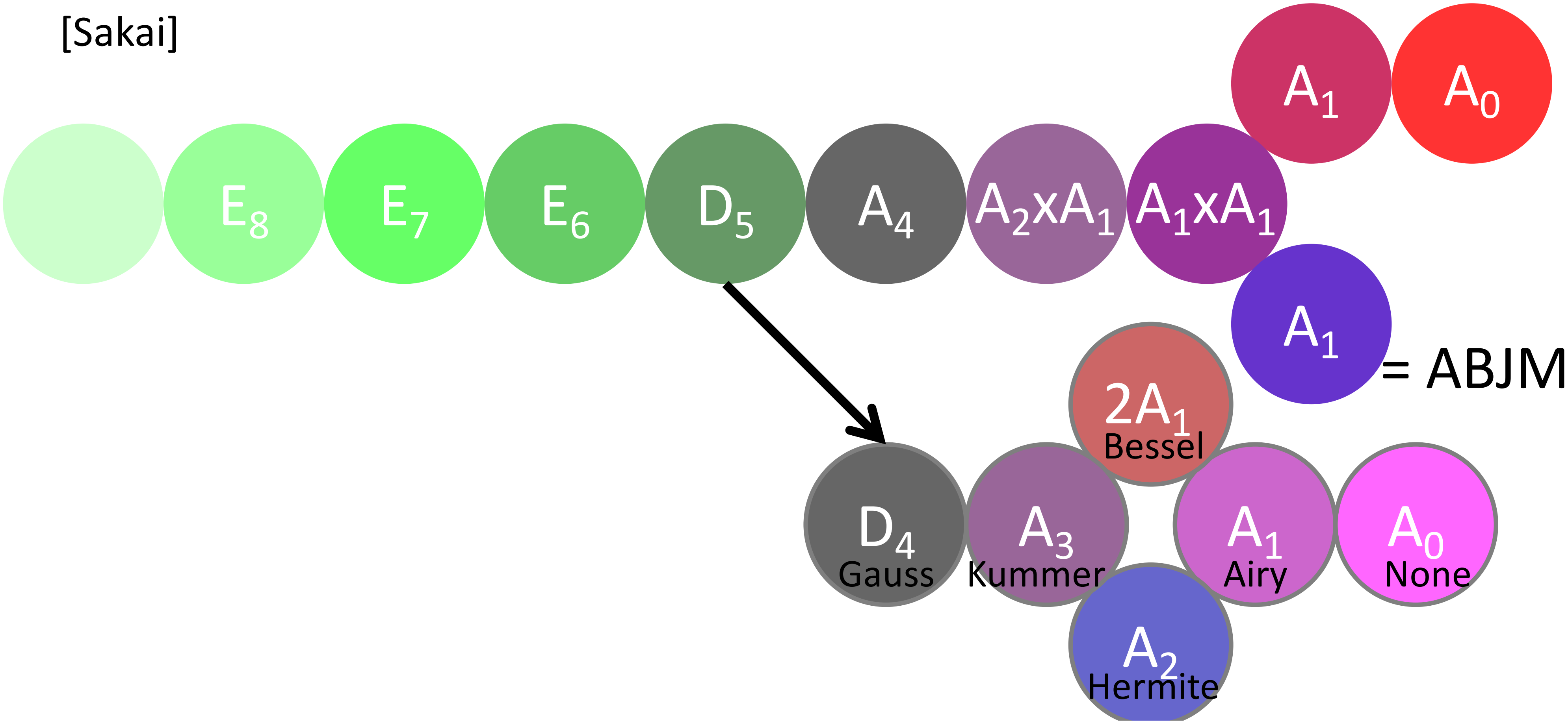
Group-Theoretical Viewpoint

- ABJM OK, But Larger Symmetry is Clearer

In terms of Space of Initial Data for Painleve eq

[Sakai]



As simple generalizations

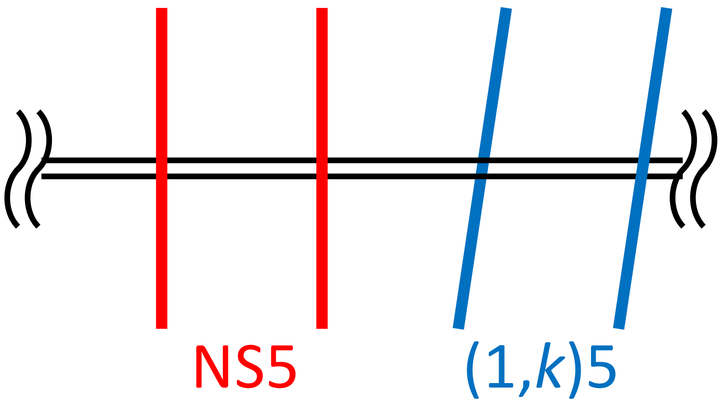
Spectral Det

$\mathrm{Det}\,(\,1 + z\,H^{-1}\,)$

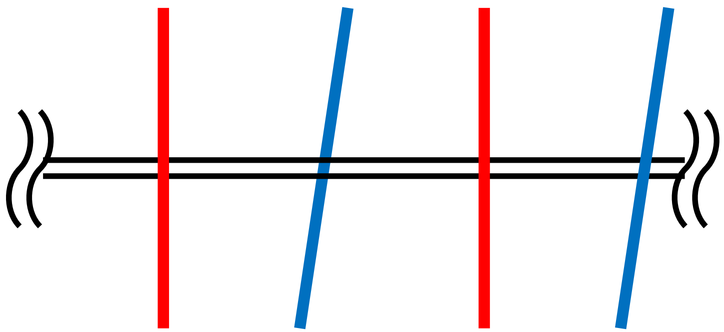
$H = \mathcal{Q}^2\,\mathcal{P}^2$

$H = \mathcal{Q}\,\mathcal{P}\,\mathcal{Q}\,\mathcal{P}$

(2,2) Model [M-Nosaka]



