Spin depolarization in high energy storage rings

Stochastic differential equations, reduced Bloch equation and invariant spin field approximation

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Ongoing projects

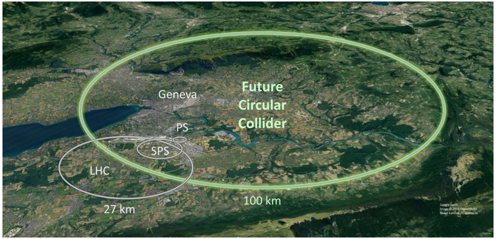
- Spin dynamics in high energy storage rings
  - James Ellison, Klaus Heinemann (UNM)
  - Desmond Barber (DESY,UNM)
- Spin tracking and spin matching software for electron storage rings. Polarization at ESR-EIC, —-in future FCC-ee and CEPC
  - James Ellison, Klaus Heinemann (UNM)
  - Desmond Barber (DESY,UNM)
  - Georg Hoffstaetter, David Segan (Cornell)
- High order Hermite particle-in-cell method
  - Daniel Appelo (MSU)
  - Allen Alvarez Loya (UC)

Outline

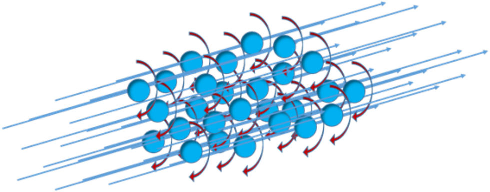
Spin dynamics in electron storage rings

- Spin polarized particle accelerators.
- Spin–orbit motion: SDEs and Bloch equation approach.
- Spin–orbit motion: perturbation approximation.
- Radiative single resonance model (rSRM).
- Numerical experiments.

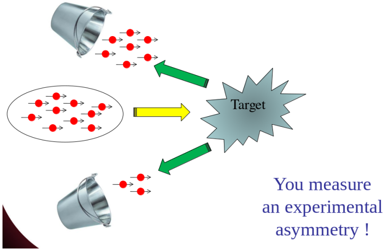
Future spin–polarized colliders



https://cds.cern.ch



Electrons (positrons) carry an intrinsic quantum angular momentum and we imagine that they spin around an axis.



- The polarization of a beam is the average of the spin vectors of the beam.
- Spin polarization determines the quality of a beam for conducting certain spin–sensitive collision experiments and for beam energy measurements.

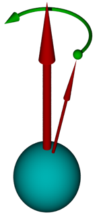
Main questions:

(Q1) Can one get high polarization?

(Q2) What are the theoretical limits of the polarization?

Spin–orbit dynamics

- The spin angular momentum vectors of electrons couple to the electric and magnetic fields in storage ring and this causes the spins to precess, i.e. to rotate according to Thomas-BMT equation.
- Electrons moving in the magnetic fields in storage rings emit streams of photons in the beam direction known as synchrotron radiation (modelled by adding noise and damping).
- Without the synchrotron radiation, the spin motion would be deterministic along trajectories.
- The photon emission also affects the spin motion and this non–precession change of spin can lead to the build–up of spin polarization due to asymmetric spin-flip rates (Sokolov–Ternov process).
- The depolarization can be viewed as a consequence of the fact that photon emission is stochastic and puts noise into the particle trajectories which then feeds through to the spin precession via the spin–orbit coupling embodied in the Thomas–BMT equation (spin diffusion).
- Problems at high energy arise because spin precession increases with energy.



Derbenev-Kondratenko formulas

Analytical estimates of the attainable equilibrium polarization are based on the Derbenev–Kondratenko (D–K) formalism*
The value of the equilibrium polarization in an $e^{\pm}$ storage ring is given by

$$P_{\rm dk} = \mp \frac{8}{5\sqrt{3}} \frac{\oint ds \langle \frac{1}{|\rho(s)|^3} \hat{b} \cdot (\hat{n} - \frac{\partial \hat{n}}{\partial \delta}) \rangle_s}{\oint ds \langle \frac{1}{|\rho(s)|^3} (1 - \frac{2}{9} (\hat{n} \cdot \hat{s})^2 + \frac{11}{18} \left(\frac{\partial \hat{n}}{\partial \delta} \right)^2) \rangle_s}$$

- $\langle\ \rangle_s$ denotes an average over phase space at azimuth s,m
- $\hat{s}$ is the direction of motion
- $\hat{b} = (\hat{s} \times \dot{\hat{s}})/|\dot{\hat{s}}|$
- $\hat{b}$ is the magnetic field direction
- $\hat{n}(s, y)$ is a unit 1–turn periodic 3–vector field over the phase space satisfying the Thomas–BMT equation along particle trajectories. It is called the *invariant spin field*, see later.

* Y. Derbenev, A. Kondratenko, Polarization kinetics of particles in storage rings, Sov. Phys.JETP 37 (1973) 968.

