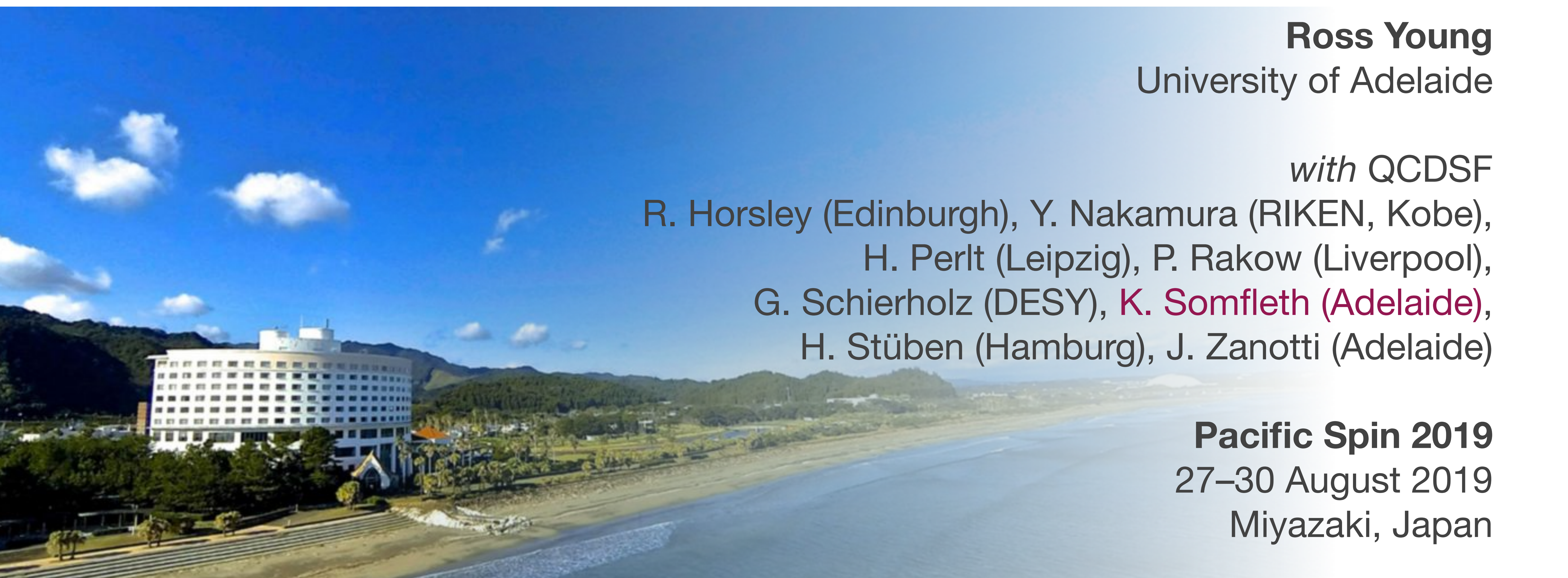
THE UNIVERSITY of ADELAIDE

SPECIAL RESEARCH CENTRE FOR THE SUBATOMIC STRUCTURE OF MATTER

# Nucleon virtual Compton amplitude in lattice QCD

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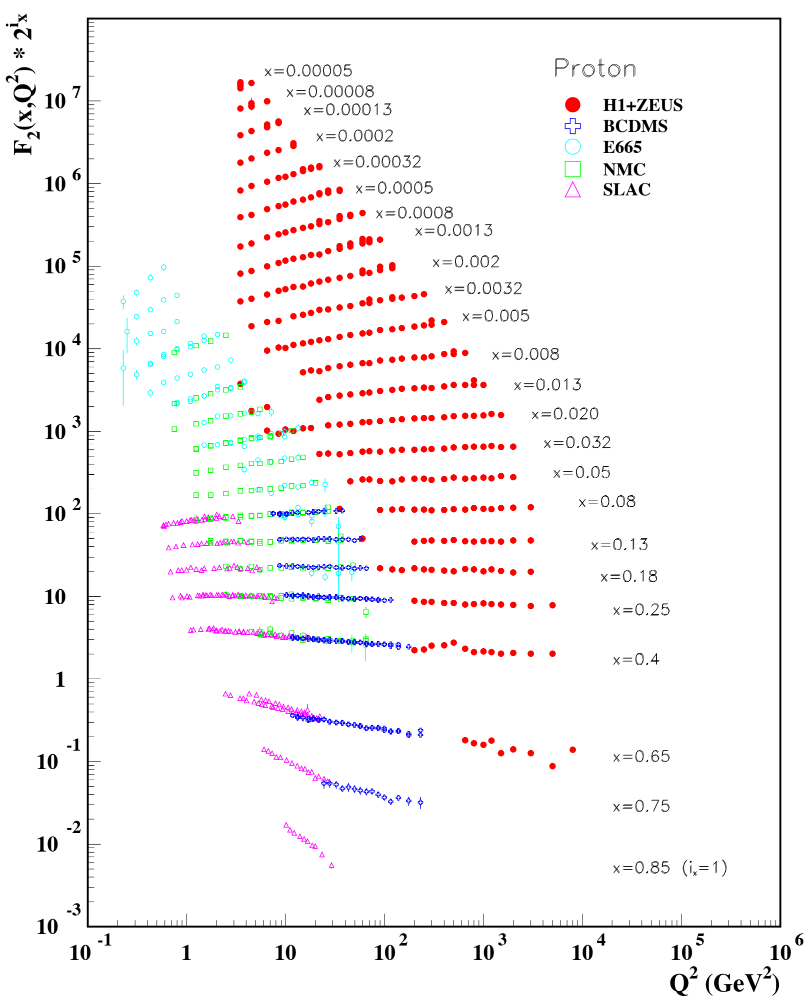
**Pacific Spin 2019**
27–30 August 2019
Miyazaki, Japan

# Beyond leading twist

**Twist-4 operators**
**Theoretical foundations to inform $Q^2$ cuts of empirical parton fits.**




$\gamma$

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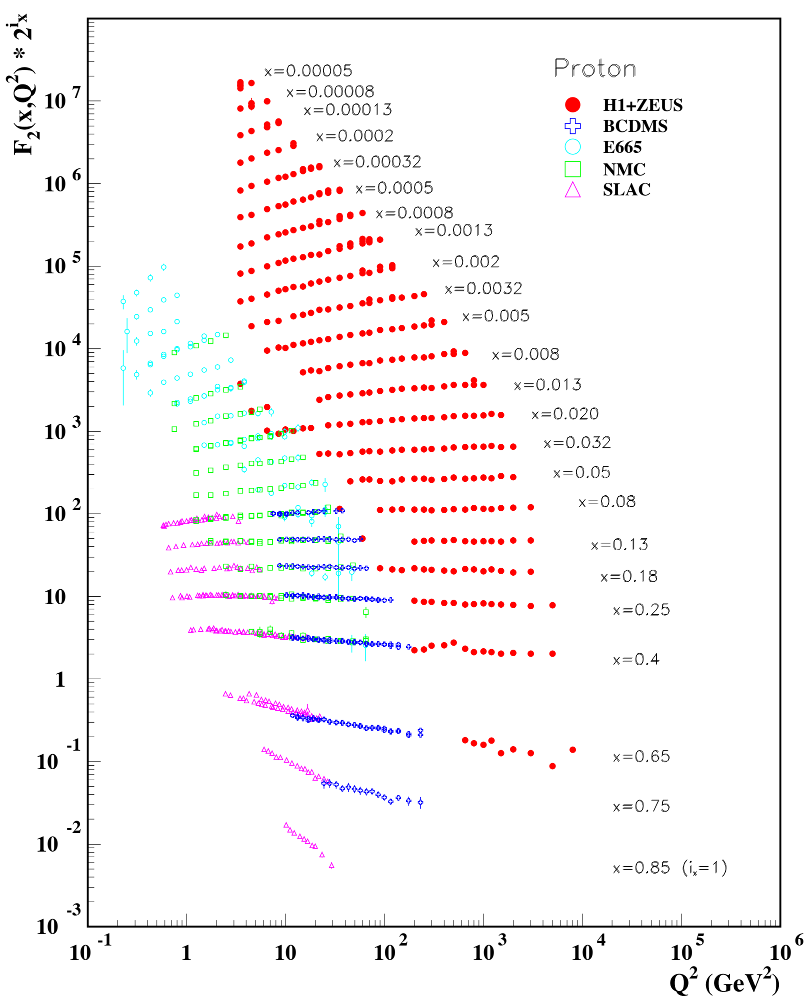


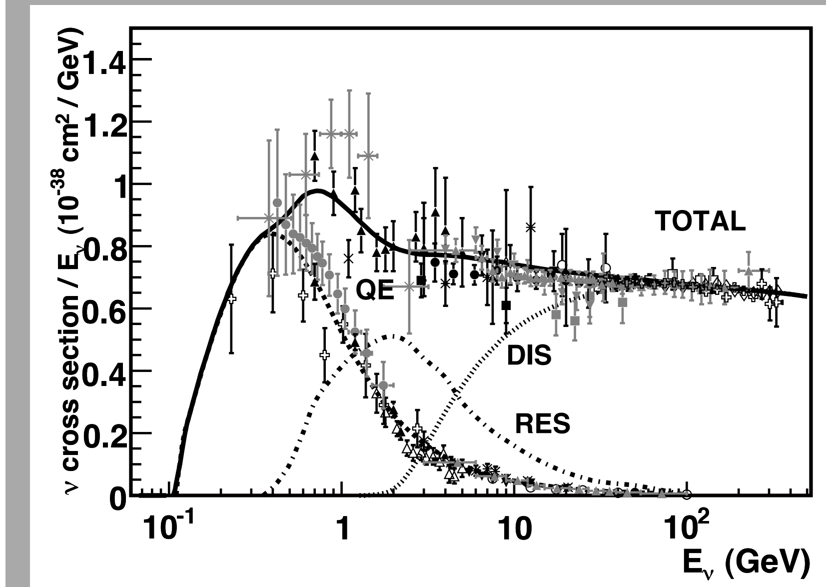


**Neutrino-nucleus cross sections**
**Precise theoretical input required for next-generation neutrino oscillation program**

$\gamma$

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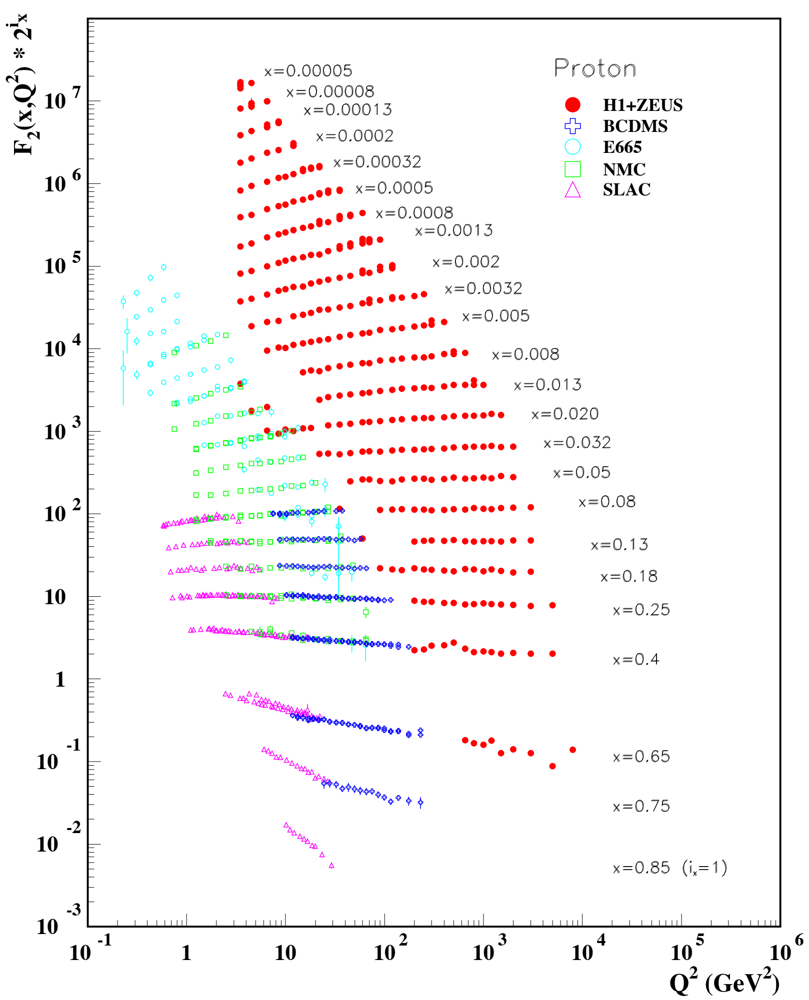
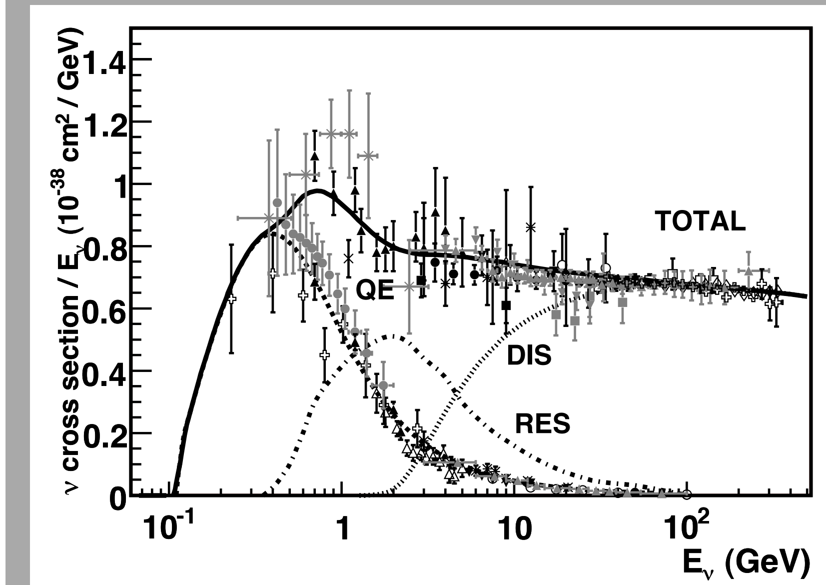
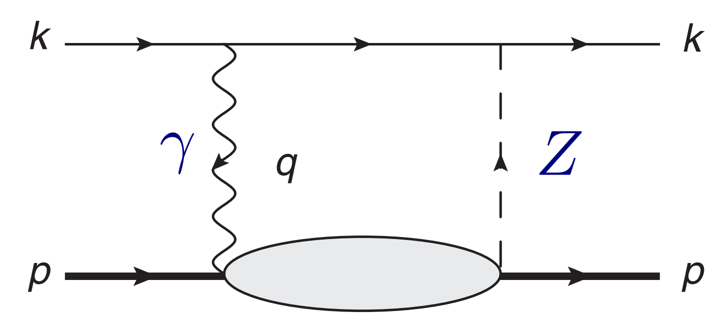
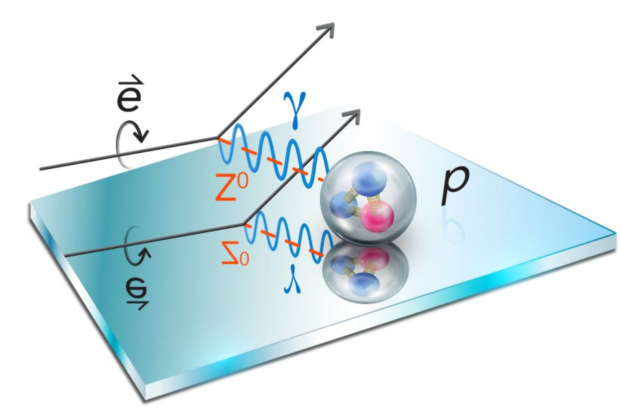



**Neutrino-nucleus cross sections**
Precise theoretical input required for next-generation neutrino oscillation program

**Radiative corrections**
Searches for new physics in the proton weak charge.
Require knowledge of gamma-*Z* interference structure functions.



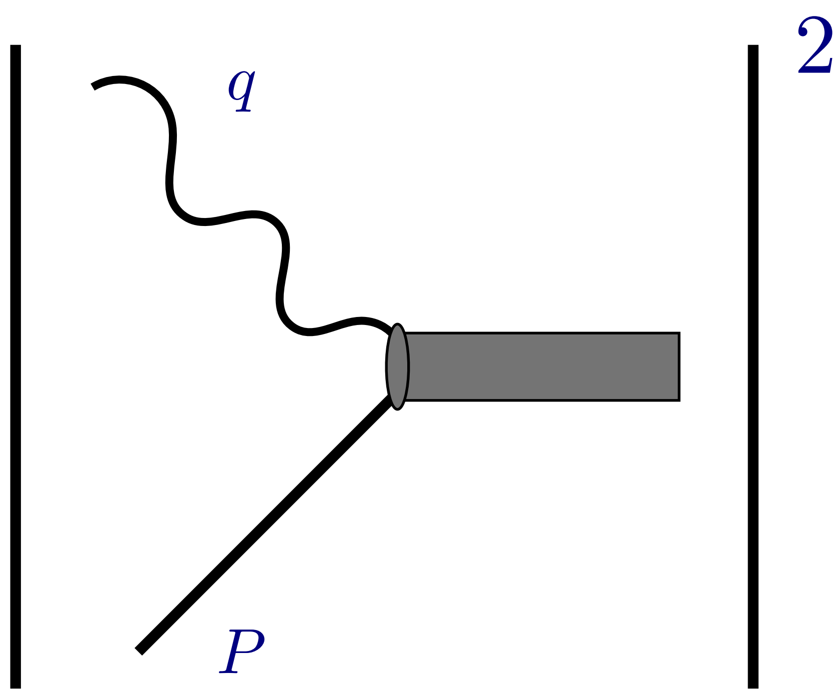



# Outline

- Inelastic structure functions and the forward Compton scattering tensor
- Empirical Compton amplitude
- Feynman-Hellmann in lattice QCD
    - Spin matrix elements; disconnected spin
    - Advert: vector form factors at large momentum transfer
- Compton amplitude as an energy shift: Feynman-Hellmann@2$^{nd}$ order
- Numerical results

# Inelastic structure functions and the forward Compton scattering tensor

# Inelastic scattering




Cross section ~ Hadron tensor

$$W_{\mu\nu} \sim \int d^4x\, e^{iq.x} \langle p | \left[ J_\mu(x), J_\nu(0) \right] | p \rangle$$

Structure functions $F_{1,2}(P.q, Q^2)$

$$F_i = \frac{1}{2\pi} \mathrm{Im}\, T_i$$


Forward Compton amplitude

$$T_{\mu\nu} \sim \int d^4x\, e^{iq.x} \langle p | \mathrm{T} J_\mu(x) J_\nu(0) | p \rangle$$