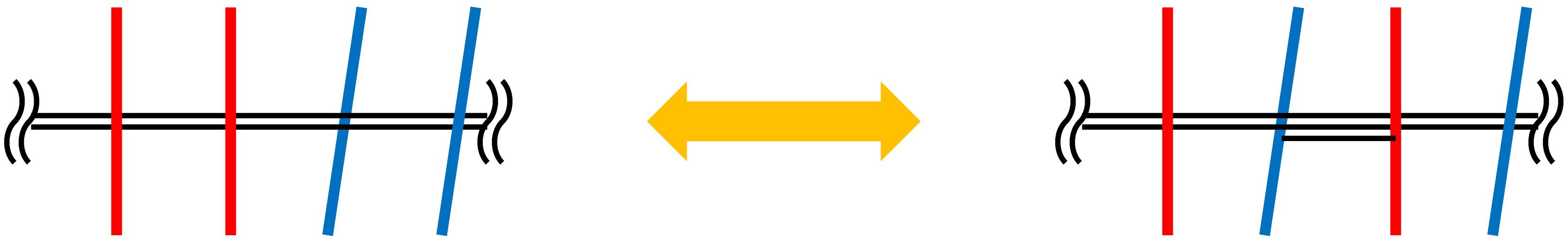
(1,1,1,1) Model [Honda-M]



... & Their Rank Deformations

- connected with Hanany-Witten Transitions

[Hanany-Witten 1997] As Explained Later



Spectral Operators

- For (2,2) Model

$H = \mathcal{Q}^2 \mathcal{P}^2$

$= (Q^{1/2}+Q^{-1/2})^2 (P^{1/2}+P^{-1/2})^2$

$= Q^1P^1+2P^1+Q^{-1}P^1+2Q^1+4+2Q^{-1}+Q^1P^{-1}+2P^{-1}+Q^{-1}P^{-1}$

- For (1,1,1,1) Model

$H = \mathcal{Q}\,\mathcal{P}\,\mathcal{Q}\,\mathcal{P}$

$= (Q^{1/2}+Q^{-1/2})(P^{1/2}+P^{-1/2})(Q^{1/2}+Q^{-1/2})(P^{1/2}+P^{-1/2})$

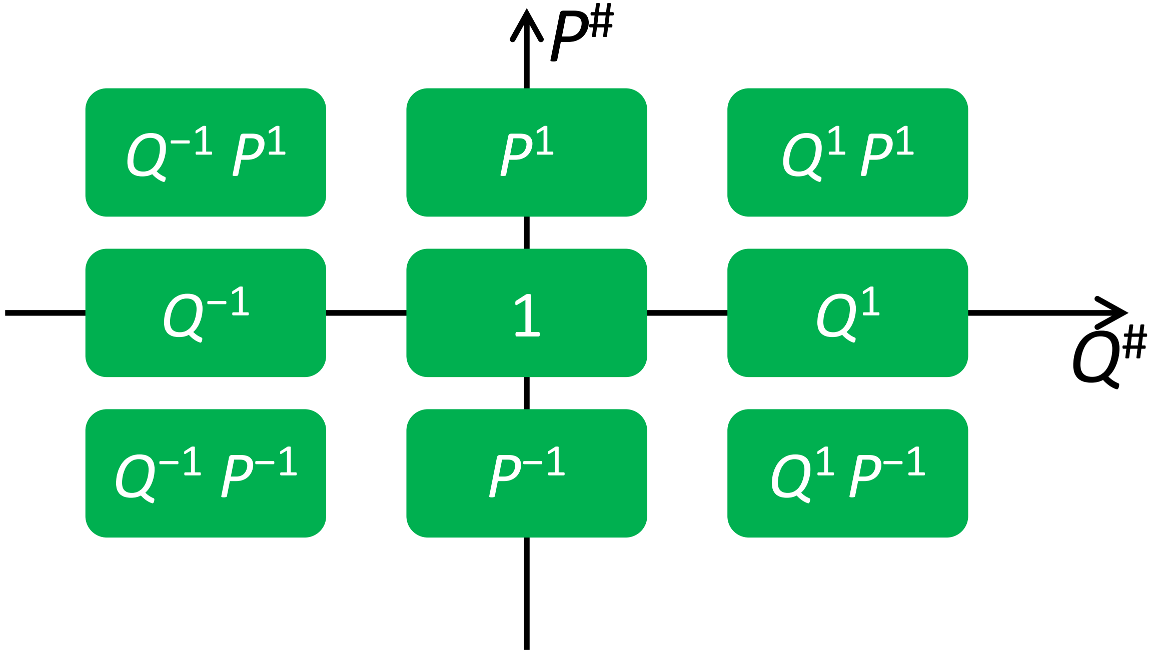
$= q^{-1/4}\, Q^1P^1 + (q^{1/4} + q^{-1/4})\, P^1 + q^{1/4}\, Q^{-1}P^1 + \ldots$

(Since $P^\alpha Q^\beta = q^{-\alpha\beta}\, Q^\beta P^\alpha$, $q = e^{i\hbar} = e^{2\pi i k}$)

Spectral Operators

- For Either Case

$H =$



Well-known Newton Polygon of D5[=so(10)] Curve

Deformations

- Rank Deformations = Still D5 Curve ?

- Deformations by Relative Ranks

$$C_B = \{\text{Brane Configurations}\},\ \dim C_B = 3$$

- Deformations of D5 Curve

$$C_P = \{\text{Point Configurations}\},\ \dim C_P = 5$$

- Embedding of C_B in C_P ?

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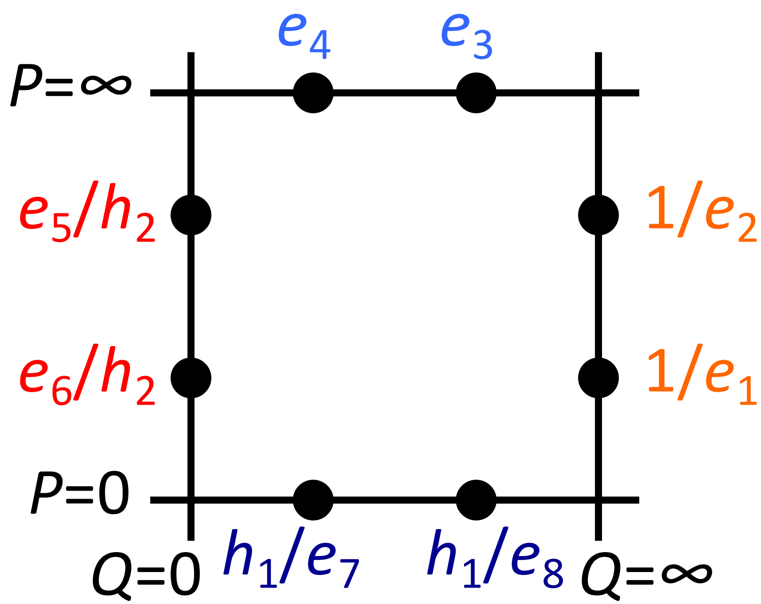
Parameterization

Parameterize D5 Curve

by "Asymptotic Values" "Point Configurations"

$$
\begin{aligned}
H/\alpha = \quad & Q\,P && -(e_3+e_4)\,P && +\,e_3e_4\,Q^{-1}\,P \\
& -(e_1{}^{-1}+e_2{}^{-1})\,Q && +\,E/\alpha && -\ldots\,Q^{-1} \\
& +(e_1e_2)^{-1}\,Q\,P^{-1} && -\ldots\,P^{-1} && +\ldots\,Q^{-1}\,P^{-1}
\end{aligned}
$$

[Kajiwara-Noumi-Yamada 2015]



Subject to Vieta's Formula

解と係数の関係 $(h_1h_2)^2 = e_1\ldots e_8$

D5 Weyl Transformation

Trivial Transformations

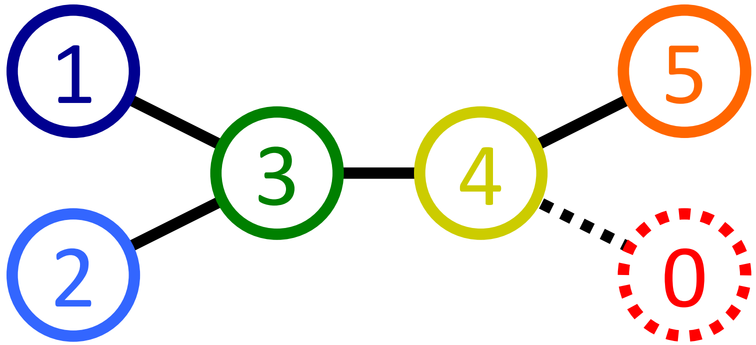
(Switching Asymptotic Values)