Lorentz-scalar functions $T_{1,2}(P.q, Q^2)$

# (Virtual) Compton amplitude

- Forward Compton amplitude

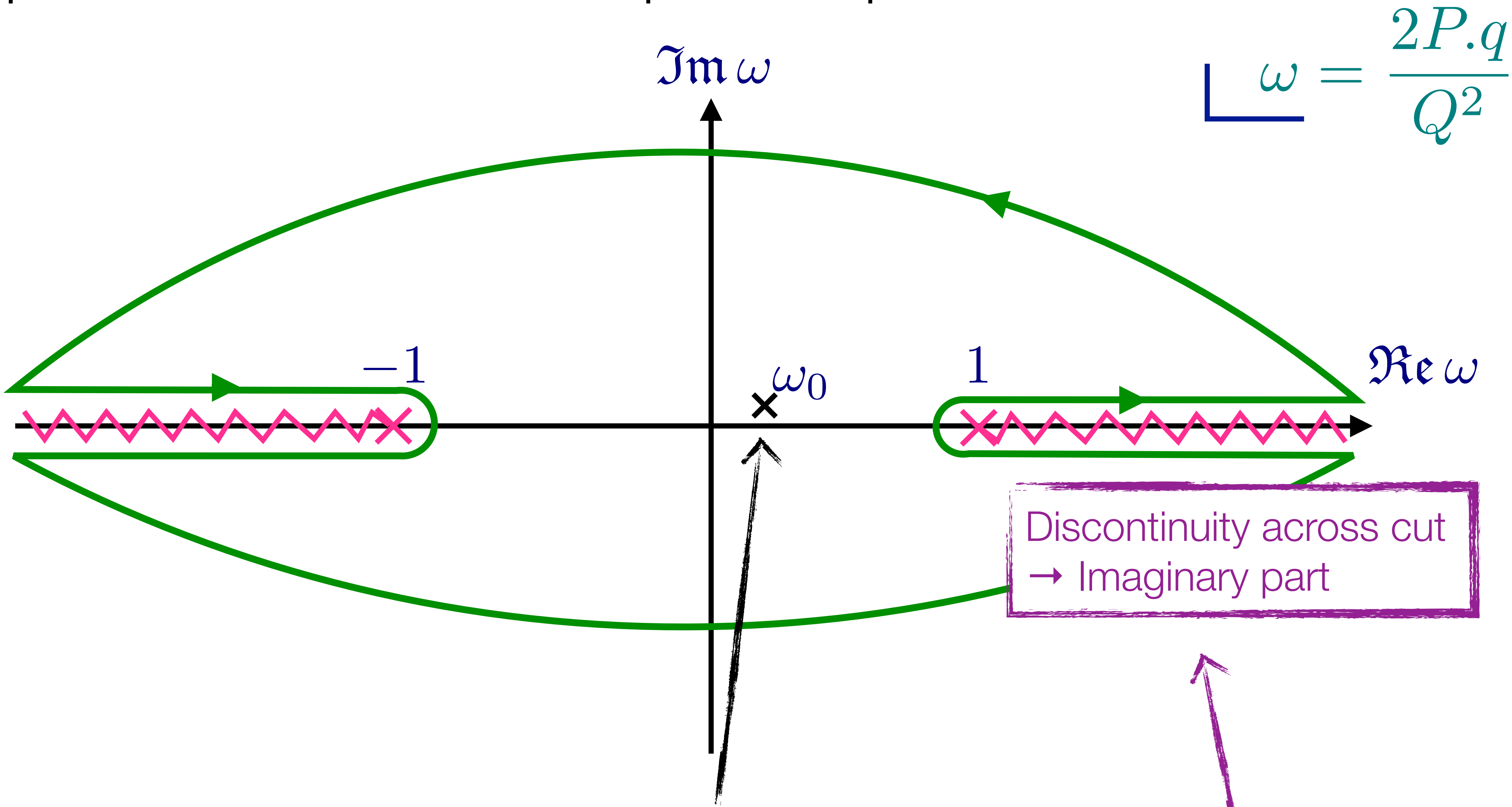
$$T^{\mu\nu}(p,q) = \rho_{ss'} \int d^4x\, e^{iq.x} \langle p,s | \mathrm{T}\left\{J^\mu(x) J^\nu(0)\right\} | p,s \rangle$$
$$= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1(P.q, Q^2) + \frac{1}{P.q}\left(p^\mu - \frac{P.q}{q^2} q_\mu\right)\left(p^\nu - \frac{P.q}{q^2} q_\nu\right) T_2(P.q, Q^2)$$

- Looking ahead to lattice results shown at end, consider simple case

$$\mu = \nu = 3,\ \ q_3 = 0,\ \ P_3 = 0$$

$$\Rightarrow T_{33}(P,q) = T_1(P.q, Q^2)$$

# Dispersion relation for Compton amplitude




Compton amplitude in unphysical region as integral over inelastic structure function

# Moments of structure functions

$$\omega = \frac{2P.q}{Q^2}$$

- Compton amplitude as integral over inelastic cut:

$$T_1(\omega, Q^2) = \frac{2\omega^2}{\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}\, T_1(\omega', Q^2)}{\omega'(\omega^2 - \omega'^2)} = 4\omega^2 \int_0^1 dx\, x \frac{F_1(x, Q^2)}{1 - (\omega x)^2}$$

subtracted dispersion relation

$$x = 1/\omega'$$

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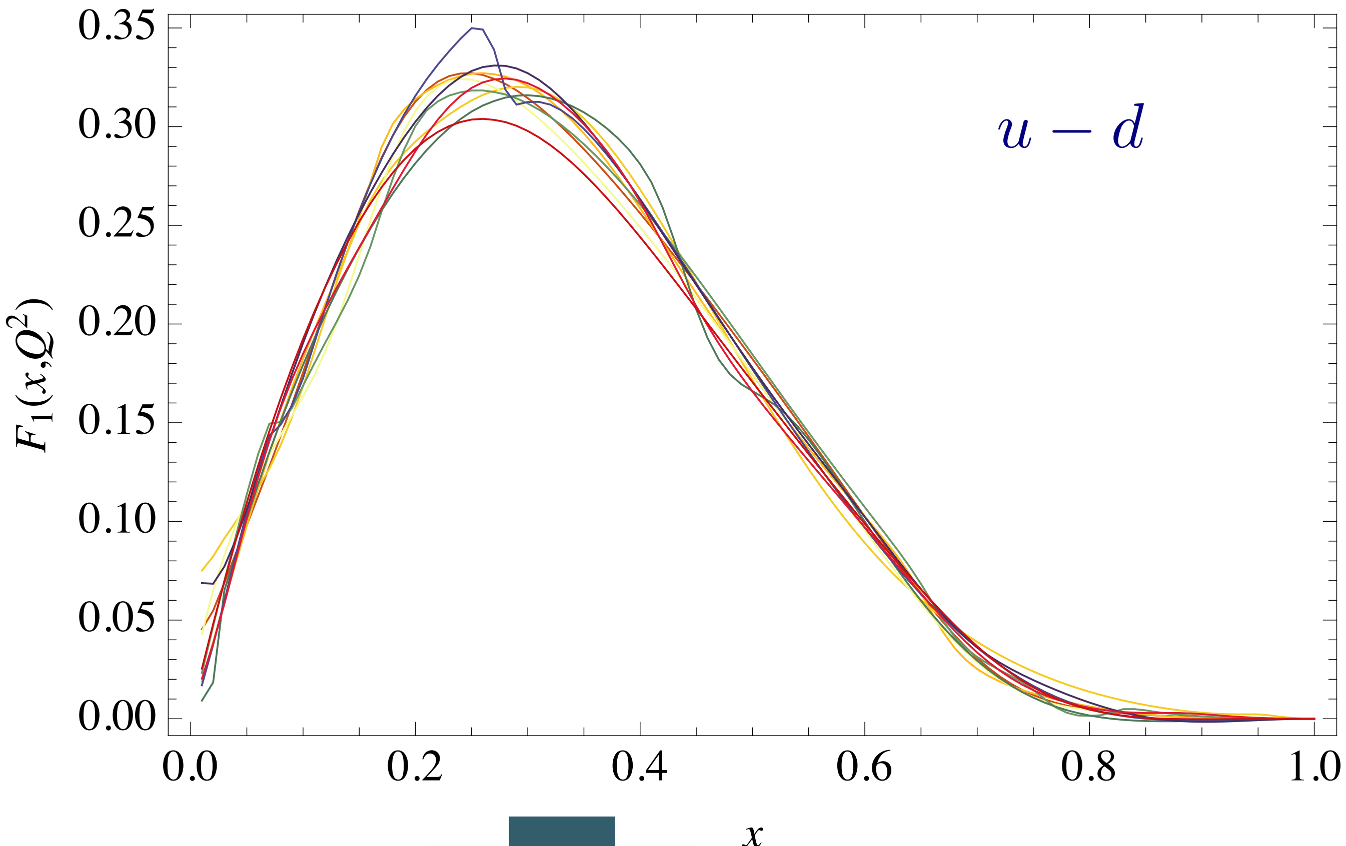
Taylor expansion

- **Moments of structure functions**

$$T_1(\omega, Q^2) = \sum_{j=1}^\infty 4\omega^{2j}\int_0^1 dx\, x^{2j-1} F_1(x, Q^2) \equiv \sum_{j=1}^\infty 4\omega^{2j}\, f_{1,2j-1}(Q^2)$$

# Empirical Compton amplitude (DIS region)

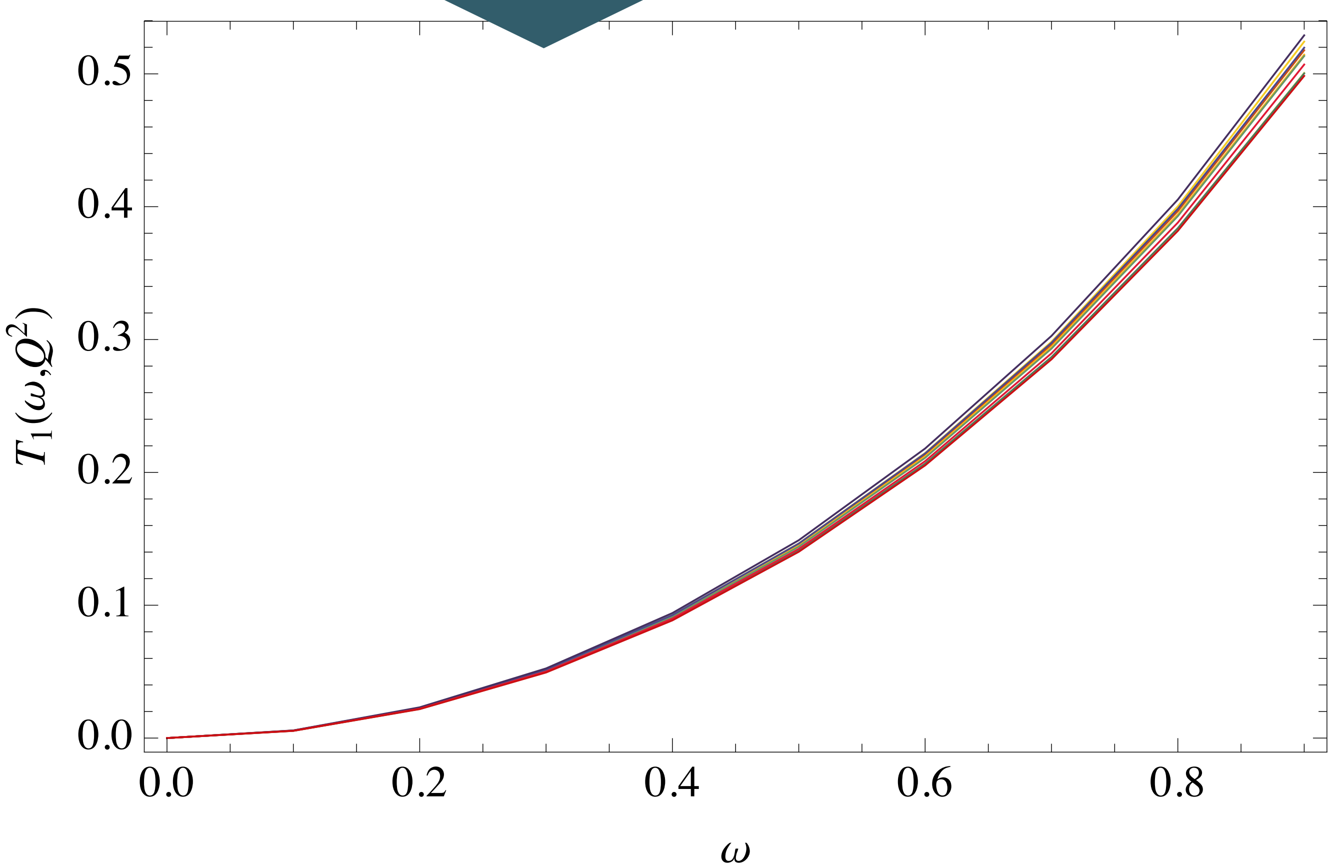
**Structure function**




NNPDF3.1NNLO
10 replicas

$Q^2 = 9\,\mathrm{GeV}^2$

**Compton amplitude**




$$T_1(\omega, Q^2) = \sum_{j=1}^{\infty} 4\omega^{2j} f_{1,2j-1}(Q^2)$$

**Coefficients of Taylor expansion are moments of structure function**

Compton amplitude as an energy shift:
Feynman-Hellmann

$$T_1(\omega, Q^2) - T_1(0, Q^2) = 4\omega^2 \int_0^1 dx\, x \frac{F_1(x, Q^2)}{1 - (\omega x)^2}$$

# Feynman-Hellmann theorem in lattice QCD

# Matrix elements from “Feynman–Hellmann”

- Feynman–Hellmann in quantum mechanics:

$$\frac{dE_n}{d\lambda} = \langle n|\frac{\partial H}{\partial \lambda}|n\rangle$$

  - matrix elements of the derivative of the Hamiltonian determined by derivative of corresponding energy eigenstates

- Lattice QCD: evaluate energy shifts with respect to weak external fields

- Modify action with external field:

$$S \to S + \lambda \int \mathrm{d}^4x\, \mathcal{O}(x)$$

real parameter

local operator, e.g. $\bar{q}(x)\gamma_5\gamma_3 q(x)$

- Calculation of matrix element ≡ hadron spectroscopy [2-pt functions only]

$$\frac{\partial E_H(\lambda)}{\partial \lambda} = \frac{1}{2E_H(\lambda)}\langle H|\mathcal{O}|H\rangle$$

# Spin content [connected]

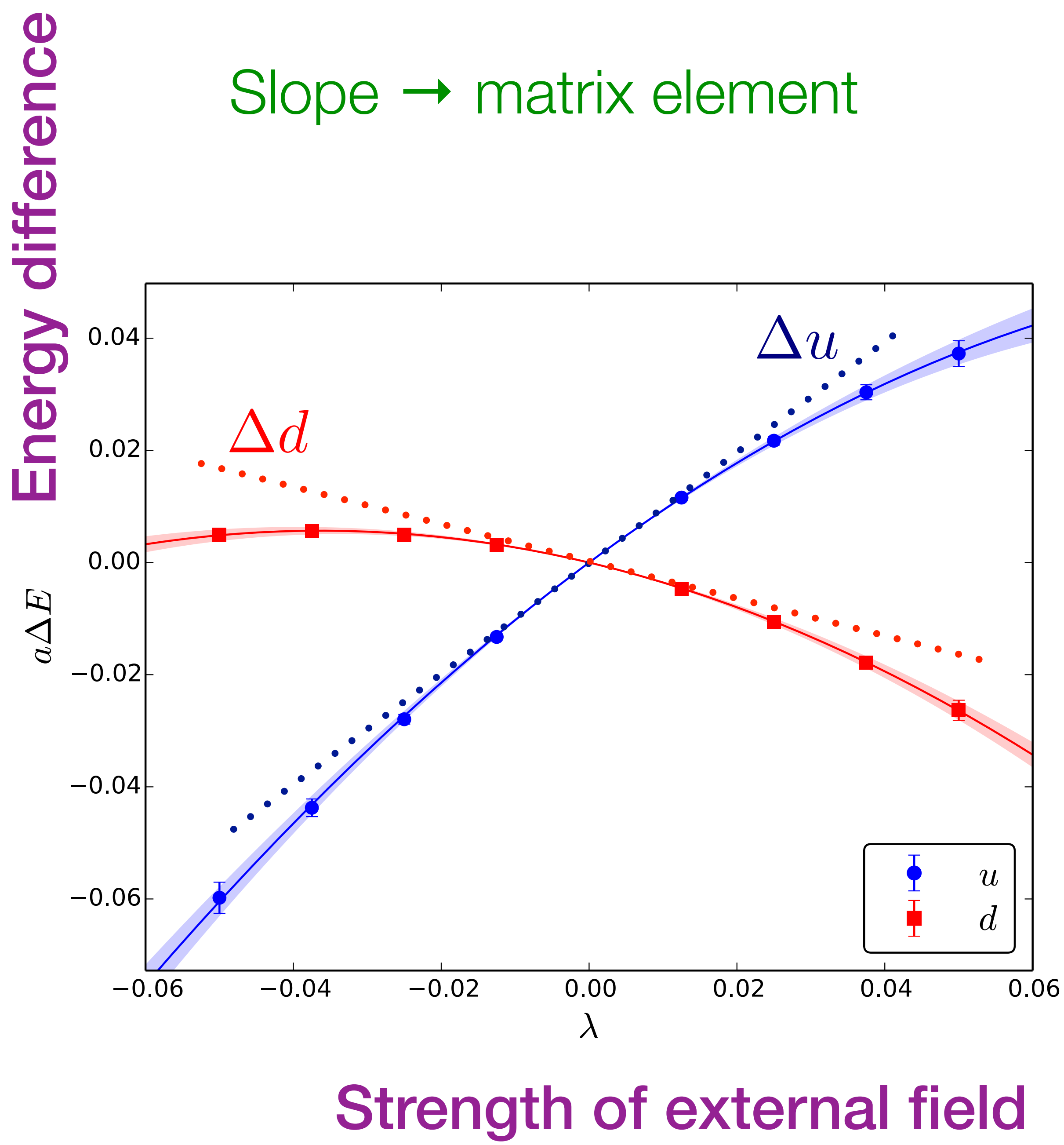
- Modify action

$$S \rightarrow S + \lambda \sum_x \bar{q}(x) i\gamma_5\gamma_3 q(x)$$

- Nucleon energy shift isolates spin content

$$\frac{\partial E_N(\lambda)}{\partial \lambda} = \frac{1}{2M_N}\langle N|\bar{q}i\gamma_5\gamma_3 q|N\rangle$$
$$= \Delta q$$

Slope → matrix element




[Chambers *et al*. PRD(2014)]

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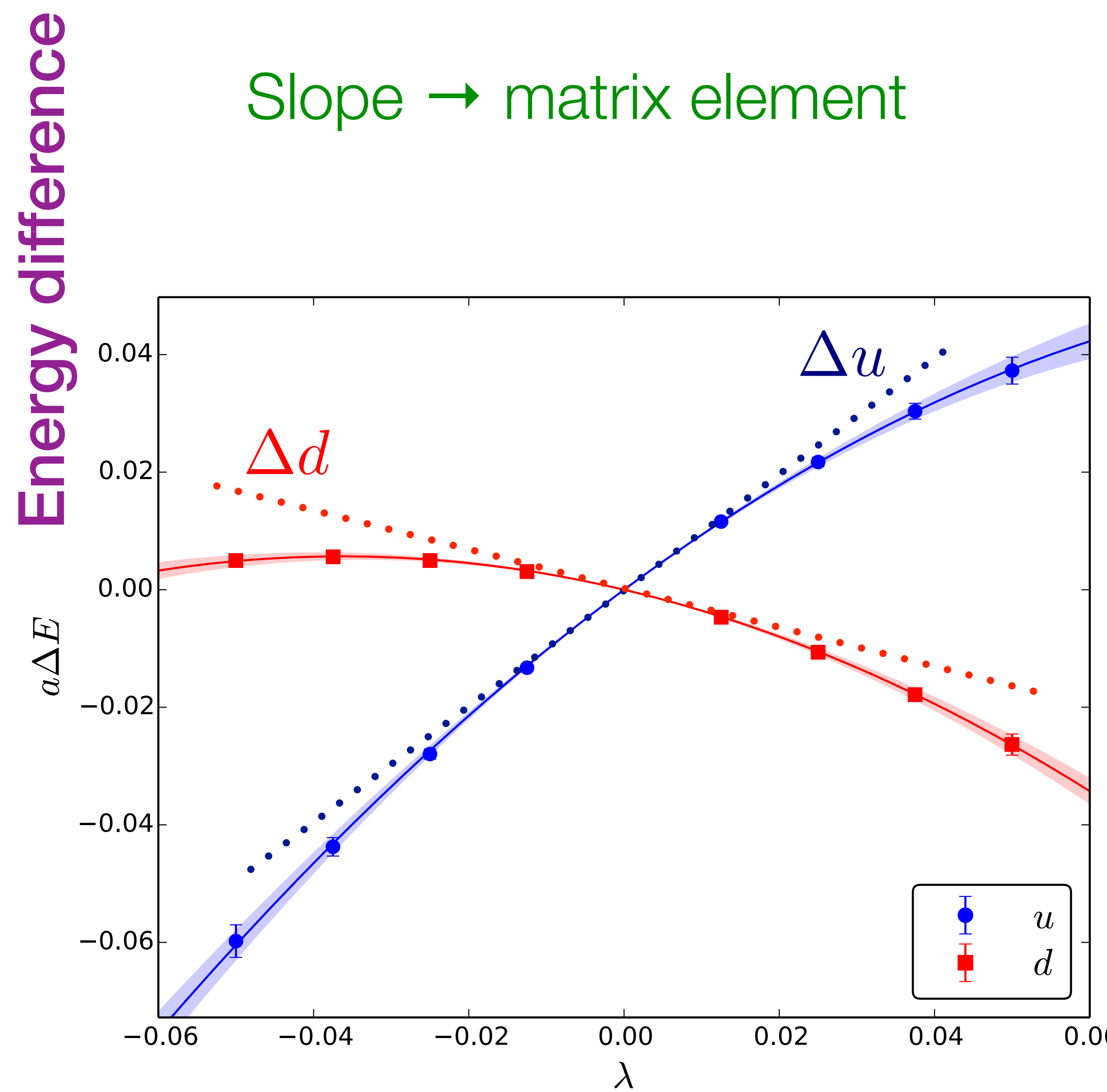