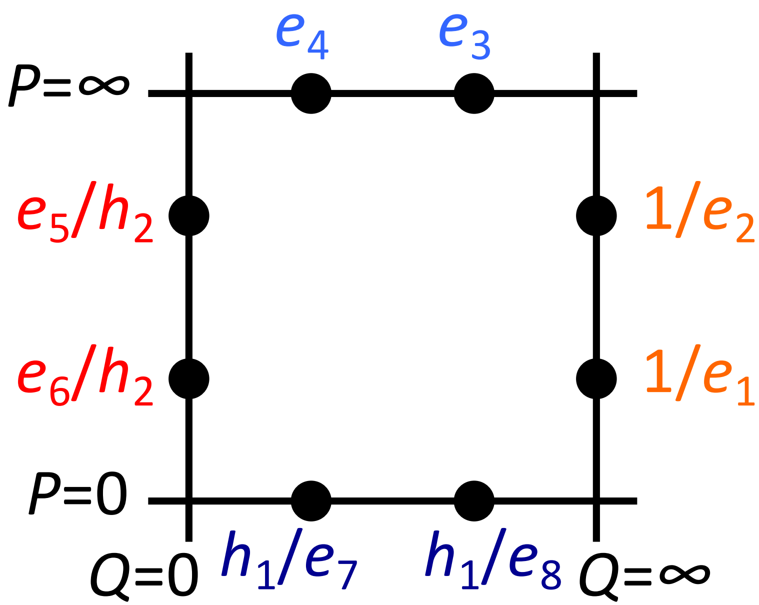
s_1: $h_1/e_7 \Leftrightarrow h_1/e_8$

s_2: $e_3 \Leftrightarrow e_4$

s_5: $1/e_1 \Leftrightarrow 1/e_2$

s_0: $e_5/h_2 \Leftrightarrow e_6/h_2$





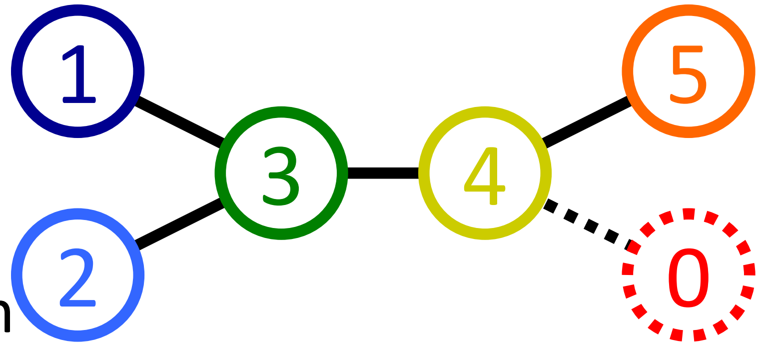
D5 Weyl Transformation

Nontrivial s_3 and s_4

by Suitable Similarity Transf.

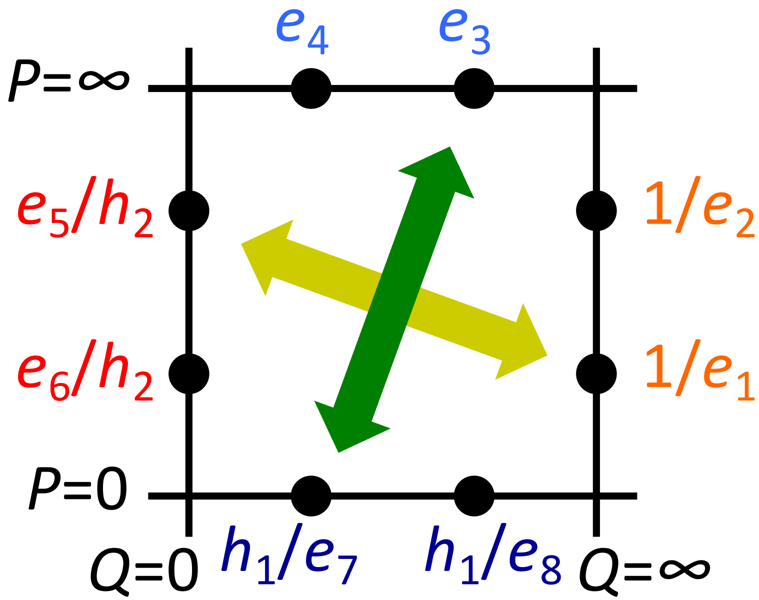
$Q' = GQG^{-1}$, $P' = GPG^{-1}$, G = Dilogarithm

[Hasegawa 2007]



s_3: $e_3 \Leftrightarrow h_1/e_7$

s_4: $1/e_1 \Leftrightarrow e_5/h_2$



Totally, D5 Weyl Transf.

Gauge Fixing

- Redundancies in Parametrization

- $(h_1,h_2,e_1,\ldots,e_8)$: 10 Parameters for 8 Asymptotic Values

- Similarity Transformation to Rescale Q & P

$$(Q,P)\sim(AQ,P),\ (Q,P)\sim(Q,BP)$$

- Totally, **4** Gauge Fixing Conditions

$$e_2 = e_4 = e_6 = e_8 = 1$$

- 6 Parameters Subject to 1 Vieta's Constraint

$$\rightarrow 5 \text{ DOF } (h_1,h_2,e_1,e_3,e_5)$$

$\boldsymbol{s_1}, \boldsymbol{s_2}, \boldsymbol{s_3}, \boldsymbol{s_4}, \boldsymbol{s_5} : (h_1,h_2,e_1,e_3,e_5) \rightarrow (*, *, *, *, *)$