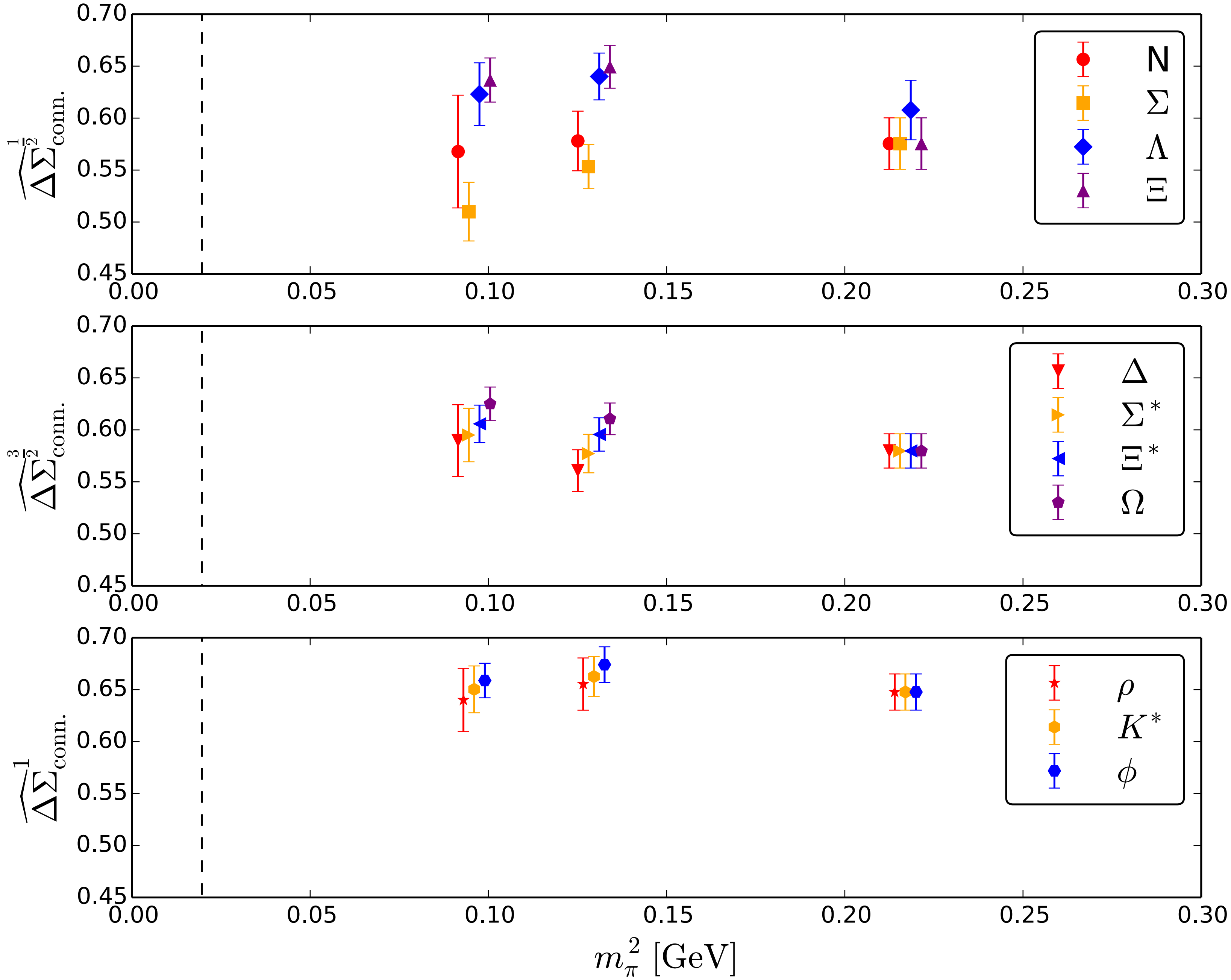

[Chambers *et al*. PRD(2014)]

**3-pt function → 2-pt function**

# Connected Spin Contributions

[Chambers *et al*. PRD(2014)]

- Connected spin factions in various hadrons




(Connected) Spin Fraction Universal ~60%

"Heavy" quarks: $m_q \sim m_s$

# “Disconnected” quark spin

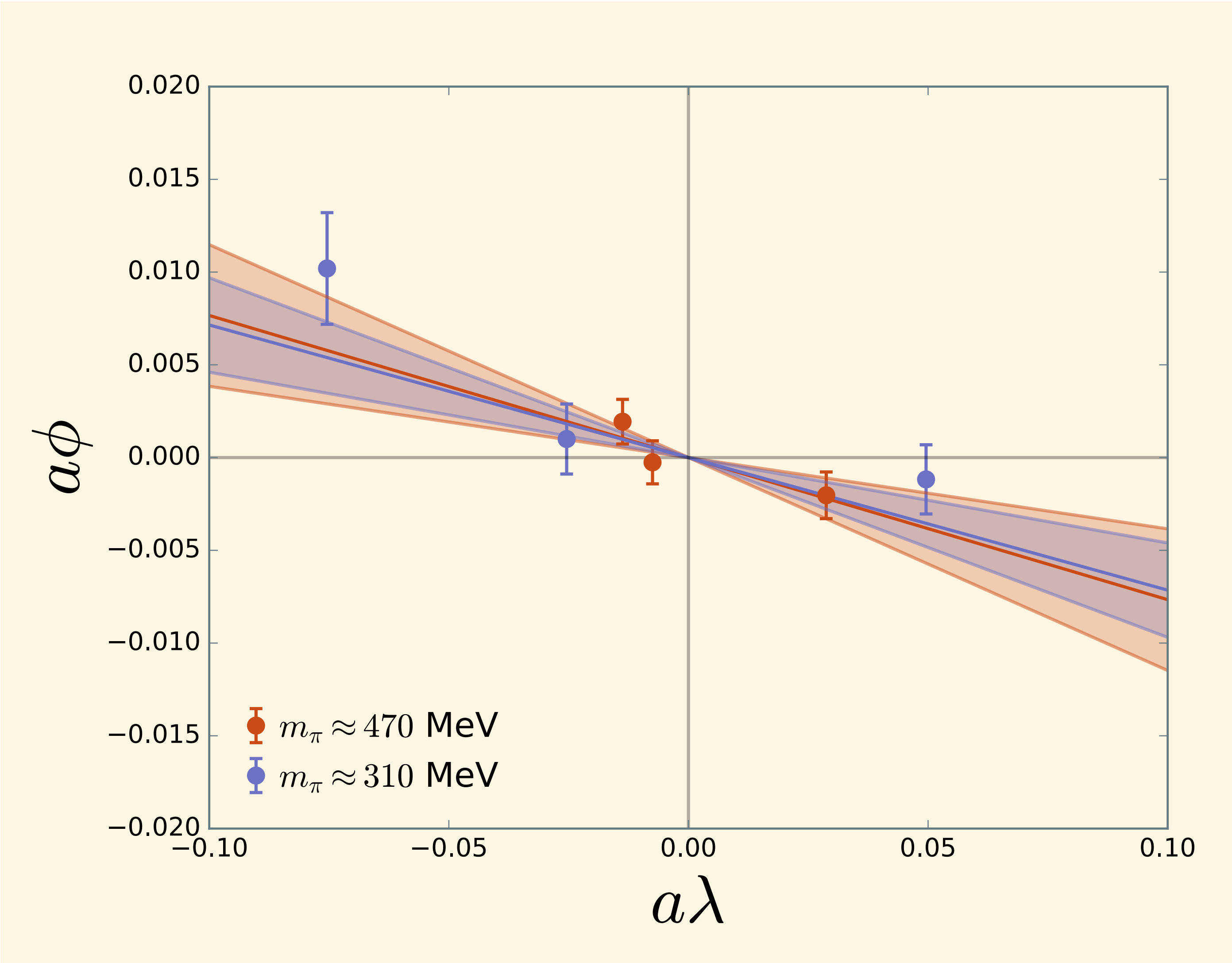
- External field operator

$$\mathcal{L} \to \mathcal{L} + i\lambda\bar{q}\gamma_\mu\gamma_5 q$$

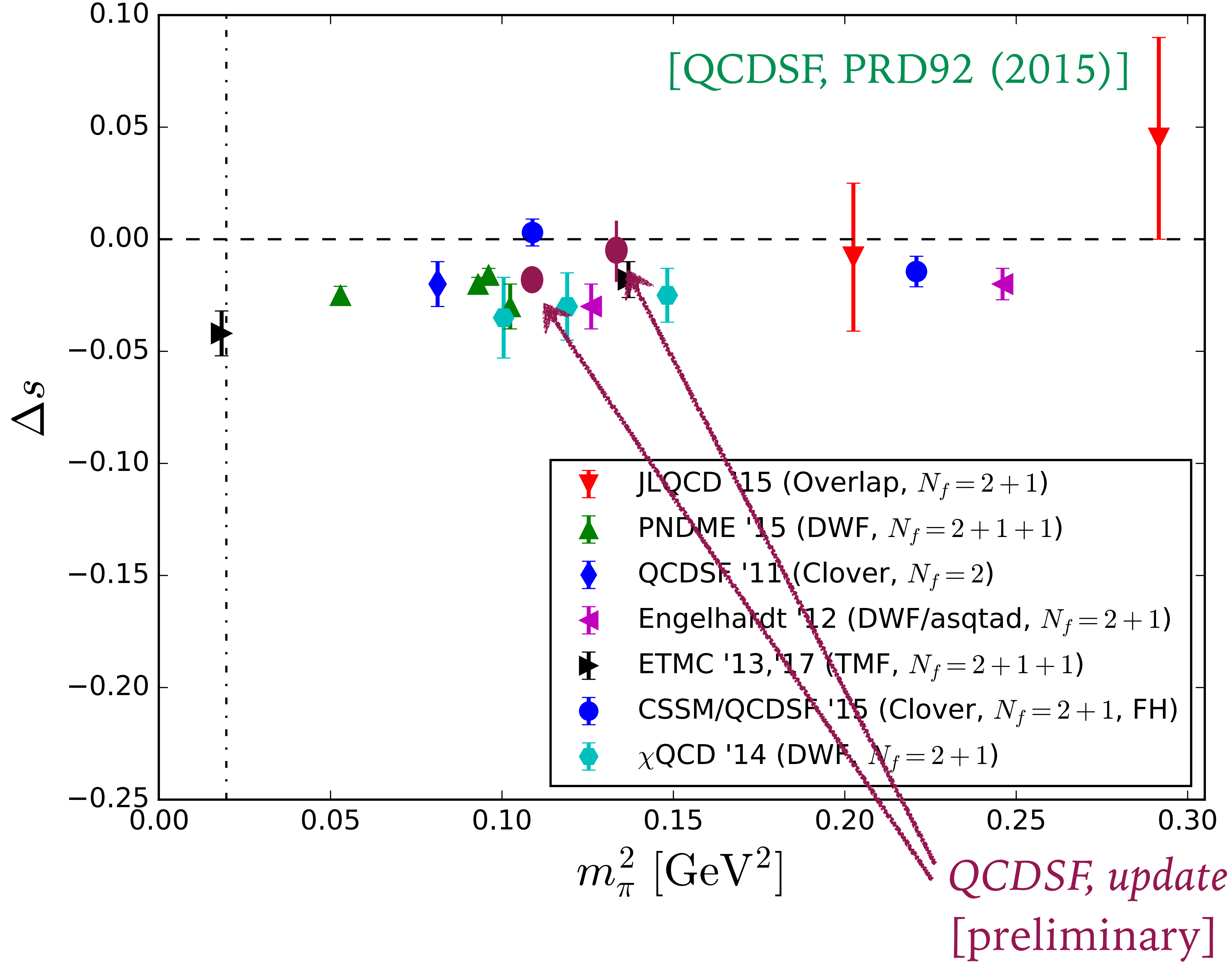
$\gamma_5$-Hermitian for Monte Carlo

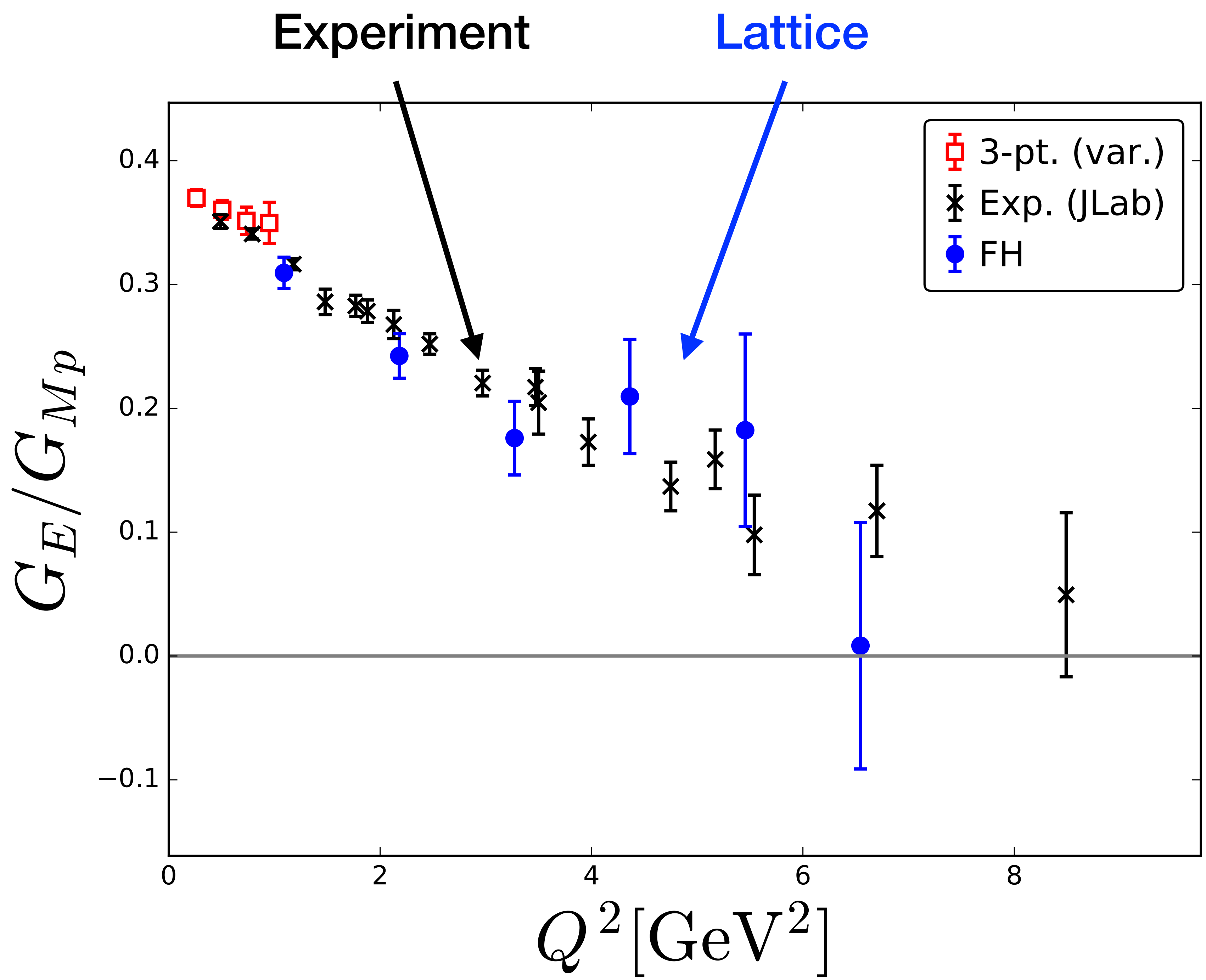
- Generate modified gauge field ensembles
- Calculate unmodified correlation functions
- Matrix element in phase of correlator:

$$\left.\frac{\partial\phi_N}{\partial\lambda}\right|_{\lambda=0} \propto \Delta q_{\text{disc.}}$$



# Disconnected spin contribution






[Chambers *et al*. PRD96(2017)

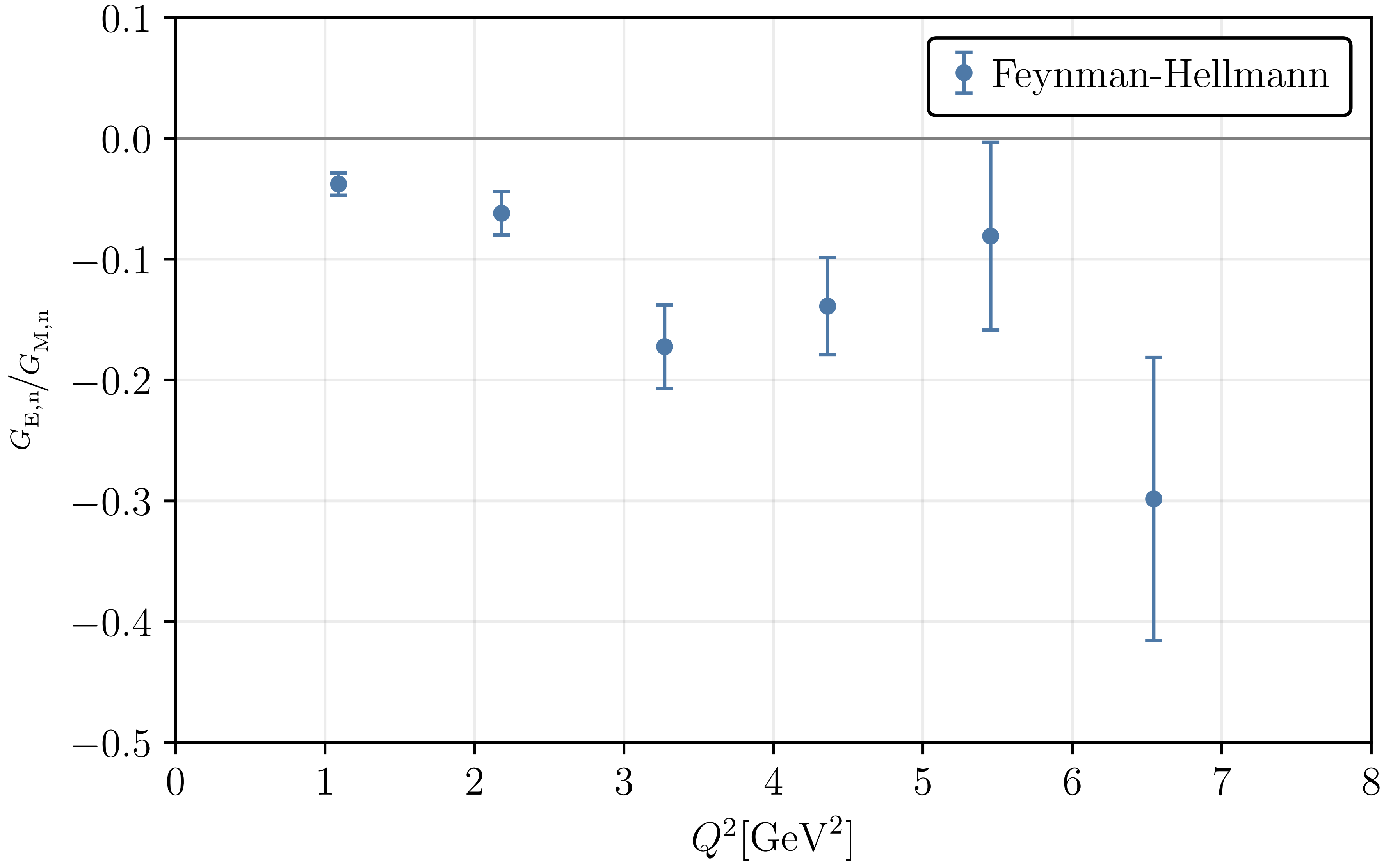
# Proton form factors

Feynman-Hellmann with momentum transfer

# Neutron GE/GM

PhD Thesis, A. Chambers (2018)

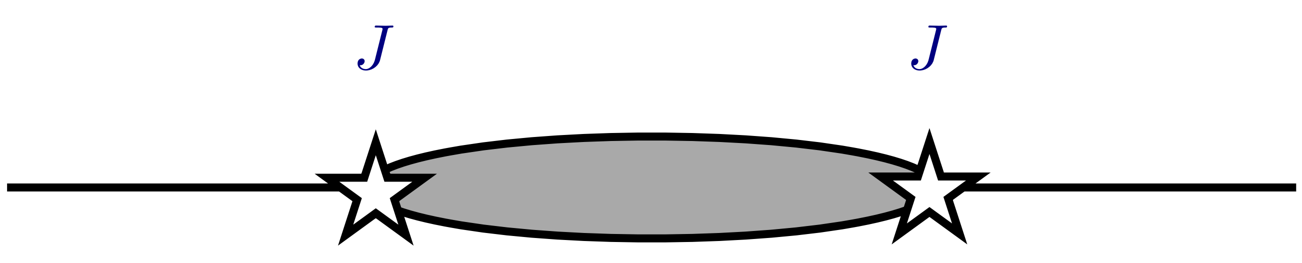
without $\mu_n$ factor




**Statistical errors only; expect systematics to dominate**

# Feynman-Hellmann (2nd order)

$$T_1(\omega, Q^2) - T_1(0, Q^2) = 4\omega^2 \int_0^1 dx\, x \frac{F_1(x, Q^2)}{1-(\omega x)^2}$$



# Feynman–Hellmann (2nd order)

- Quantum mechanics: 2nd order perturbation theory

$$E = E_0 + \lambda \langle N|V|N\rangle + \lambda^2 \sum_{X \neq N} \frac{\langle N|V|X\rangle\langle X|V|N\rangle}{E_0 - E_X} + \ldots$$

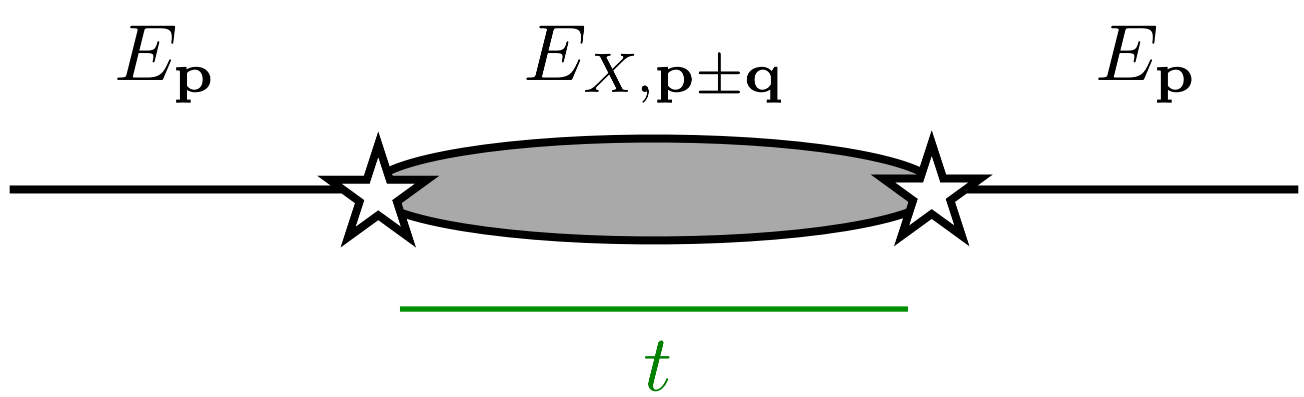
  - Only get a linear term for elastic case $\boldsymbol{\omega}=1$ [Breit frame]

- Insert a weak spatially-varying vector current, e.g.
  -
$$S \to S_0 + \lambda \int d^4y \left(e^{i\mathbf{q}\cdot\mathbf{y}} + e^{-i\mathbf{q}\cdot\mathbf{y}}\right) \bar{q}(y)\gamma_3 q(y)$$

- Second-order energy shifts isolate forward Compton amplitude $(\mathbf{q}^2 > |2\mathbf{p}.\mathbf{q}|)$

$$\frac{\partial^2 E_{\mathbf{p}}}{\partial \lambda_{\mathbf{q}}^2} = -\frac{1}{E_{\mathbf{p}}} \int d^4x \, e^{iq.x} \langle \mathbf{p}|\mathrm{T} J(x) J(0)|\mathbf{p}\rangle$$

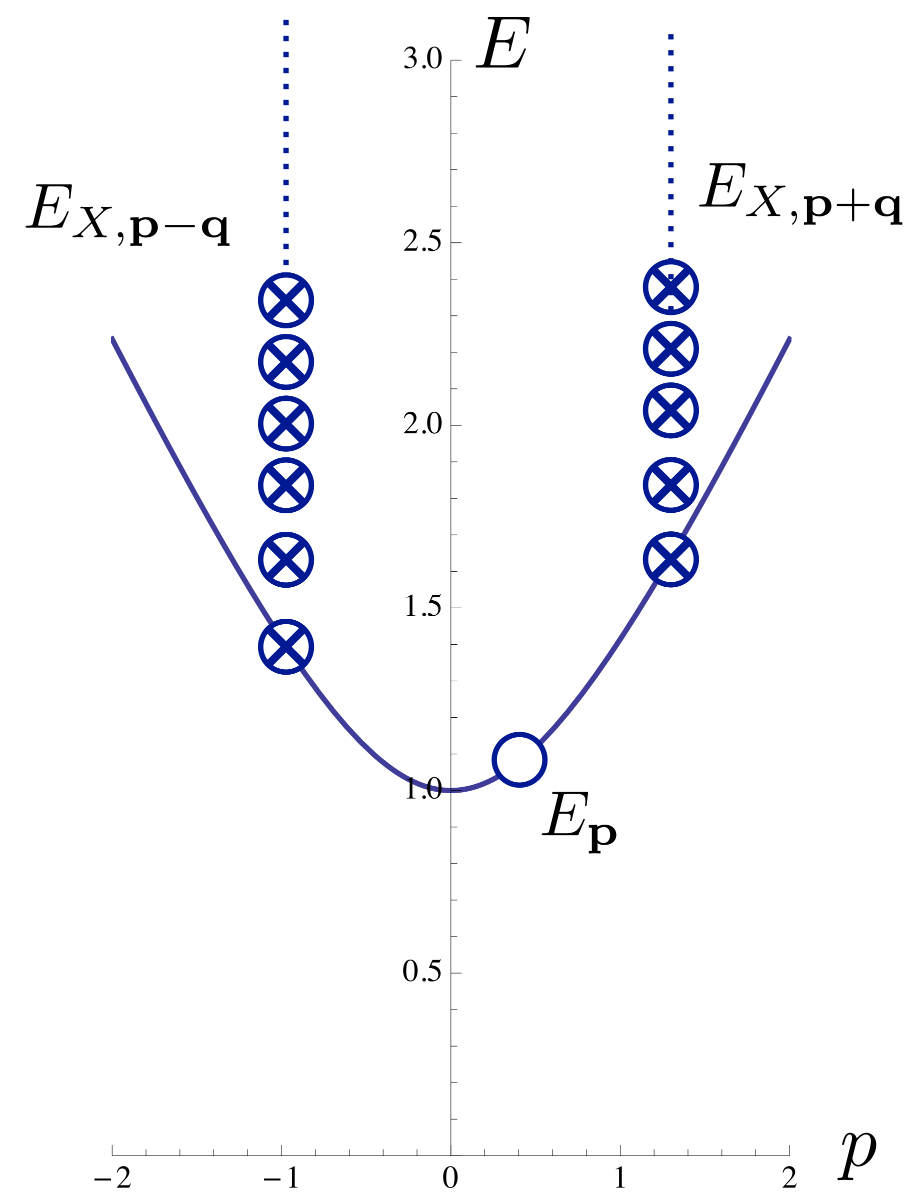
# Two current insertions




Euclidean decay of intermediate state
FH: integrate over all times

$$\int_0^T dt\, e^{-t(E_{X,\mathbf{p}+\mathbf{q}}-E_{\mathbf{p}})} = \frac{1-e^{-T(E_{X,\mathbf{p}+\mathbf{q}}-E_{\mathbf{p}})}}{E_{X,\mathbf{p}+\mathbf{q}}-E_{\mathbf{p}}}$$

$$\frac{\partial^2 E_{\mathbf{p}}}{\partial\lambda^2} \sim \sum_X \frac{\langle p|J|X,\mathbf{p}+\mathbf{q}\rangle\langle X,\mathbf{p}+\mathbf{q}|J|p\rangle}{E_{X,\mathbf{p}+\mathbf{q}}-E_{\mathbf{p}}} + (\mathbf{q}\to-\mathbf{q})$$




$E_{\mathbf{p}} < E_X$
Intermediate states cannot go "on-shell" for $\omega$<1

# Kinematic variation of $\omega$

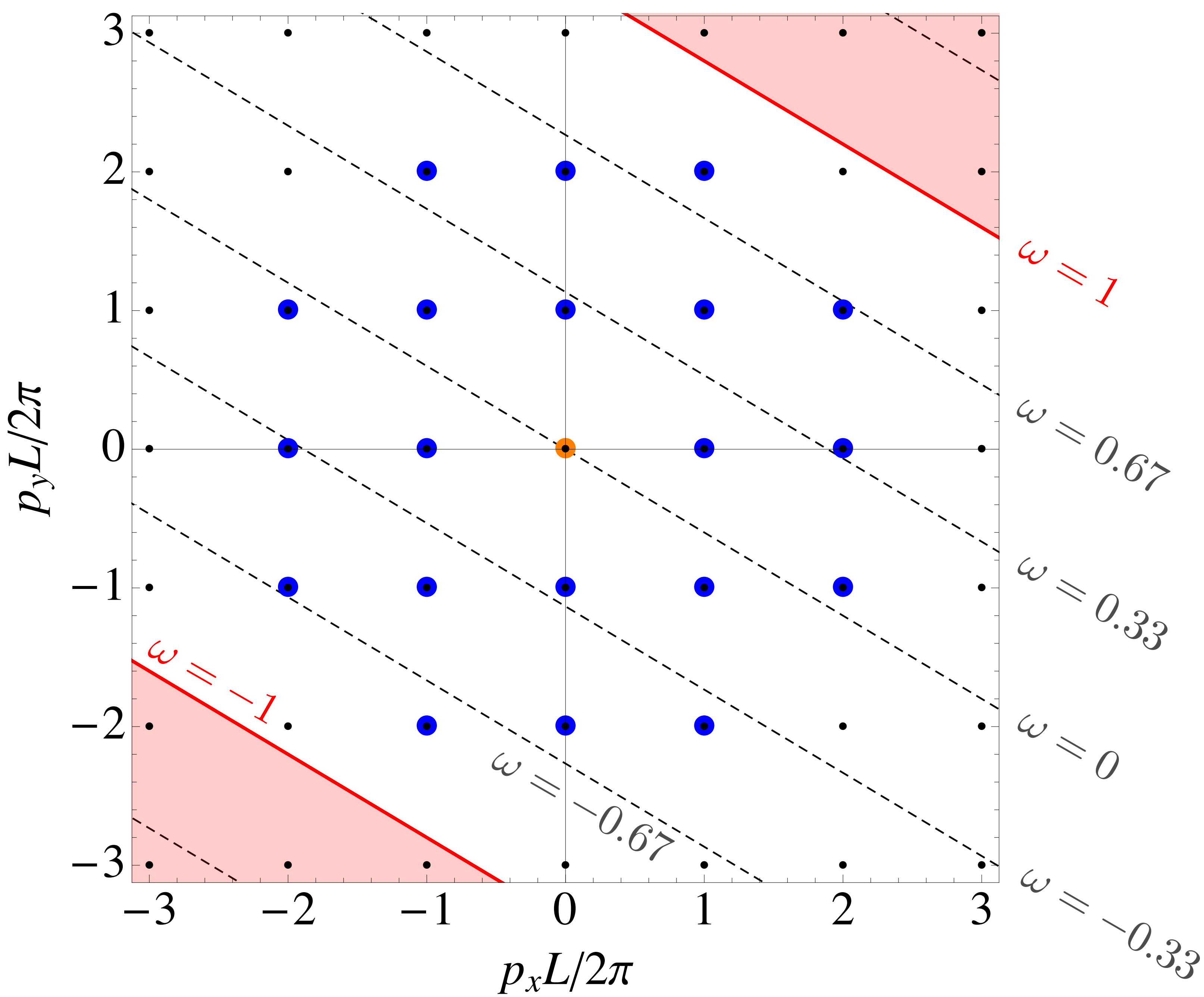
**External momentum**

$$\vec{q} = (3,5,0)\,\frac{2\pi}{L}$$

**Can access different $\omega$ by varying the nucleon momenta**

$$\omega = \frac{2P.q}{Q^2} = \frac{2\vec{P}.\vec{q}}{\vec{q}^2}$$

$q_4 = 0$




**Blue dots:** different nucleon Fourier momenta

# Numerical results

**Lattice specs:**
NP-improved clover
$m_\pi$ ~ 470 MeV

SU(3) symmetric
a~0.074 fm
$32^3$x64
(+ *a couple extras*)

# Lattice QCD: Energy eigenstates

- QCD path integral: discretise Euclidean spacetime; derivatives to finite difference; gluon field encoded in gauge links; fermion actions & chiral symmetry; etc.
- For each snapshot of the gluons, we compute propagation amplitude

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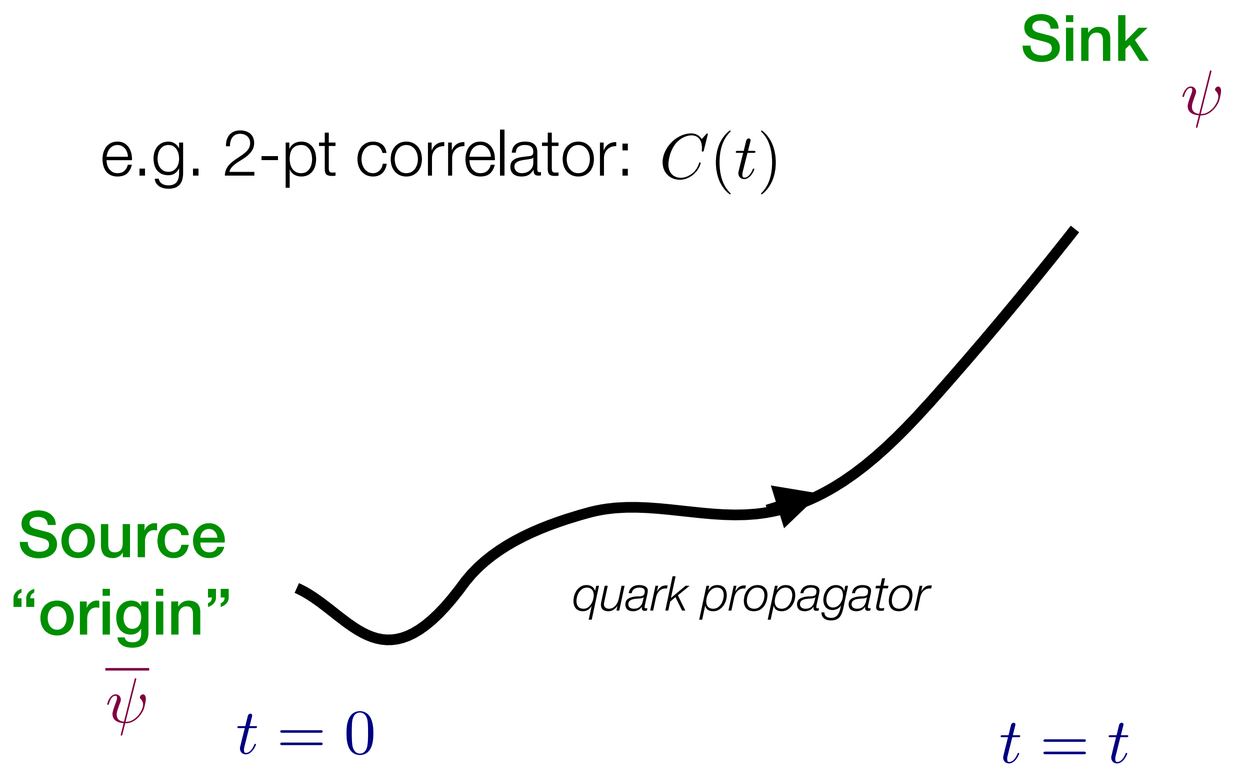

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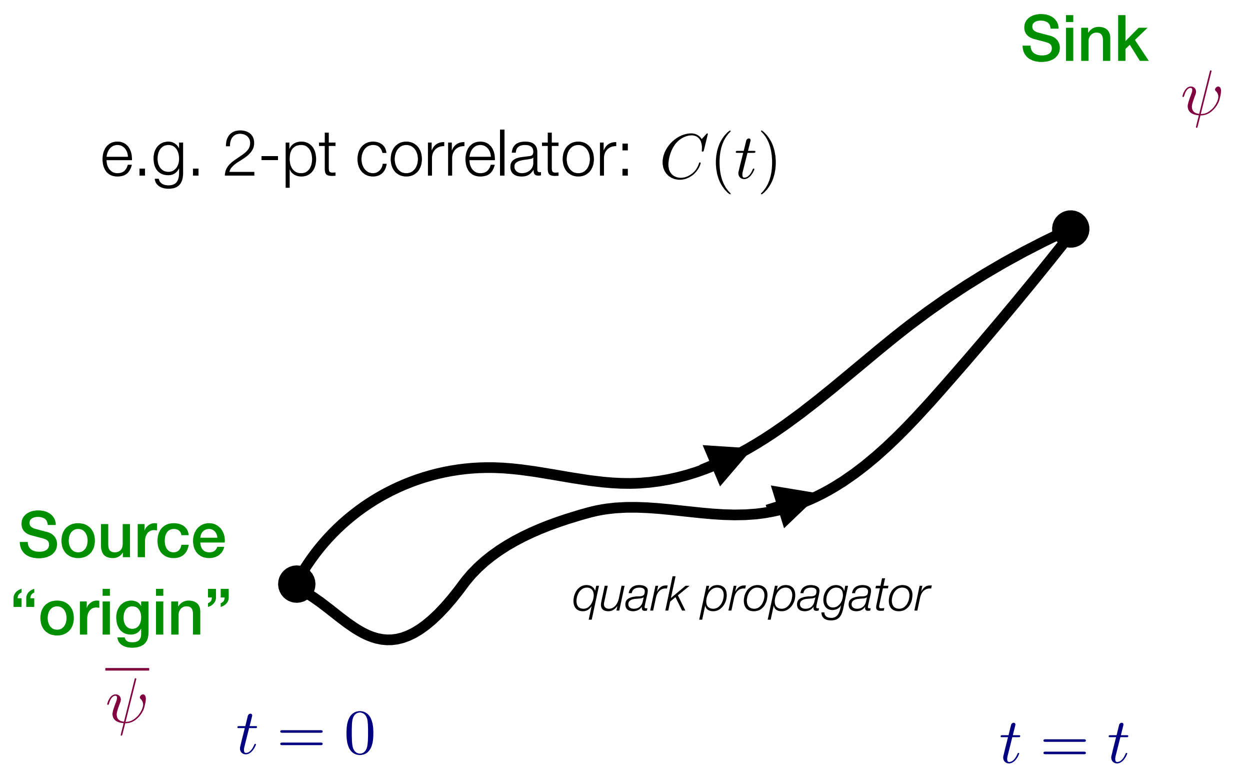

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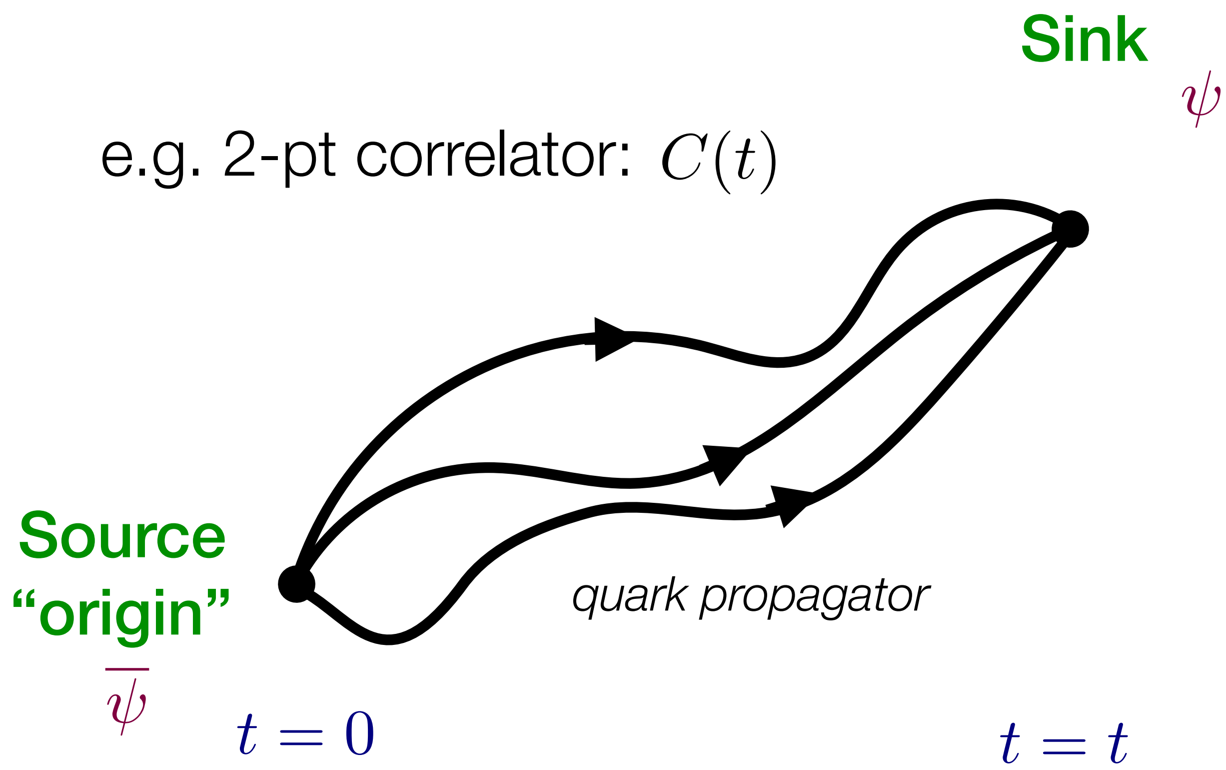