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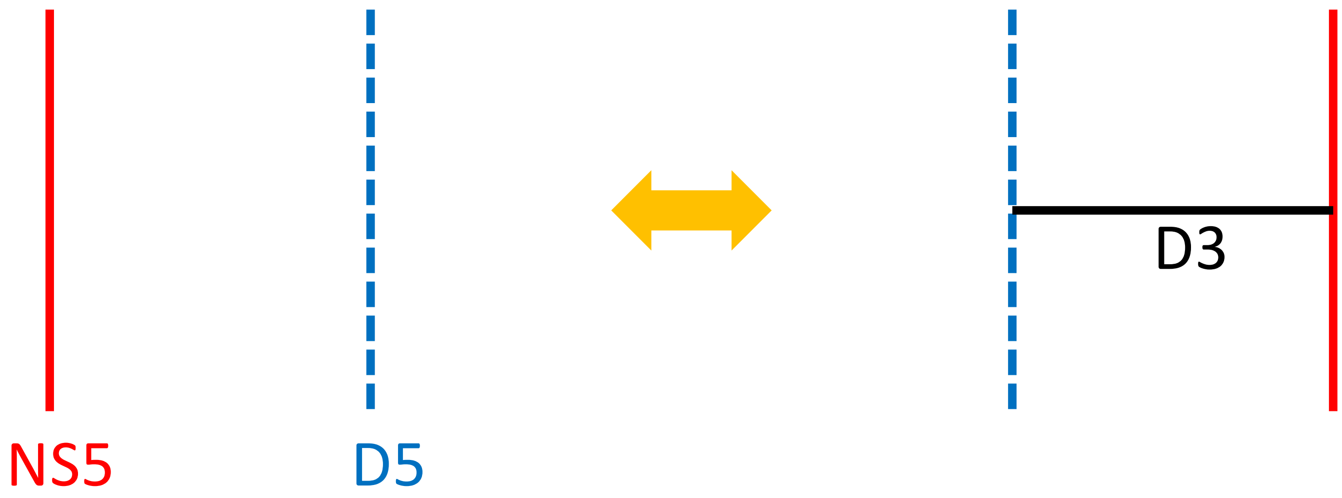
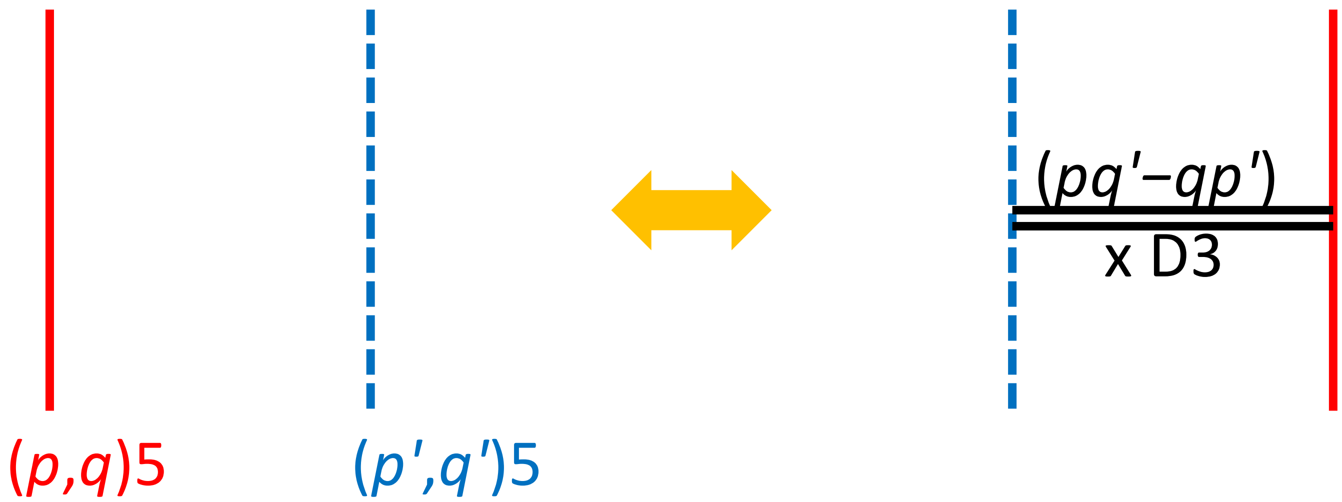
Hanany-Witten Transitions



- Exchanging NS5 & D5, An Extra D3 Generated

[Hanany-Witten 1997]

More Generally,

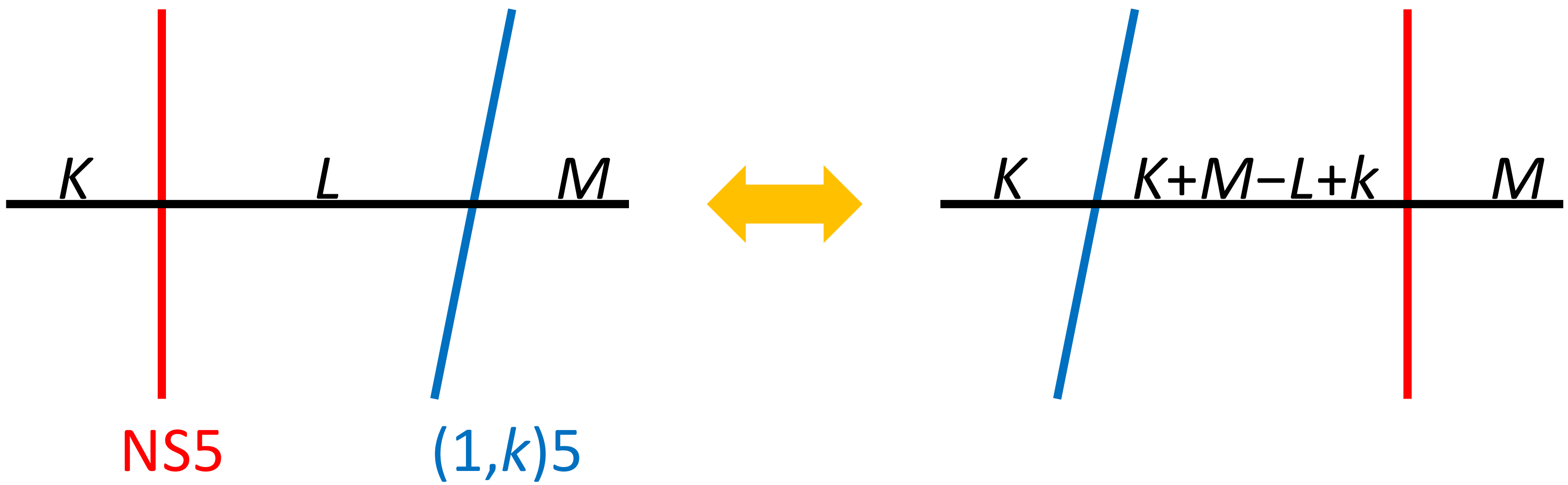


$(p,q)5$: Bound States of p x NS5 & q x D5

- Understood from Charge Conservations

$$q_{RR} = [\#(D5)_L - \#(D5)_R]/2 + [\#(D3)_L - \#(D3)_R]$$

For our case



- Overall Rank Irrelevant: $K, L, M \rightarrow K, L, M + N$
- Application to ABJM or its Generalizations

- "Locally", OK

- "Globally", What Left/Right on S^1 ???