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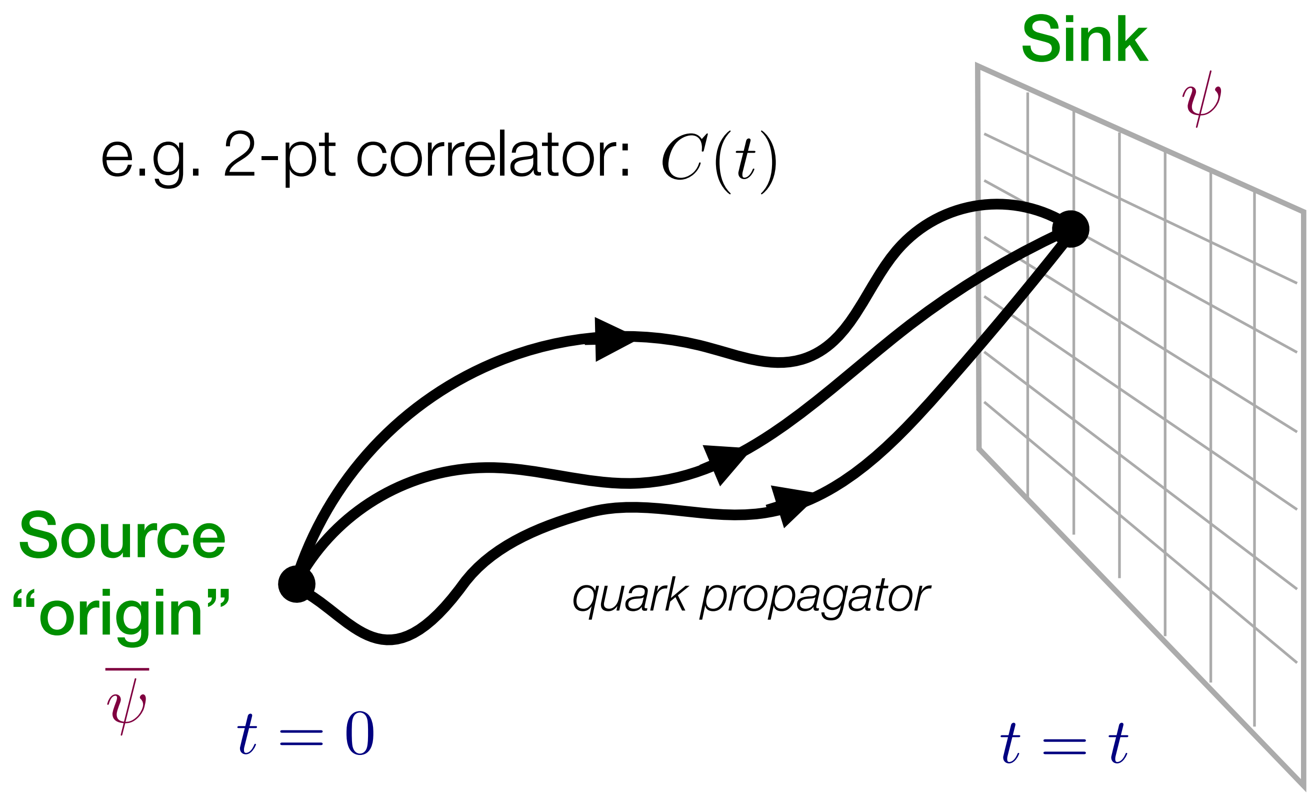


Fourier project onto 3-volume at sink
⇒ definite 3-momentum; e.g. $\mathbf{p}' = 0$

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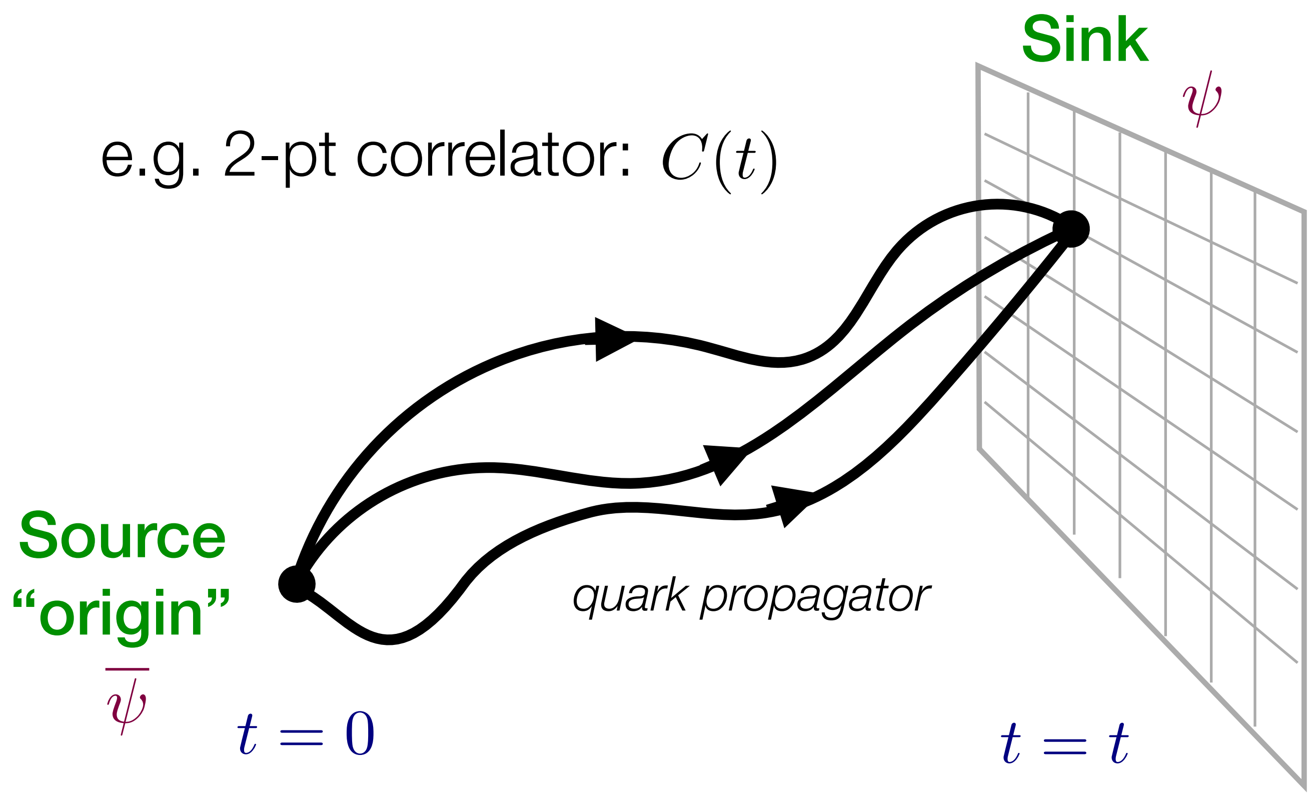


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Euclidean time evolution: $\exp(-Ht)$

$$\langle C(t) \rangle = \sum_n \left| \langle n | \overline{\psi} | \text{vac.} \rangle \right|^2 e^{-E_n t}$$

lowest energy state dominates at large *t*

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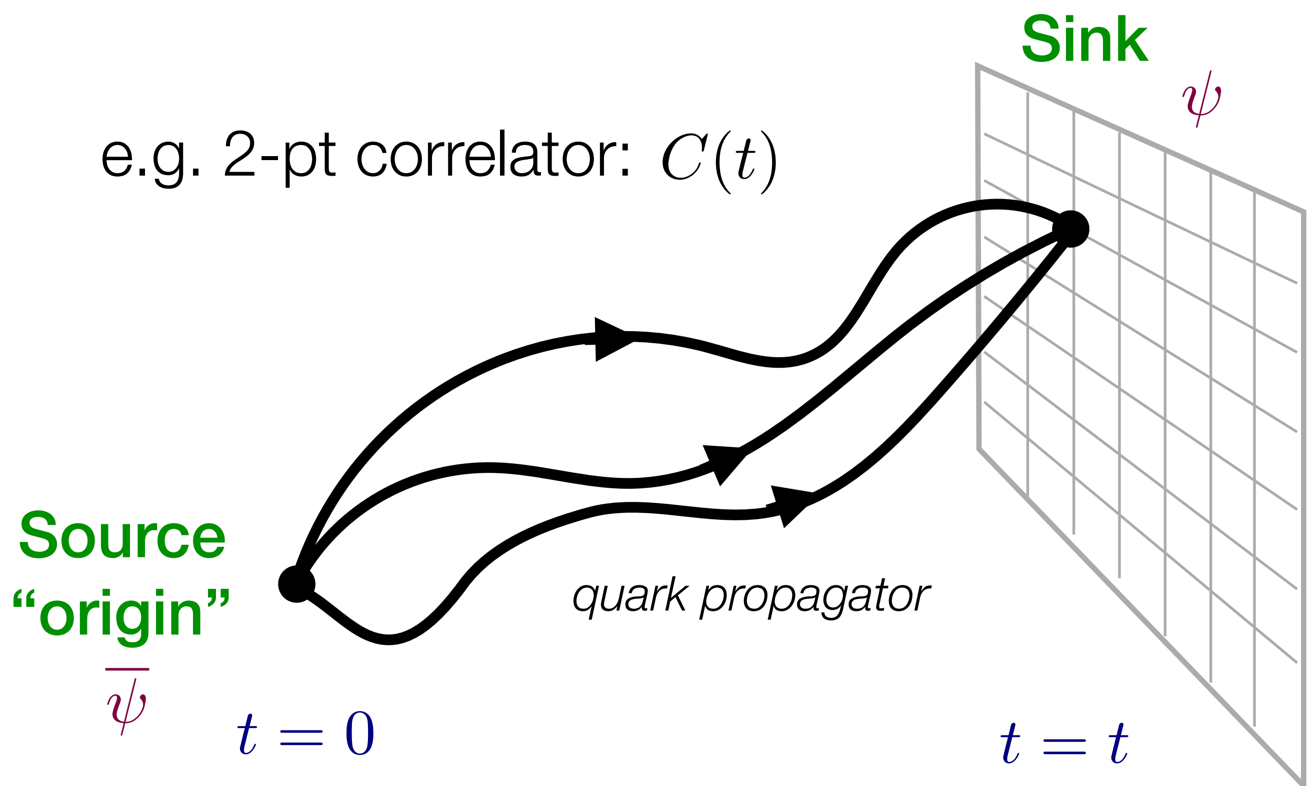


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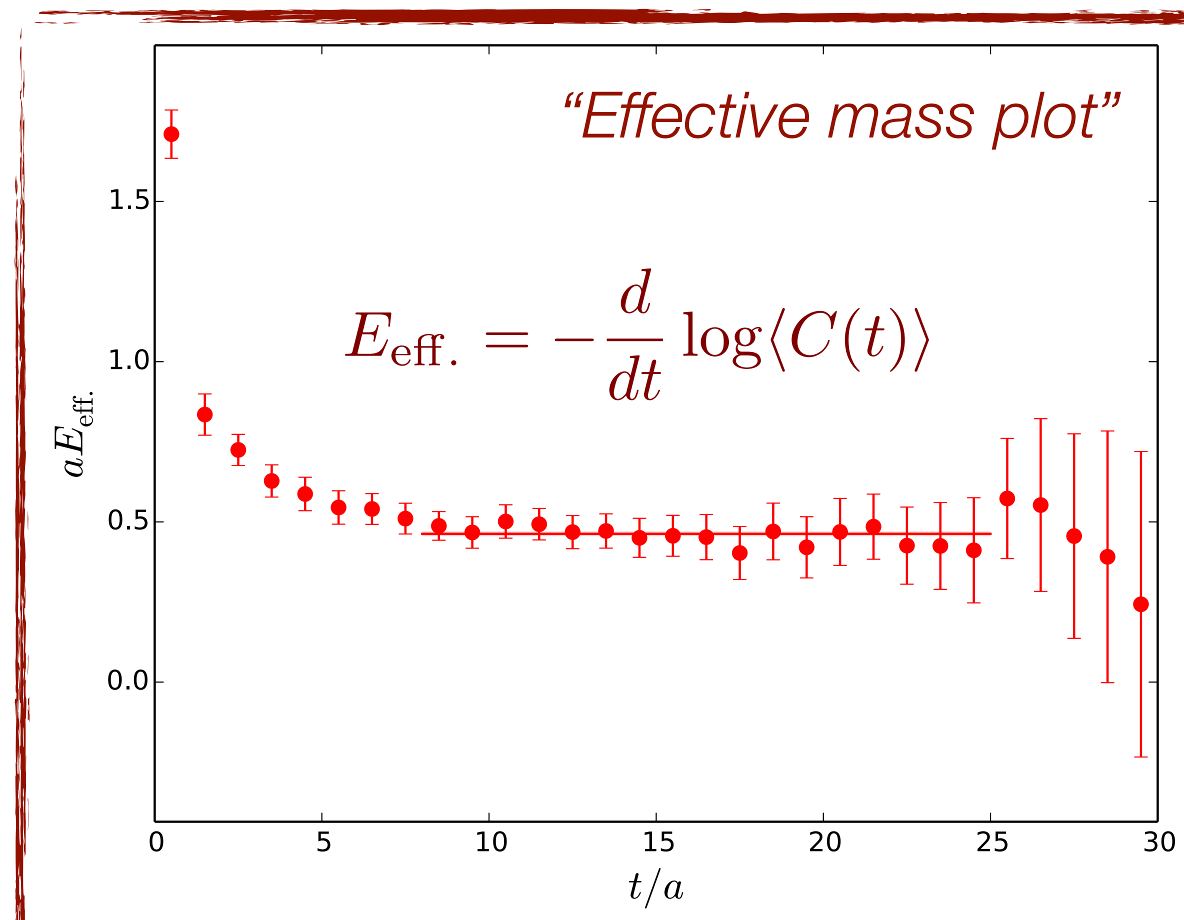


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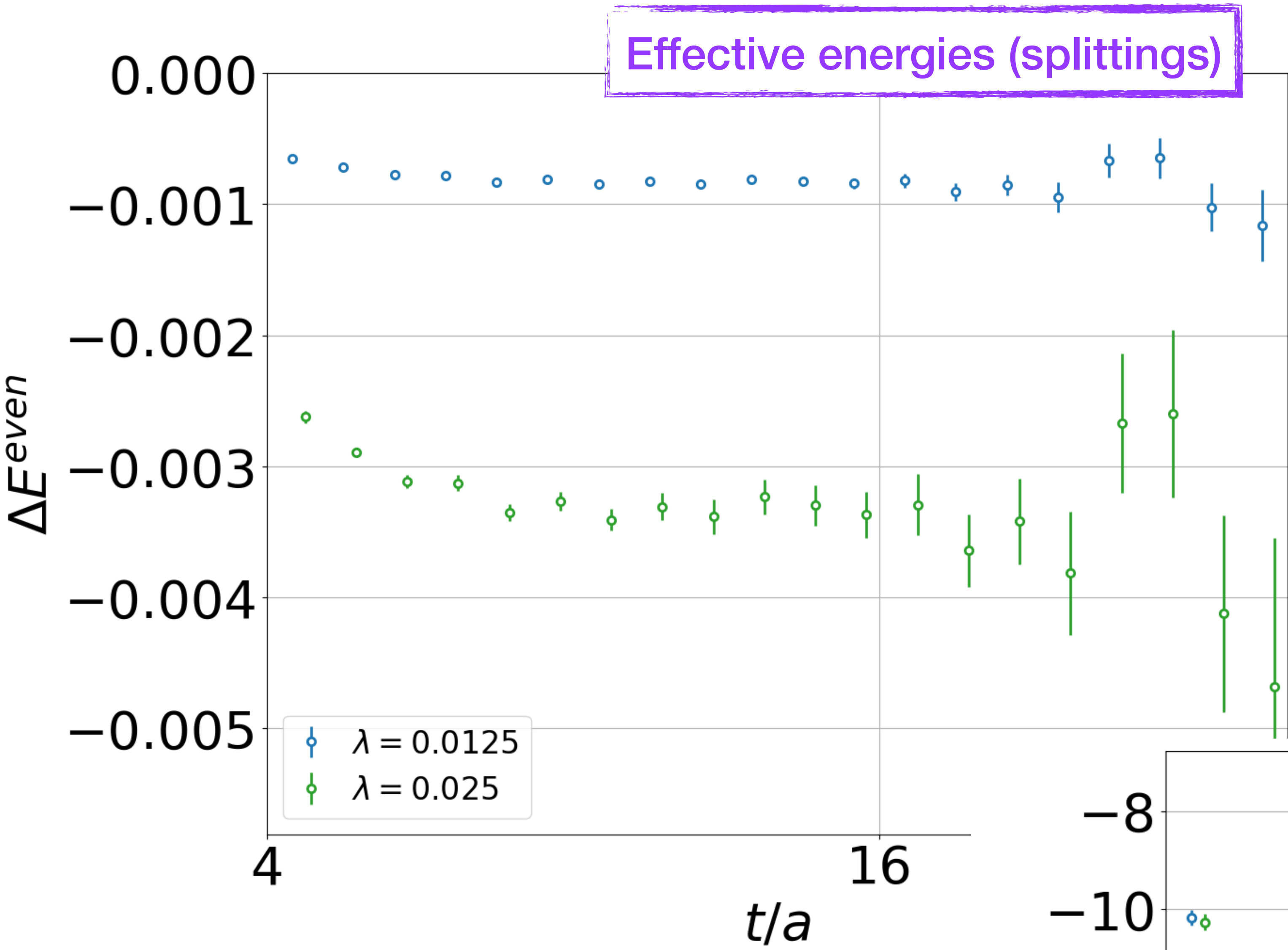
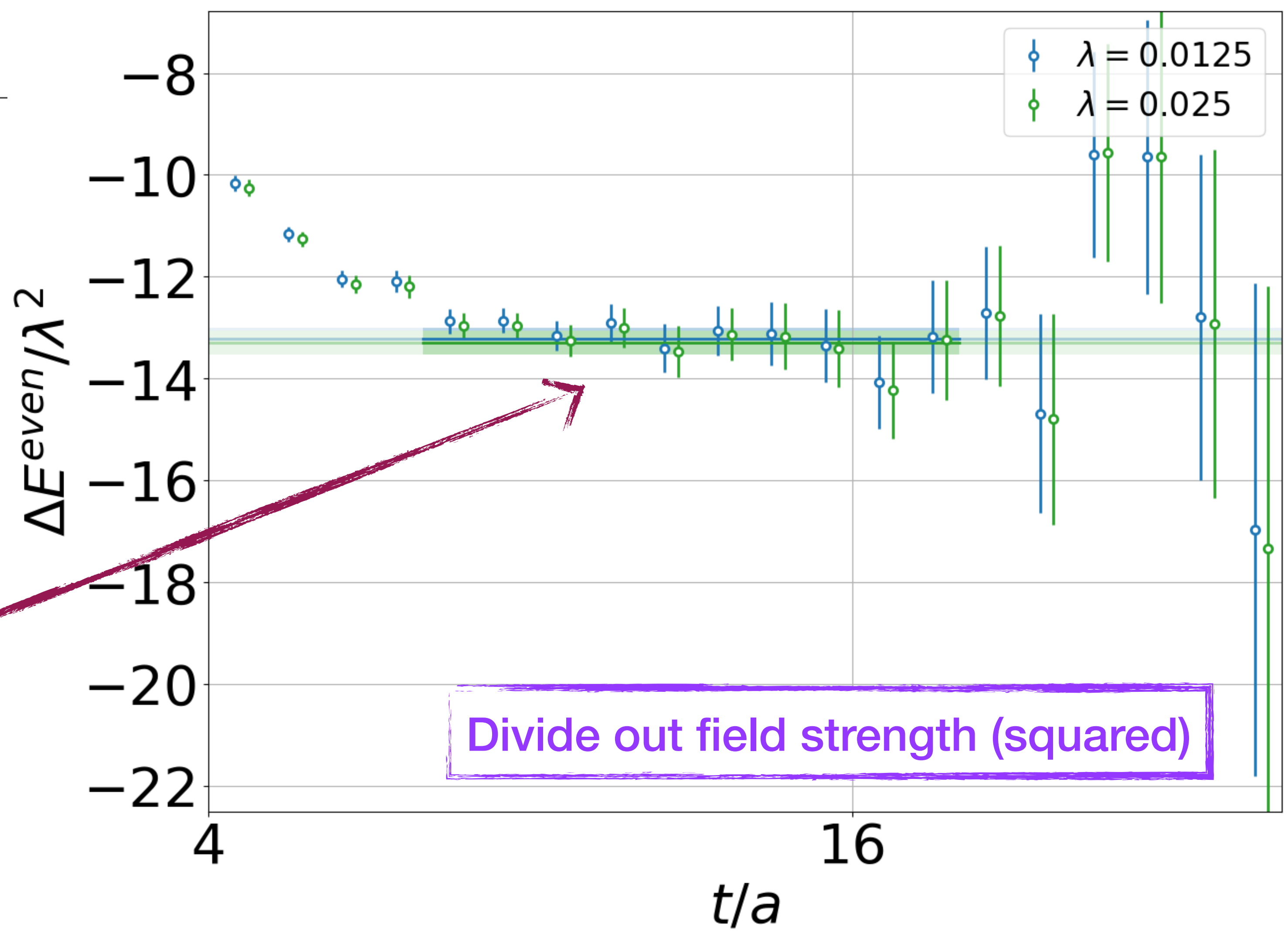
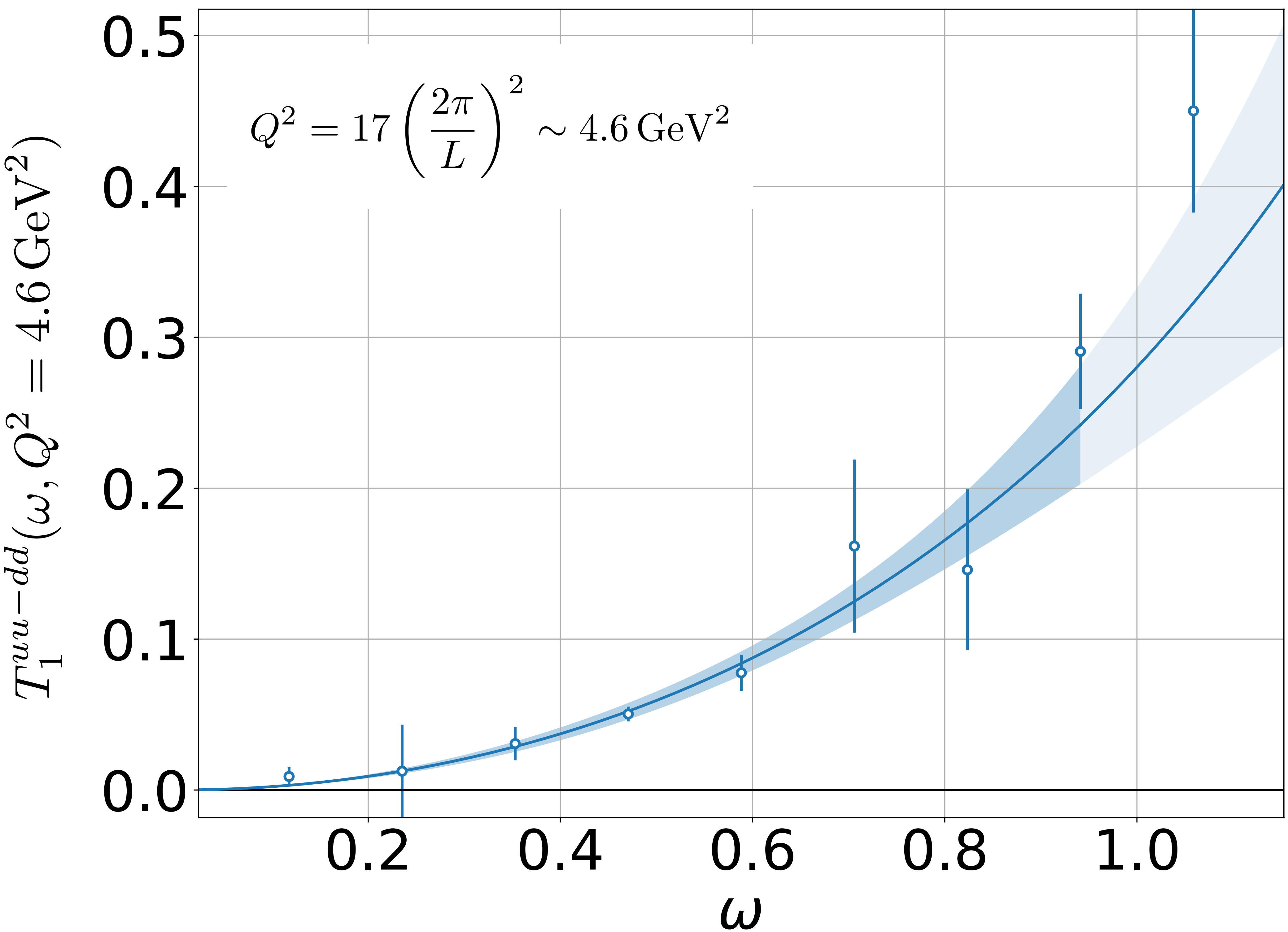



Quadratic energy shift realised (almost) exactly point-by-point in effective mass

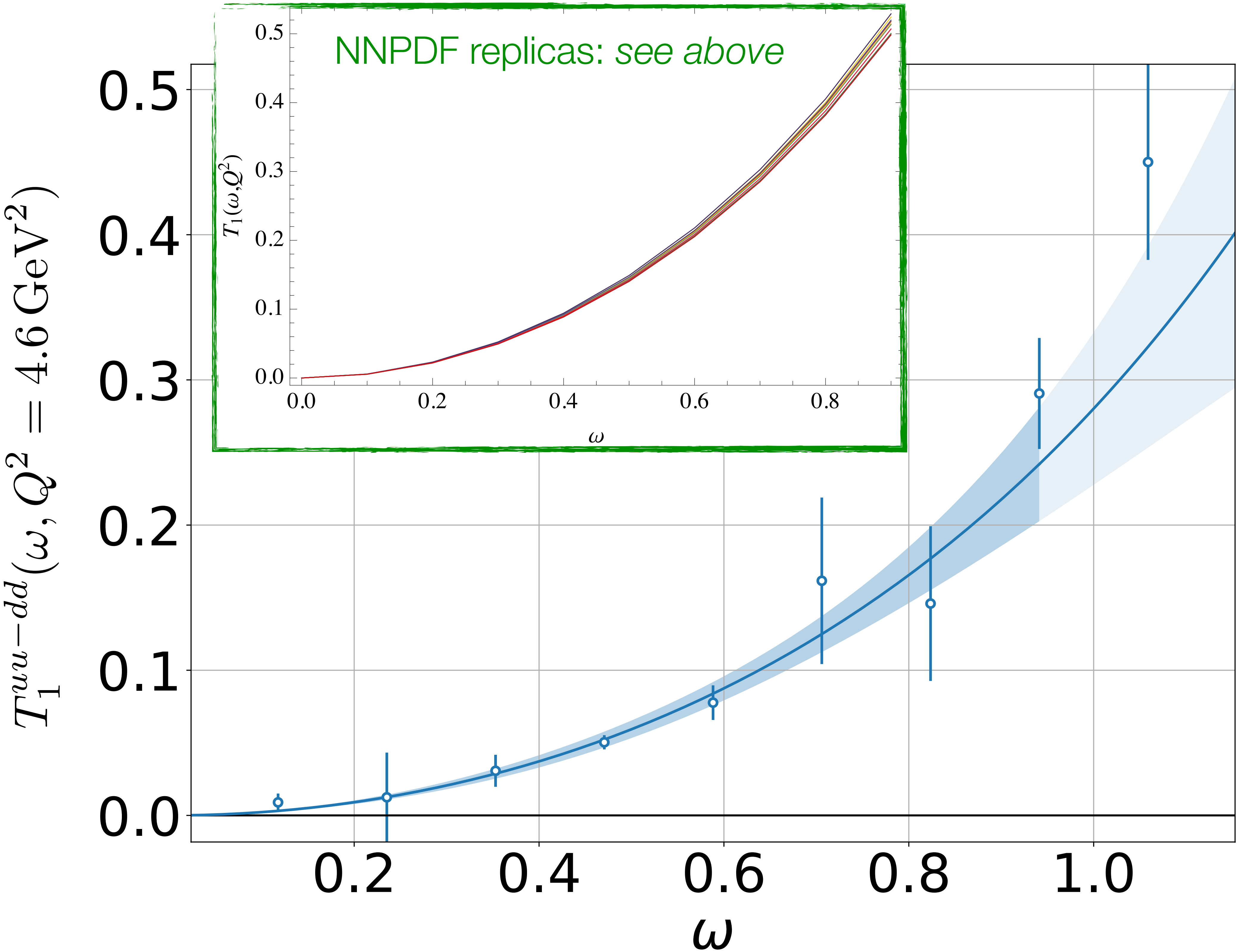





Compton amplitude

Single external momentum

$$\frac{\vec{q}L}{2\pi} = (4, 1, 0)$$

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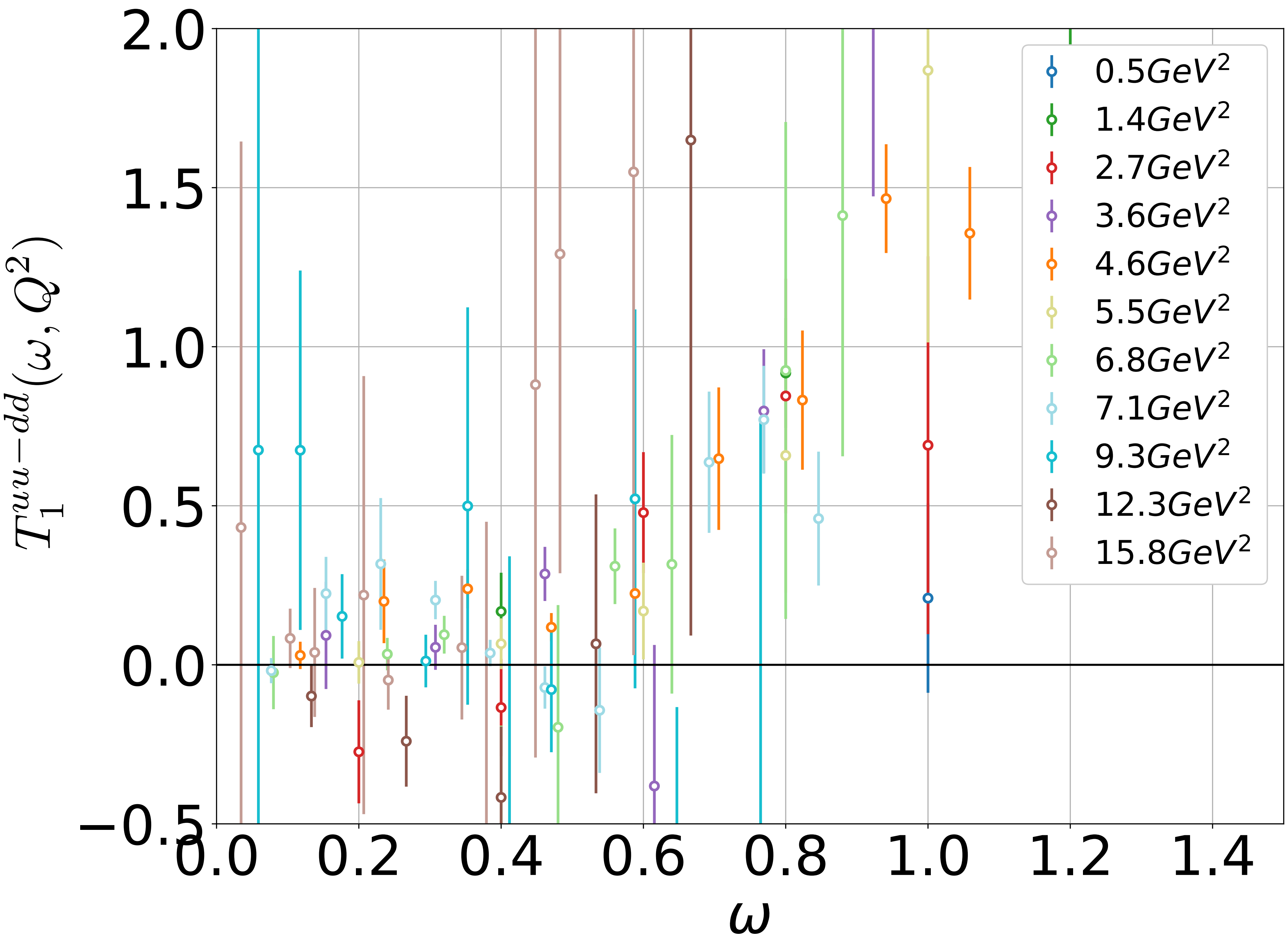
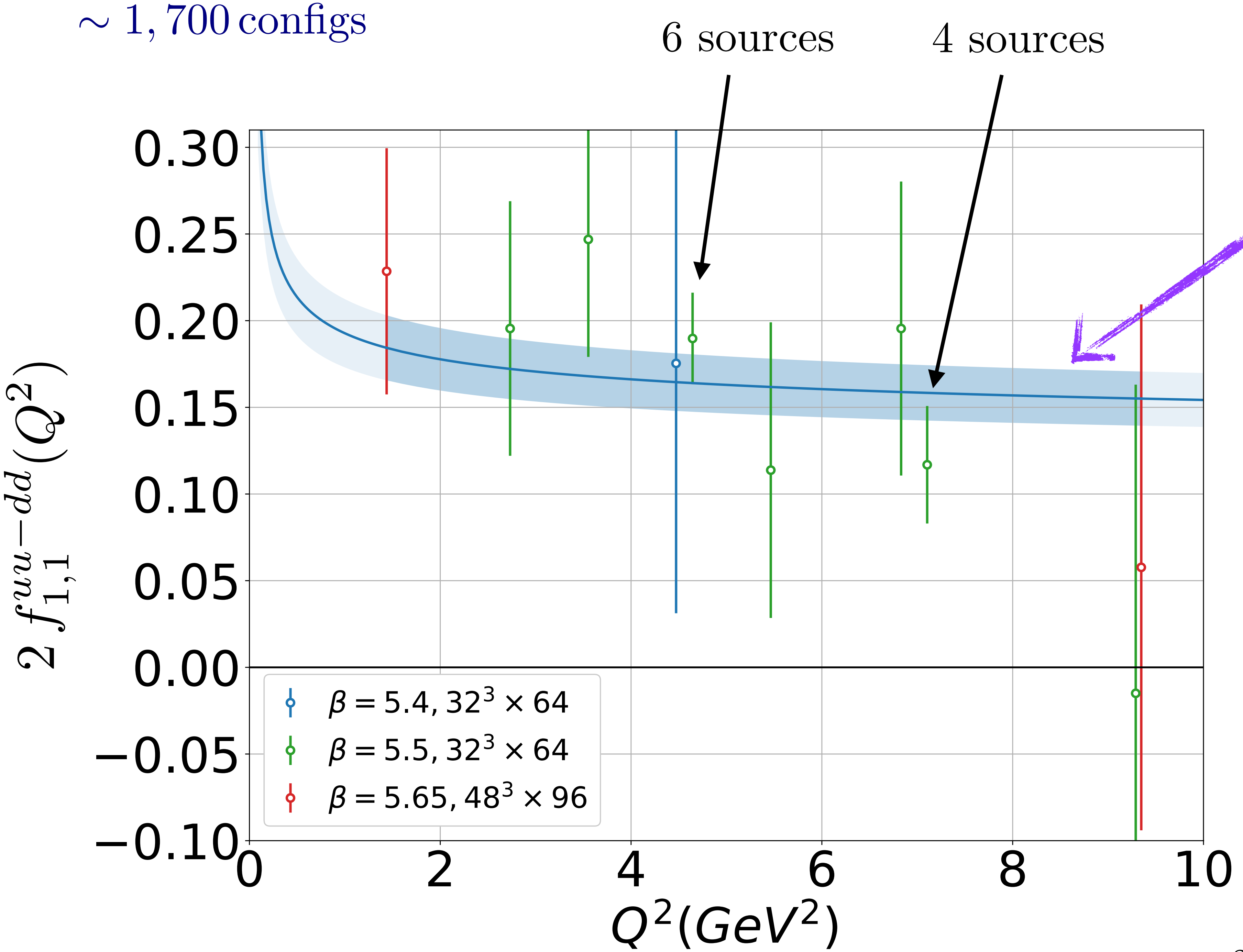
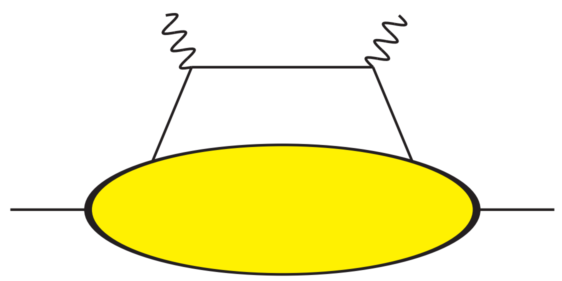



Compton amplitude

Lots of external momenta

$\sim 1,700$ configs



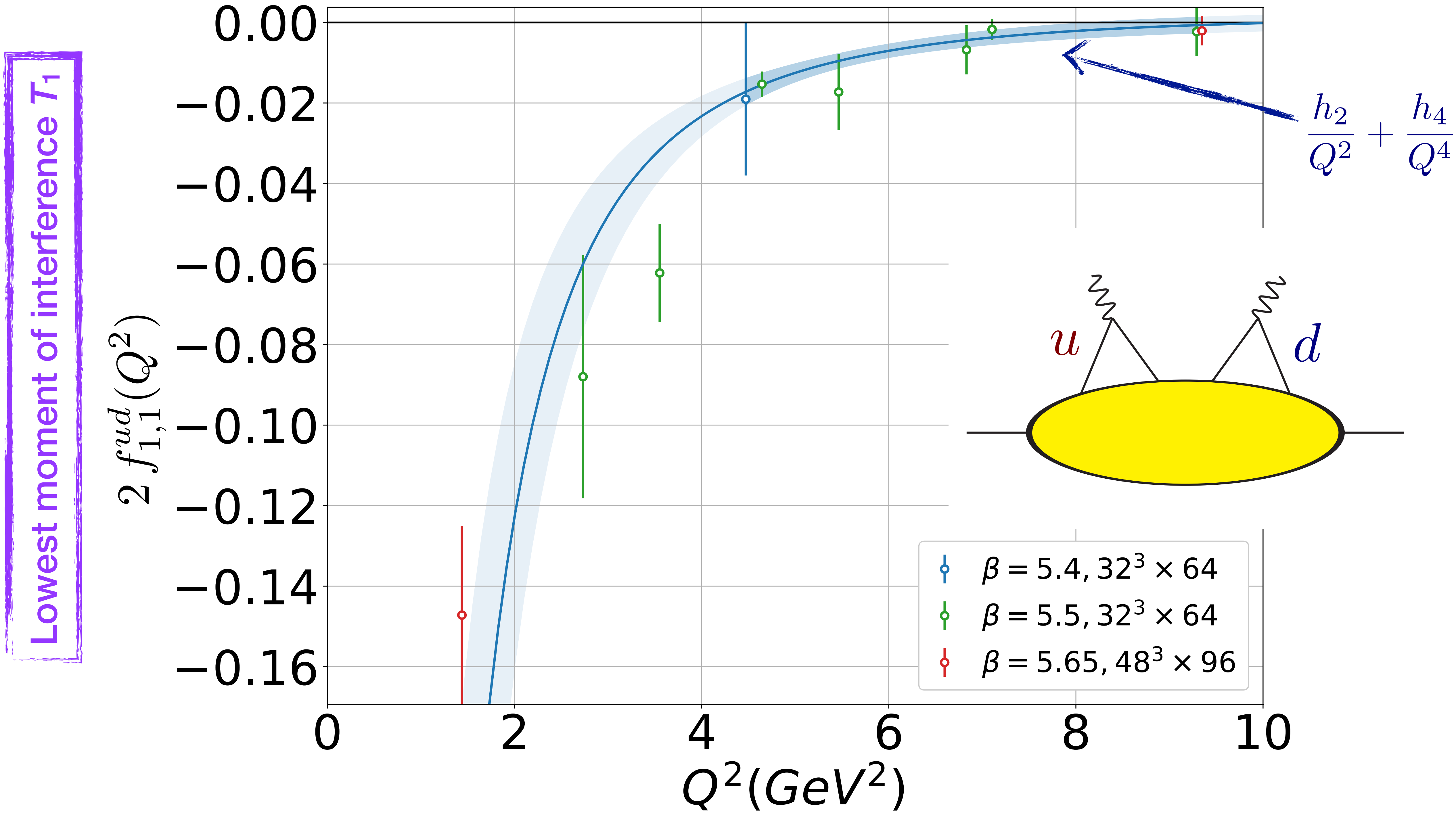



$$f_{1,1}(Q^2) \sim \frac{1}{2}\langle x \rangle\,(1 + \log)$$

$$T_1(\omega, Q^2) = \sum_{j=1}^{\infty} 4\omega^{2j}\, f_{1,2j-1}(Q^2)$$

# Scaling: Lowest moment

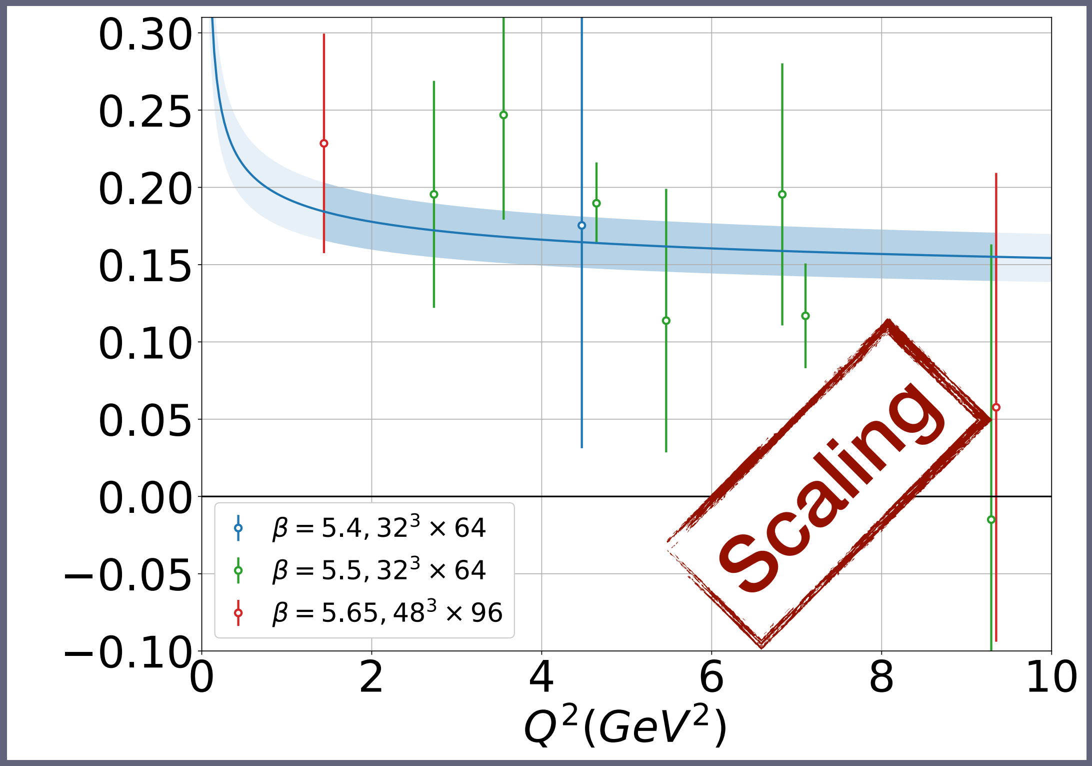
- Compatible with scaling
- Trend *not in*consistent with pQCD
- too early to assess higher twist…