Question

A Question on Hanany-Witten Transitions for Brane System:

What by "Left/Right" in the T-duality circle S^1 ?

Similar Question in Matrix Model:

Cut the T-duality Circle Open & Assign $\mathcal{P}^{-1} \leftrightarrow$ NS5, $\mathcal{Q}^{-1} \leftrightarrow (1,k)5$

Where to cut?

Similarity Transformations Are Irrelevant ?

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(Point Config vs Brane Config)
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Ubiquitous

- Mechanics

  Fix A Reference Frame. Then, Change of Frames.

- Geometry

  Fix A Local Chart. Then, Transition Functions.

Similarly,

- Fix a CUT in the T-duality Circle

For Brane Configurations:

No Exchange of 5-branes Over the Cut

For Fredholm Determinants:

No More Uncritical Similarity Transformations

- Change of CUTs

Quantum Curves

- Prepare Canonical Operators $\mathcal{Q}_4$, $\mathcal{Q}_3$, $\mathcal{P}_1$, $\mathcal{P}_2$
- Fix $H = \mathcal{Q}_4\mathcal{Q}_3\mathcal{P}_1\mathcal{P}_2$ as Reference
- Other Spectral Operators, e.g. $H = \mathcal{P}_2\mathcal{Q}_4\mathcal{Q}_3\mathcal{P}_1$ from Reference

Quantum Curves

- $\mathcal{P}_2\mathcal{Q}_4\mathcal{Q}_3\mathcal{P}_1$ from $\mathcal{Q}_4\mathcal{Q}_3\mathcal{P}_1\mathcal{P}_2$, ($P^\alpha Q^\beta = q^{-\alpha\beta}\, Q^\beta P^\alpha$)
- Read off Asymptotic Values

$$\{e_1^{-1}, e_2^{-1}\} = \{q,1\}, \{h_2^{-1}e_5, h_2^{-1}e_6\} = \{q^{-1},1\}, \ldots$$

But Which is Which? Note that

$P^{n/2}\mathcal{Q} = (q^{-n/4}Q^{1/2}+q^{n/4}Q^{-1/2})P^{n/2}$, $P^{-n/2}\mathcal{Q} = (q^{n/4}Q^{1/2}+q^{-n/4}Q^{-1/2})P^{-n/2}$

Imply that, when $e_{1/2}{}^{-1}$ is $-q^{n/2}$, $h_2{}^{-1}e_{5/6}$ is $-q^{-n/2}$, and vice versa

Namely ($e_1{}^{-1}$ & $h_2{}^{-1}e_5$), ($e_2{}^{-1}$ & $h_2{}^{-1}e_6$) are correlated

- $\mathcal{Q}_4$, $\mathcal{Q}_3$, $\mathcal{P}_1$, $\mathcal{P}_2$ are respectively responsible for

(e_4 & $h_1e_8{}^{-1}$), (e_3 & $h_1e_7{}^{-1}$), ($e_1{}^{-1}$ & $h_2{}^{-1}e_5$), ($e_2{}^{-1}$ & $h_2{}^{-1}e_6$)

Quantum Curves

For 14 Spectral Operators of Different Order

Reference

Type	Quantum curve	$(\bar{h}_1, \bar{h}_2, e_1, e_3, e_5)$
$\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{P}}$	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2$	$(q, q^{-1}, 1, 1, 1)$
$\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}$	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2$	$(q, q^{-1}, q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_2$	$(1, q^{-1}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1$	$(q, 1, q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1$	$(1, 1, q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{\frac{1}{2}})$
$\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}$	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3$	$(q, 1, 1, q, 1)$
	$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4$	$(q^{-1}, 1, 1, q^{-1}, 1)$
$\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}$	$\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2$	$(1, q^{-1}, q^{-1}, 1, q^{-1})$
	$\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1$	$(1, q, q, 1, q)$
$\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}$	$\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3$	$(1, 1, q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4$	$(q^{-1}, 1, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_3$	$(1, q, q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}})$
	$\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4$	$(q^{-1}, q, q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{\frac{1}{2}})$
$\widehat{\mathcal{P}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}$	$\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3$	$(q^{-1}, q, 1, 1, 1)$

Quantum Curves

- First Observation

All 14 Models Lie in 3-Dimensional Subspace !

(The First Sign of Success for Our Working Hypothesis "Fixing A Reference")

- Label as $(h_1, h_2, e_1, e_3, e_5) = (ef/h,\ hf/e,\ f/e,\ ef,\ f/e)$

- Translate into $(h,\ e,\ f)$

Quantum Curves

For 14 Spectral Operators of Different Order

Reference

Type	Quantum curve	$(\overline{h}_1, \overline{h}_2, e_1, e_3, e_5)$	(h, e, f)
$\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{P}}$	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2$	$(q, q^{-1}, 1, 1, 1)$	$(q^{-1}, 1, 1)$
$\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}$	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2$	$(q, q^{-1}, q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{-\frac{1}{2}})$	$(q^{-\frac{1}{2}}, q^{\frac{1}{2}}, 1)$
	$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_2$	$(1, q^{-1}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$	$(q^{-\frac{1}{2}}, 1, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1$	$(q, 1, q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}})$	$(q^{-\frac{1}{2}}, 1, q^{\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1$	$(1, 1, q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{\frac{1}{2}})$	$(q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, 1)$
$\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}$	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3$	$(q, 1, 1, q, 1)$	$(1, q^{\frac{1}{2}}, q^{\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4$	$(q^{-1}, 1, 1, q^{-1}, 1)$	$(1, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$
$\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}$	$\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2$	$(1, q^{-1}, q^{-1}, 1, q^{-1})$	$(1, q^{\frac{1}{2}}, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1$	$(1, q, q, 1, q)$	$(1, q^{-\frac{1}{2}}, q^{\frac{1}{2}})$
$\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}$	$\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3$	$(1, 1, q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{-\frac{1}{2}})$	$(q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1)$
	$\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4$	$(q^{-1}, 1, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$	$(q^{\frac{1}{2}}, 1, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_3$	$(1, q, q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}})$	$(q^{\frac{1}{2}}, 1, q^{\frac{1}{2}})$
	$\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4$	$(q^{-1}, q, q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{\frac{1}{2}})$	$(q^{\frac{1}{2}}, q^{-\frac{1}{2}}, 1)$
$\widehat{\mathcal{P}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}$	$\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3$	$(q^{-1}, q, 1, 1, 1)$	$(q, 1, 1)$