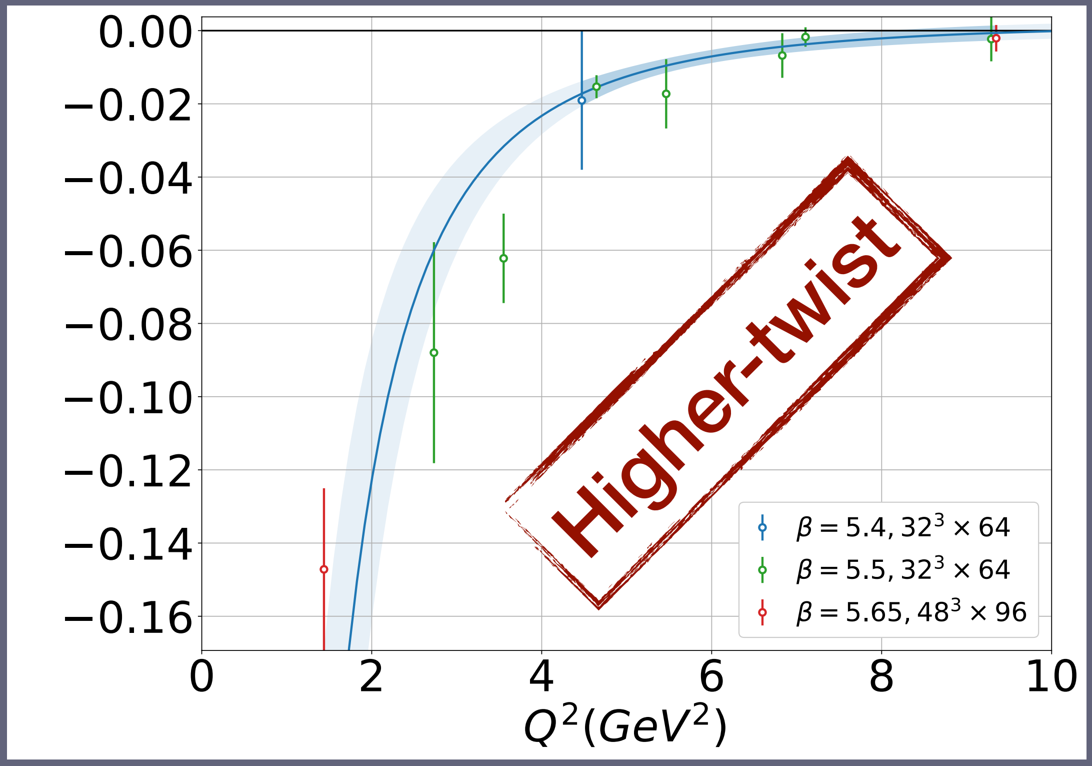



# *ud* Interference: Higher twist

Pure higher-twist effect

Vanishes asymptotically, as expected




# Compton outlook

**Opportunities**

- Scaling on the lattice
- New insight into twist expansion
- Study transition between resonant to shallow-inelastic scattering
- New tool to study spin-dependent Compton

…

Physical Compton amplitude, can vary kinematics directly

**Still some work to do:**

- “Inversion” problem: reconstructing PDFs
- Subtraction term not yet understood
- Understanding continuum limit
- Better statistics

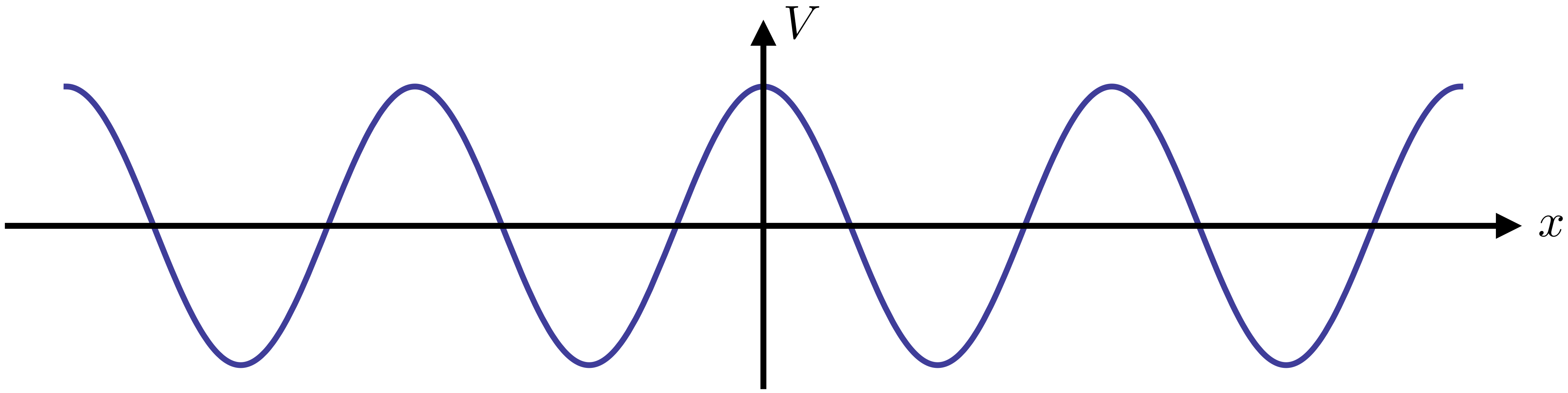
…

# Back-up slides

# Feynman–Hellman with momentum transfer

# Warm up: Periodic potential, 1-D QM

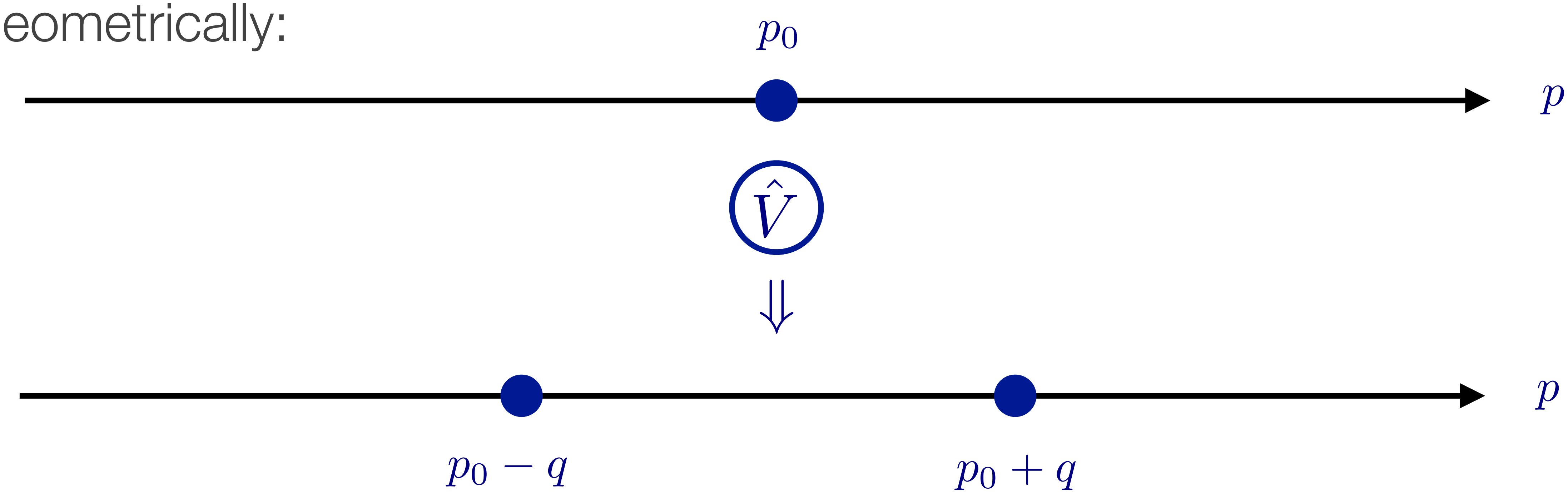
- Almost free particle $H_0|p\rangle = \frac{p^2}{2m}|p\rangle$
- Subject to weak external periodic potential $V(x) = 2\lambda V_0 \cos(qx)$




$$\hat{V}|p\rangle = \lambda V_0|p+q\rangle + \lambda V_0|p-q\rangle$$

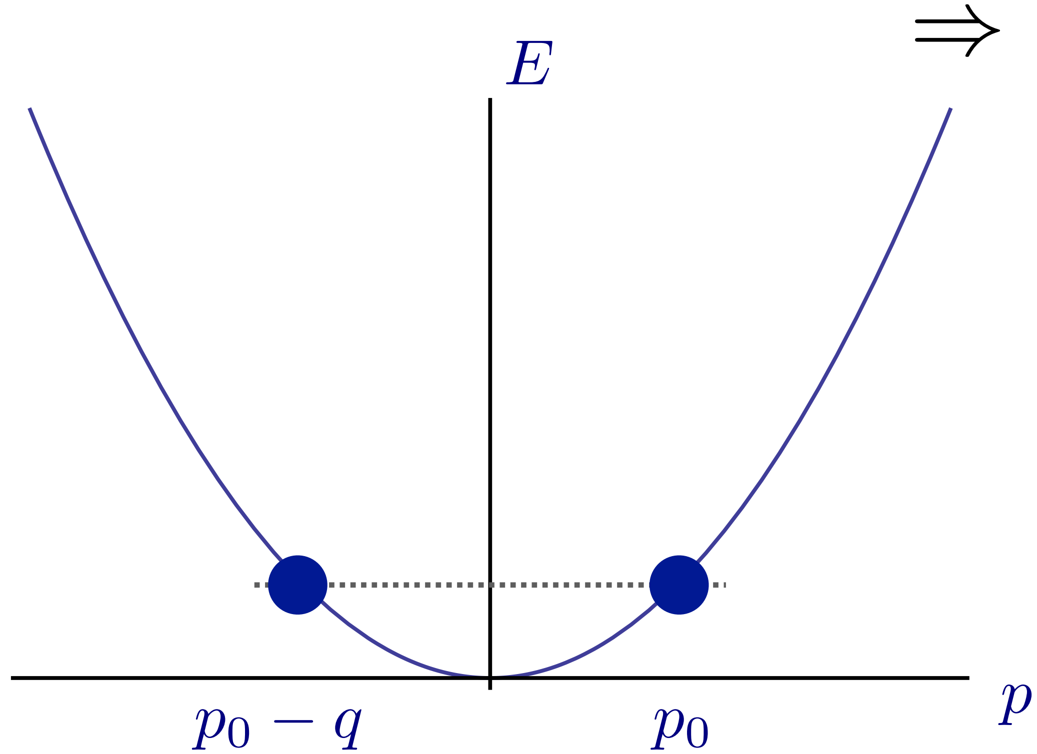
# Warm up: Periodic potential, 1-D QM

- Geometrically:




$$\Rightarrow \langle p|\hat{V}|p\rangle = 0$$

No first order energy shifts?




If $p_0 = \pm q/2$
$\Rightarrow$ transition between degenerate states

# Degenerate perturbation theory

- Exact degeneracy: $p = q/2$

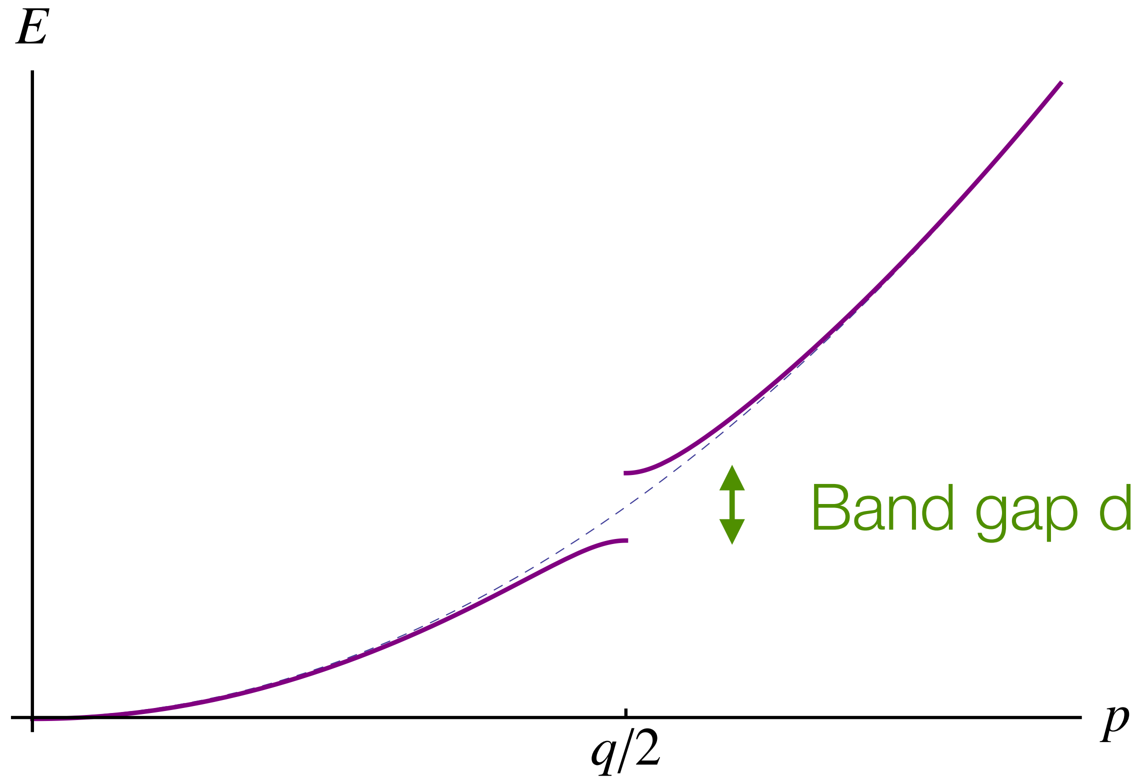
$$H = \begin{pmatrix} \frac{p^2}{2m} & \lambda V_0 \\ \lambda V_0 & \frac{p^2}{2m} \end{pmatrix} \qquad H\left\{|q/2\rangle \pm |-q/2\rangle\right\} = \left(E_{q/2} \pm \lambda V_0\right)\left\{|q/2\rangle \pm |-q/2\rangle\right\}$$

- Consider mixing on almost-degenerate states $p \sim q/2$

$$H = \begin{pmatrix} \frac{p^2}{2m} & \lambda V_0 \\ \lambda V_0 & \frac{(p-q)^2}{2m} \end{pmatrix}$$

Eigenvalues

$$\frac{p^2 + (p-q)^2}{4m} \pm \sqrt{\frac{q^2(q-2p)^2}{16m^2} + \lambda^2 V_0^2}$$




Band gap determined by strength of perturbation

$$\Delta E = 2\lambda V_0$$

# External momentum field on the lattice

- Modify Lagrangian with external field containing a spatial Fourier transform [constant in time]

$$\mathcal{L}(y) \to \mathcal{L}_0(y) + \lambda 2 \cos(\vec{q}.\vec{y})\overline{q}(y)\gamma_\mu q(y)$$

- Project onto "back-to-back" momentum state: $|\vec{q}/2\rangle + |-\vec{q}/2\rangle$

**"Breit frame" kinematics**

- E.g. pion form factor

$$\langle \pi(\vec{p}')|\overline{q}(0)\gamma_\mu q(0)|\pi(\vec{p})\rangle = (p+p')_\mu F_\pi(q^2)$$

- "Feynman-Hellmann"

$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \frac{(p+p')_\mu}{2E} F_\pi(q^2)$$

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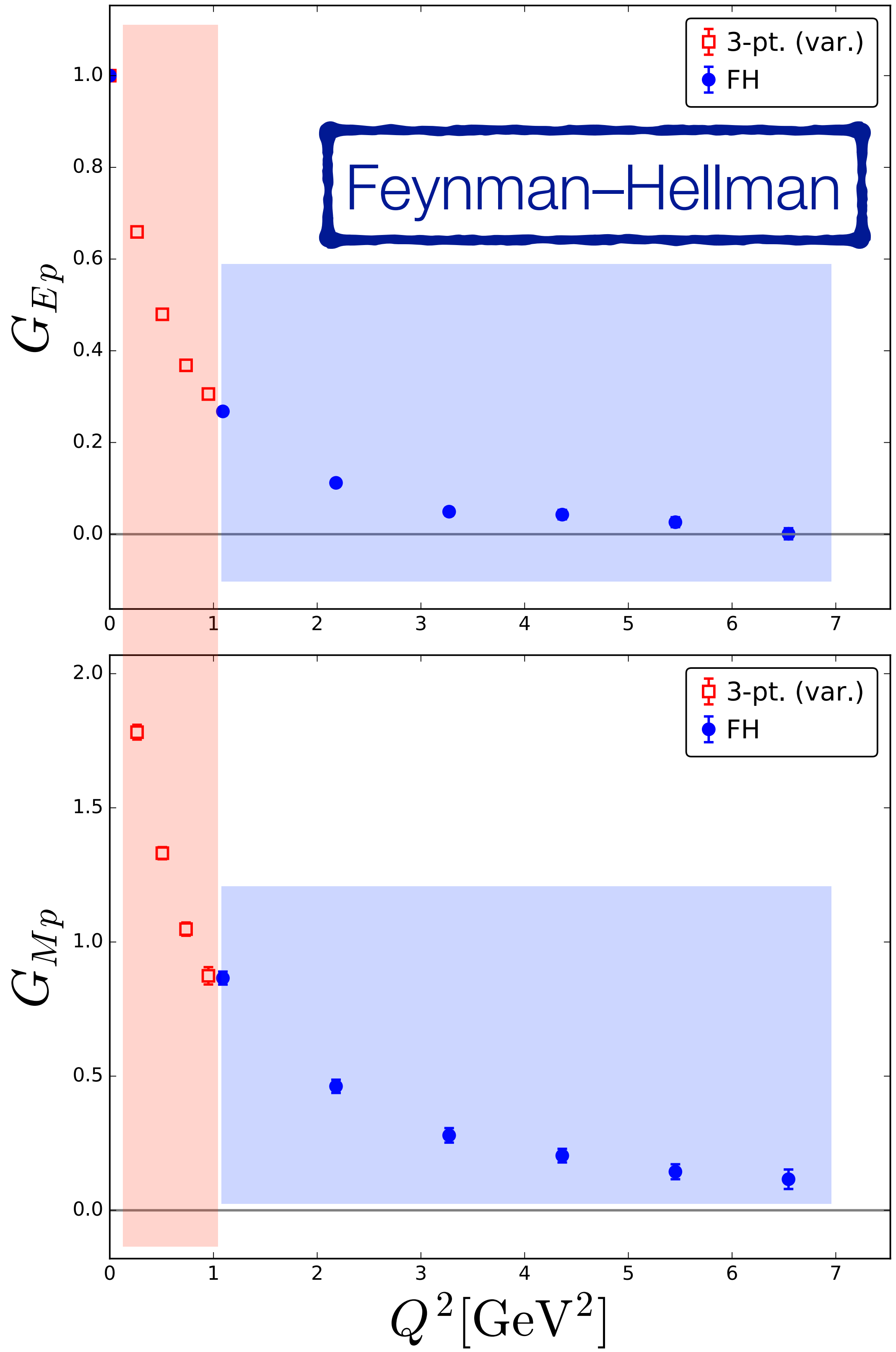
- "Feynman-Hellmann"

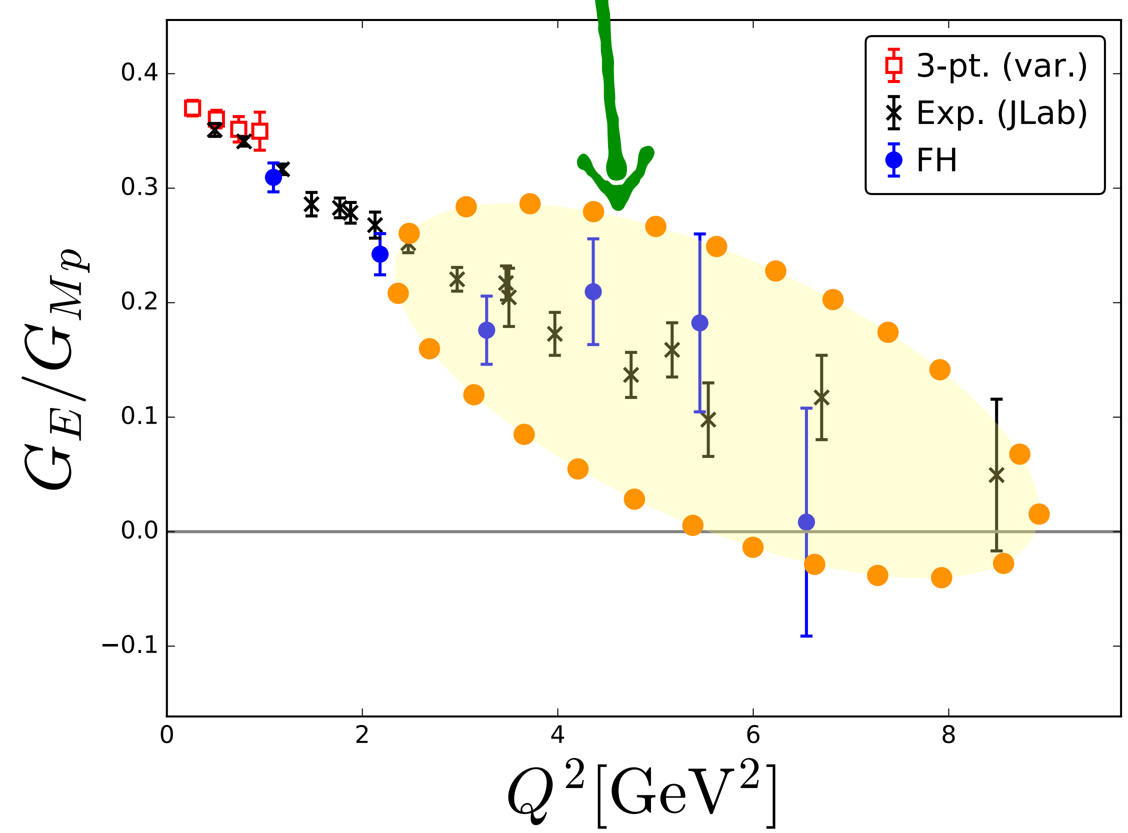
$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \frac{(p+p')_\mu}{2E} F_\pi(q^2) \quad \xrightarrow{\mu=4} \quad \left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = F_\pi(q^2)$$

# Proton Form Factors

3-pt functions




Phenomenologically-interesting region.
Domain dominated by model calculations…
previously prohibitive to study in lattice QCD.

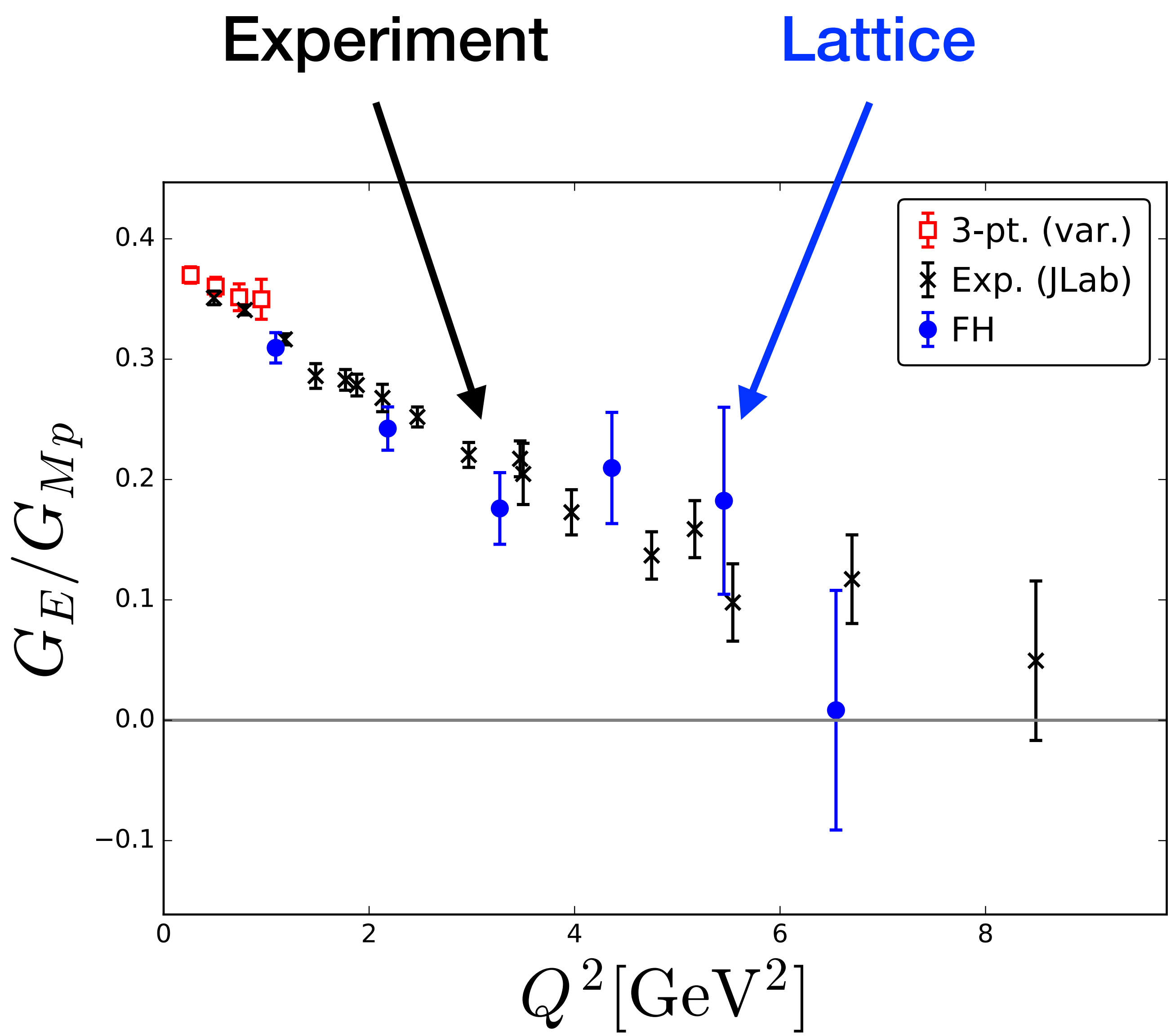



[Chambers *et al*. arXiv:1702.01513]

# Proton form factors

[my comments]

- One volume
  - Not worried (yet)
- One quark mass
  - Surprised that we see a similar trend as experiment
- One lattice spacing
  - We should investigate further




[Chambers *et al*. arXiv:1702.01513]

# Second-order “Feynman-Hellmann”
# (with external momentum)

# Feynman–Hellmann (2nd order)

- Two-point correlator

$$\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}} \frac{1}{\mathcal{Z}(\lambda)} \int \mathcal{D}\phi\, \chi(x)\chi^\dagger(0) e^{-S(\lambda)} = \sum_N \frac{|{}_\lambda\langle\Omega|\chi|N,\mathbf{p}\rangle_\lambda|^2}{2E_{N,\mathbf{p}}(\lambda)} e^{-E_{N,\mathbf{p}}(\lambda)x_0}$$

Integral over all fields

only interested in perturbative shift of ground-state energy

$$\simeq A_\mathbf{p}(\lambda) e^{-E_\mathbf{p}(\lambda)x_0}$$

"Momentum" quantum# at finite field

$$|N,\mathbf{p}\rangle_\lambda$$

$$\mathbf{p} \equiv \mathbf{p} + n\mathbf{q},\ n \in \mathbb{Z}$$

# Feynman–Hellmann (2nd order)

- Differentiate spectral sum

$$\frac{\partial}{\partial\lambda}\sum_N \frac{|{}_\lambda\langle\Omega|\chi|N,\mathbf{p}\rangle_\lambda|^2}{2E_N(\mathbf{p},\lambda)} e^{-E_{N,\mathbf{p}}(\lambda)x_4} = \sum_N \left[\frac{\partial A_{N,\mathbf{p}}(\lambda)}{\partial\lambda} - A_{N,\mathbf{p}}(\lambda)x_4\frac{\partial E_{N,\mathbf{p}}}{\partial\lambda}\right] e^{-E_{N,\mathbf{p}}(\lambda)x_4}$$

$$\to \left[\frac{\partial A_{\mathbf{p}}(\lambda)}{\partial\lambda} - A_{\mathbf{p}}(\lambda)x_4\frac{\partial E_{\mathbf{p}}}{\partial\lambda}\right] e^{-E_{\mathbf{p}}(\lambda)x_4}$$

- And again

**Not Breit frame, $\omega$<1 ⇒ 0**

$$\frac{\partial^2}{\partial\lambda^2}\left[\cdots\right] = \sum_N \left[\frac{\partial^2 A_{N,\mathbf{p}}(\lambda)}{\partial\lambda^2} - 2\frac{\partial A_{N,\mathbf{p}}(\lambda)}{\partial\lambda}x_4\frac{\partial E_{N,\mathbf{p}}(\lambda)}{\partial\lambda} - A_{N,\mathbf{p}}(\lambda)x_4\frac{\partial^2 E_{N,\mathbf{p}}(\lambda)}{\partial\lambda^2} + A_{N,\mathbf{p}}(\lambda)x_4^2\left(\frac{\partial E_{N,\mathbf{p}}(\lambda)}{\partial\lambda}\right)^2\right]$$

$$\to \left[\frac{\partial^2 A_{\mathbf{p}}(\lambda)}{\partial\lambda^2} - A_{\mathbf{p}}(\lambda)x_4\frac{\partial^2 E_{\mathbf{p}}}{\partial\lambda^2}\right] e^{-E_{\mathbf{p}}(\lambda)x_4}$$

**Quadratic energy shift**

Watch for temporal enhancement $\sim x_4\, e^{-E_{\mathbf{p}}x_4}$

# Feynman–Hellmann (2nd order)

- **Differentiate path integral**

$$\frac{\partial}{\partial\lambda}\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}}\frac{1}{\mathcal{Z}(\lambda)}\int\mathcal{D}\phi\,\chi(x)\chi^\dagger(0)e^{-S(\lambda)}$$

$$=\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}}\frac{1}{\mathcal{Z}(\lambda)}\int\mathcal{D}\phi\,\chi(x)\chi^\dagger(0)\left[-\frac{\partial S}{\partial\lambda}-\frac{1}{\mathcal{Z}(\lambda)}\frac{\partial\mathcal{Z}}{\partial\lambda}\right]e^{-S(\lambda)},$$

"Disconnected" operator insertions;
drop for simplicity

- Differentiate again, take zero-field limit and note: $\dfrac{\partial^2 S}{\partial\lambda^2}=0$

$$\frac{\partial^2}{\partial\lambda^2}[\cdots]\bigg|_{\lambda=0}=\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}}\frac{1}{\mathcal{Z}_0}\int\mathcal{D}\phi\,\chi(x)\chi^\dagger(0)\left(\frac{\partial S}{\partial\lambda}\right)^2 e^{-S_0}$$

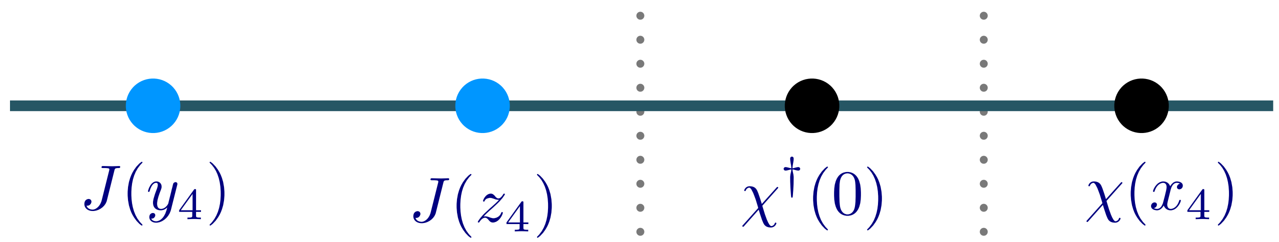
Current insertions integrated over 4-volume

$$\frac{\partial S}{\partial\lambda}=\int d^4y\,2\cos(\mathbf{q}.\mathbf{y})\overline{q}(y)\gamma_\mu q(y)$$

# Field time orderings

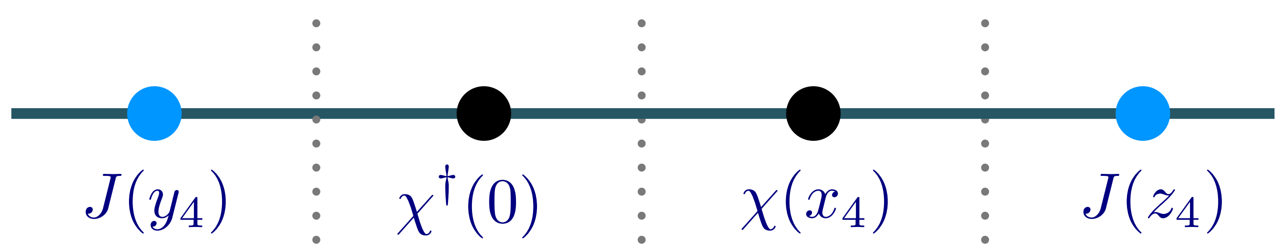
ignore finite T


- Current insertion possibilities
  - Both currents “outside” (together)




$$\langle \chi(x)\chi^\dagger(0)\mathrm{T}(J(y)J(z))\rangle, \quad y_4, z_4 < 0 < x_4$$
$$\sim e^{-E_X x_4}, \quad E_X \gtrsim E_\mathbf{p}$$

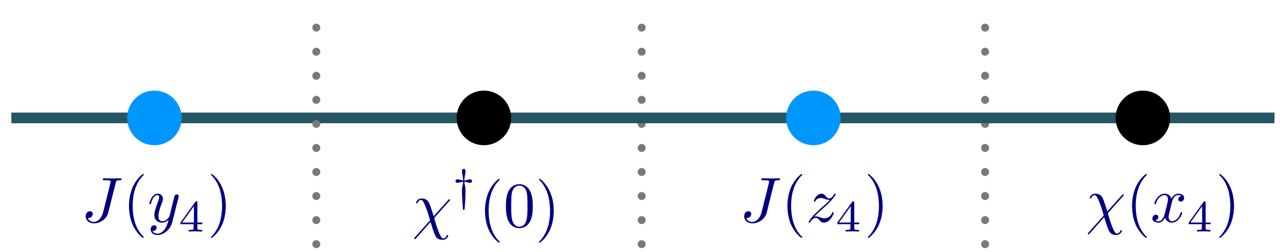
  - Both currents “outside” (opposite)




$$\langle J(z)\chi(x)\chi^\dagger(0)J(y)\rangle, \quad y_4 < 0 < x_4 < z_4$$
$$\sim e^{-E_X x_4}, \quad E_X \gtrsim E_\mathbf{p}$$

$E_X = E_\mathbf{p} \Rightarrow$ changes amplitudes

  - One current “inside”

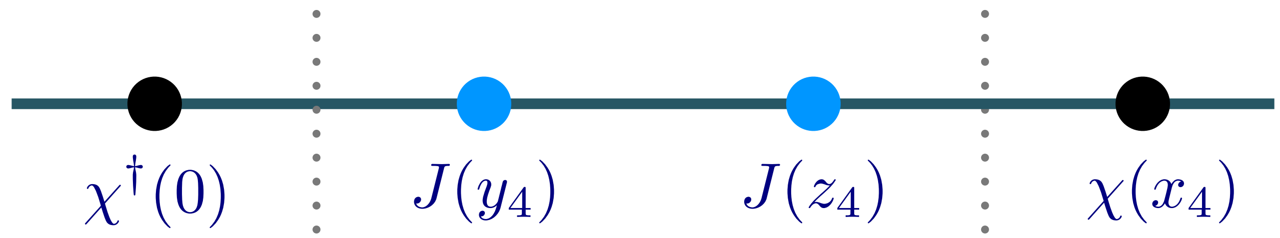



$$\langle \chi(x)J(z)\chi^\dagger(0)J(y)\rangle, \quad y_4 < 0 < z_4 < x_4$$
$$\sim \frac{\partial E_\mathbf{p}}{\partial \lambda} x_4\, e^{-E_\mathbf{p} x_4} \to 0$$

linear energy shift
(and changed amplitude)

# Field time orderings

- Both currents between creation/annihilation




$$\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}} \frac{1}{\mathcal{Z}_0} \int \mathcal{D}\phi\, \chi(x)\chi^\dagger(0) \left(\frac{\partial S}{\partial \lambda}\right)^2 e^{-S_0}$$

$$= \sum_{N,N'} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{N,\mathbf{k}}} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2E_{N',\mathbf{k}'}} \int d^3x \int d^4z \int d^4y\, e^{-i\mathbf{p}.\mathbf{x}} \left(e^{i\mathbf{q}.\mathbf{z}} + e^{-i\mathbf{q}.\mathbf{z}}\right) \left(e^{i\mathbf{q}.\mathbf{y}} + e^{-i\mathbf{q}.\mathbf{y}}\right)$$

$$\times \langle \Omega | \chi(x) | N, \mathbf{k} \rangle \langle \mathbf{k} | \mathrm{T} J(z) J(y) | \mathbf{k}' \rangle \langle N', \mathbf{k}' | \chi^\dagger(0) | \Omega \rangle,$$

$$\vdots$$

$$\rightarrow \frac{A_{\mathbf{p}}}{2E_{\mathbf{p}}} x_4 e^{-E_{\mathbf{p}} x_4} \int d^4\xi \left(e^{iq.\xi} + e^{-iq.\xi}\right) \langle \mathbf{p} | \mathrm{T} J(\xi) J(0) | \mathbf{p} \rangle$$

Note $q_4 = 0 \quad \Rightarrow \mathbf{q}.\boldsymbol{\xi} = q.\xi$

# Final steps

- Equate spectral sum and path integral representation
  - Asymptotically, we have

$$-A_{\mathbf{p}}\frac{\partial^2 E_{\mathbf{p}}}{\partial\lambda^2}x_4 e^{-E_{\mathbf{p}}x_4} = \frac{A_{\mathbf{p}}}{2E_{\mathbf{p}}}x_4 e^{-E_{\mathbf{p}}x_4}\int d^4\xi\,\left(e^{iq.\xi}+e^{-iq.\xi}\right)\langle\mathbf{p}|\mathrm{T}J(\xi)J(0)|\mathbf{p}\rangle$$

$$\frac{\partial^2 E_{\mathbf{p}}}{\partial\lambda^2} = -\frac{1}{2E_{\mathbf{p}}}\int d^4\xi\,\left(e^{iq.\xi}+e^{-iq.\xi}\right)\langle\mathbf{p}|\mathrm{T}J(\xi)J(0)|\mathbf{p}\rangle$$