Brane Configurations

- Prepare NS5 ②, ① & $(1,k)$5 ③, ④
- Fix ⟨2134⟩ as Reference
- Other Brane Configurations, e.g. ⟨1342⟩

$$\langle N①N③N④N②\rangle = \langle N①N③N②N+k④\rangle$$

$$= \langle N①N②N+2k③N+k④\rangle = \langle N②N+2k①N+2k③N+k④\rangle$$

- Label as

$$\langle N+M_2+M_3②N+M_1+2M_3①N+2M_1+M_2+M_3③N+M_1④\rangle$$

- Translate into

$$(M_1,M_2,M_3)$$

Brane Configurations

Reference

$\Delta(h,e,f)$
$= (h,e,f)$
$/ (h,e,f)|_{\mathrm{Ref}}$

Spectral operator	$\Delta(h,e,f)$	Brane configuration	(M_1, M_2, M_3)
$\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2$	$(1,1,1)$	$\langle 0 \overset{2}{\bullet} 0 \overset{1}{\bullet} 0 \overset{3}{\circ} 0 \overset{4}{\circ} \rangle$	$(0,0,0)$
$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2$	$(q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1)$	$\langle 0 \overset{2}{\bullet} 0 \overset{3}{\circ} 0 \overset{1}{\bullet} 0 \overset{4}{\circ} \rangle$	$(\frac{k}{2}, \frac{k}{2}, 0)$
[illegible]	$(q^{\frac{1}{2}}, 1, q^{-\frac{1}{2}})$	$\langle 0 \overset{2}{\bullet} 0 \overset{4}{\circ} 0 \overset{1}{\bullet} 0 \overset{3}{\circ} \rangle$	$(\frac{k}{2}, 0, -\frac{k}{2})$
[illegible]	$(q^{\frac{1}{2}}, 1, q^{\frac{1}{2}})$	$\langle 0 \overset{1}{\bullet} 0 \overset{3}{\circ} 0 \overset{2}{\bullet} 0 \overset{4}{\circ} \rangle$	$(\frac{k}{2}, 0, \frac{k}{2})$
[illegible]	$(q^{\frac{1}{2}}, q^{-\frac{1}{2}}, 1)$	$\langle 0 \overset{1}{\bullet} 0 \overset{4}{\circ} 0 \overset{2}{\bullet} 0 \overset{3}{\circ} \rangle$	$(\frac{k}{2}, -\frac{k}{2}, 0)$
[illegible]	$(q, q^{\frac{1}{2}}, q^{\frac{1}{2}})$	$\langle 0 \overset{3}{\circ} 0 \overset{2}{\bullet} 0 \overset{1}{\bullet} 0 \overset{4}{\circ} \rangle$	$(k, \frac{k}{2}, \frac{k}{2})$
$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4$	$(q, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$	$\langle 0 \overset{4}{\circ} 0 \overset{2}{\bullet} 0 \overset{1}{\bullet} 0 \overset{3}{\circ} \rangle$	$(k, -\frac{k}{2}, -\frac{k}{2})$
$\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2$	$(q, q^{\frac{1}{2}}, q^{-\frac{1}{2}})$	$\langle 0 \overset{2}{\bullet} 0 \overset{3}{\circ} 0 \overset{4}{\circ} 0 \overset{1}{\bullet} \rangle$	$(k, \frac{k}{2}, -\frac{k}{2})$
$\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1$	$(q, q^{-\frac{1}{2}}, q^{\frac{1}{2}})$	$\langle 0 \overset{1}{\bullet} 0 \overset{3}{\circ} 0 \overset{4}{\circ} 0 \overset{2}{\bullet} \rangle$	$(k, -\frac{k}{2}, \frac{k}{2})$
$\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3$	$(q^{\frac{3}{2}}, q^{\frac{1}{2}}, 1)$	$\langle 0 \overset{3}{\circ} 0 \overset{2}{\bullet} 0 \overset{4}{\circ} 0 \overset{1}{\bullet} \rangle$	$(\frac{3k}{2}, \frac{k}{2}, 0)$
$\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4$	$(q^{\frac{3}{2}}, 1, q^{-\frac{1}{2}})$	$\langle 0 \overset{4}{\circ} 0 \overset{2}{\bullet} 0 \overset{3}{\circ} 0 \overset{1}{\bullet} \rangle$	$(\frac{3k}{2}, 0, -\frac{k}{2})$
$\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_3$	$(q^{\frac{3}{2}}, 1, q^{\frac{1}{2}})$	$\langle 0 \overset{3}{\circ} 0 \overset{1}{\bullet} 0 \overset{4}{\circ} 0 \overset{2}{\bullet} \rangle$	$(\frac{3k}{2}, 0, \frac{k}{2})$
$\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4$	$(q^{\frac{3}{2}}, q^{-\frac{1}{2}}, 1)$	$\langle 0 \overset{4}{\circ} 0 \overset{1}{\bullet} 0 \overset{3}{\circ} 0 \overset{2}{\bullet} \rangle$	$(\frac{3k}{2}, -\frac{k}{2}, 0)$
$\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3$	$(q^2, 1, 1)$	$\langle 0 \overset{3}{\circ} 0 \overset{4}{\circ} 0 \overset{2}{\bullet} 0 \overset{1}{\bullet} \rangle$	$(2k, 0, 0)$

Brane Configurations

- Second Observation

$$(h, e, f) = (e^{2\pi \mathrm{i} M_1}, e^{2\pi \mathrm{i} M_2}, e^{2\pi \mathrm{i} M_3})$$

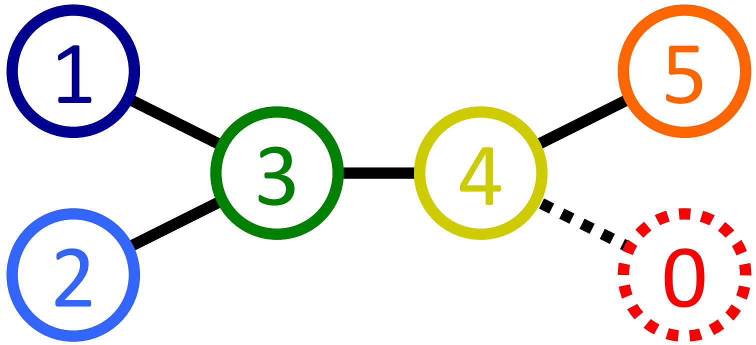
Successfully Embed 3-Dim C_{B} in 5-Dim C_{P} !

Contents

0. Introduction
1. ABJM Theory
2. Super Chern-Simons Theories
3. Quantum Curves
4. Hanany-Witten Transitions
5. QC vs HW
6. Symmetries
7. Conclusions

Symmetry

- Originally D5 Symmetry



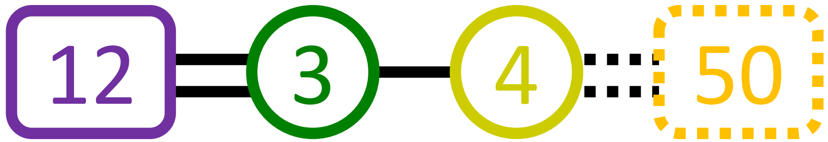
①What are Symmetries on 3-Dim Subspace?

②All Understood as Hanany-Witten Transitions?

Symmetries on C_{B}

① What are Symmetries on 3-Dim Subspace?

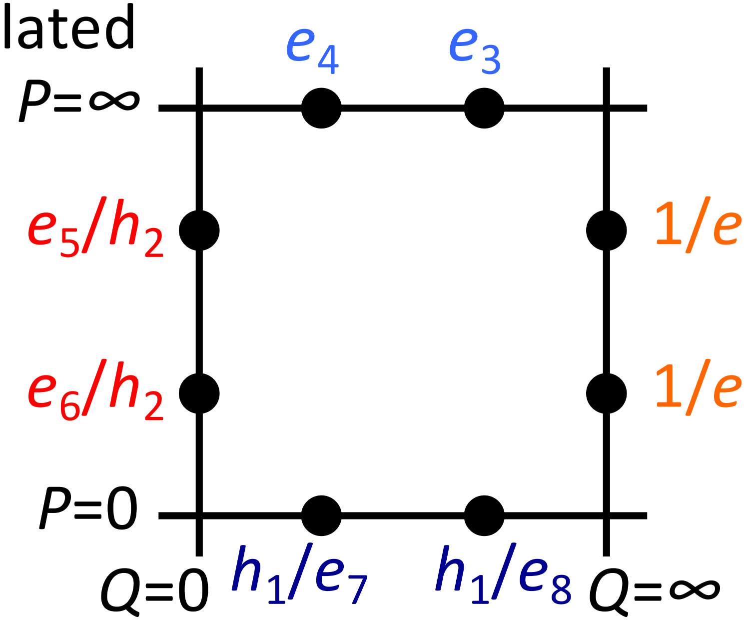
B3 = so(7)



Not Surprising

Since (e_4 & $h_1 e_8^{-1}$), (e_3 & $h_1 e_7^{-1}$),

(e_1^{-1} & $h_2^{-1} e_5$), (e_2^{-1} & $h_2^{-1} e_6$) Are Correlated



Beyond HW Transitions

② All as Hanany-Witten Transitions?

No!

Hanany-Witten Transitions Generate B2 = so(5)

What is the New Symmetry?

Beyond HW Transitions

$$s_3\!:\!\langle N_1 ② N_2 ③ N_3 ① N_4 ④ \rangle \to \langle N_1 ② N_3 ③ N_2 ① N_4 ④ \rangle$$

$$s_4\!:\!\langle N_1 ② N_2 ③ N_3 ① N_4 ④ \rangle \to \langle N_1 ② N_2 ③ N_4 ① N_3 ④ \rangle$$

Unknown Transitions? Physical Interpretations?

Summary & Conclusions

- Embedding of C_{B} in C_{P}
- "Fixing A Reference Frame"
- Beyond Hanany-Witten Transitions

  Unknown Transitions
- M2-branes Beyond Matrix Models

  Only 3-Dim in 5-Dim, What Else?

Thank you for your attention!