Invariant spin field

- The ISF is a key classifying geometric object.
- ISF is a *one turn periodic field* of unit vectors in the phase space at each azimuthal position on the ring, s.
- The unit vectors obey the T-BMT equation along particle trajectories.
- Once the ISF has been set up (e.g., by stroboscopic averaging) at some initial s_0, it can be set up all over phase space and for increasing s.
- The ISF was first introduced in the early 1970s by Derbenev and Kondratenko (DK) for calculations of electron polarisation and given the symbol n.
- We can write $n(s, x, p)$ where x and p are canonical positions and momenta.
- For protons at orbital equilibrium ISF dictates of the direction of $\vec{P}_{\rm loc}$, the local polarization, at each point in phase space.
- Note that DK did not emphasise the geometrical nature of n. Also, they only defined it for a time independent system but time dependence is essential for electrons since rf cavities are needed.
- Kaoru Yokoya showed how to include time dependence, and hence synchrotron oscillations in https://lib-extopc.kek.jp/preprints/PDF/1986/8607/8607334.pdf.
- It is important to understand that an ISF has no history! It just IS.

Phase space and polarization evolution equations

If the orbital phase space $\mathcal{P}_y$ density obeys an equation of the Fokker-Planck type

$$\partial_s \mathcal{P}_y = \mathcal{L}_{\mathrm{FP}} \mathcal{P}_y .$$

where $\mathcal{L}_{FP}$ is the Fokker-Planck operator.
How do we make a Fokker-Planck equation for spin? We need a density and we have an angular momemntum density.
Spin diffusion is described by the Bloch equation

$$\partial_s \vec{\eta} = \mathcal{L}_{\mathrm{FP}} \vec{\eta} + \vec{\Omega} \times \vec{\eta}.$$

where $\vec{\eta}(s, y)$ is the *polarization density*$\equiv 2/\hbar\times$(density in phase space per particle of spin angular momentum). In fact $\vec{\eta}(s, y) = \vec{P}_{\mathrm{loc}}(s, y)\mathcal{P}_y(s, y)$. This will be extended in the next slides.

Reduced Bloch equation

- The lab-frame full Bloch equation (Solov-Ternov is included) for the polarization density was derived in 1975 from semi-classical QED.
- The reduced beam-frame Bloch equation is obtained from lab-frame Bloch equation by neglecting Sokolov-Ternov effect and kinetic polarization effect.
- QoI: $\vec{\eta}(s, y)$ is the polarization density, proportional to angular momentum density.
- In the beam frame the reduced Bloch equation takes form

$$\partial_s \vec{\eta} = -\mathcal{D}_{\mathrm{drift}} \vec{\eta} + \frac{1}{2} \mathcal{D}_{\mathrm{diffusion}} \vec{\eta} + \Omega(s, y) \times \vec{\eta}.$$

$$\mathcal{D}_{\mathrm{drift}} \vec{\eta} := \begin{bmatrix} \nabla \cdot \mathcal{A}(s) y \eta_1 \\ \nabla \cdot \mathcal{A}(s) y \eta_2 \\ \nabla \cdot \mathcal{A}(s) y \eta_3 \end{bmatrix}, \quad \mathcal{D}_{\mathrm{diffusion}} := \nabla^{\mathrm{T}} B(s) B^{\mathrm{T}}(s) \nabla$$

- Here we see many difference scales:
  - $\mathcal{A}(s)$ incorporates the deterministic orbital dynamics, the Hamiltonian for the radiation-less orbital motion, guiding a rapidly varying orbital revolution process (rapid) and the radiation damping (slow).
  - $B(s)$ models the strengths for stochastic radiation.
  - Finally $\Omega(s, y)$ is the Thomas-BMT precession with variation rates dependent on the reference energy of the beam, and action of a particle.
- We can derive the reduced Bloch equation from the *stochastic differential equations*.

Linearized model in beam frame

Reduced stochastic differential equations

$$Y' = \mathcal{A}(s)Y + B(s)\xi(s), \qquad Y(0) = Y_0,$$

$$\vec{S}' = [\Omega_0(s) + \sum_{j=1}^{2d} \Omega_j(s) Y_j]\vec{S}, \qquad \vec{S}(0) = \vec{S}_0,$$

where $Y \in \mathbb{R}^{2d}, \vec{S} \in \mathbb{R}^3$, coefficients are 2π-periodic in s, $B(s) \in \mathbb{R}^{2d\times m}$ and ξ is vector white noise ($d = 1, 2$ or 3 is the number of degrees of freedom).

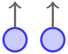
- Reduced SDEs ignore Sokolov–Ternov effect.
- Linearization in Y is simplest approximation which captures the main spin effects.
- Spin motion is non–linear.
- Numerical complexity is the same in one includes Sokolov–Ternov effect.

QoI: Beam-frame polarization vector

$$\vec{P}(s) := \langle \vec{S}(s) \rangle = \int_{\mathbb{R}^{2d}} \left[\int_{\mathbb{R}^3} \vec{s}\, \mathcal{P}_{ys}(s, y, \vec{s})\ d\vec{s} \right] dy$$

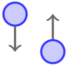
$\mathcal{P}_{ys}$ is the joint probability density.

Example 1: two particles with equal spins



$\vec{P} = \frac{1}{2}((1,0,0) + (1,0,0)) = (1,0,0)$ 100% polarization

Example 2: two particles with opposite spins



$\vec{P} = \frac{1}{2}((-1,0,0) + (1,0,0)) = (0,0,0)$ 0% polarization

Polarization density

The polarization density $\vec{\eta}$ is given by

$$\vec{\eta}(s, y) = \left[\int_{\mathbb{R}^3} \vec{s}\, \mathcal{P}_{ys}(s, y, \vec{s})\ ds \right] \propto \text{spin angular momentum density.}$$

Reduced Bloch equation (RBE) for polarization density

Splitting $\mathcal{A}(s) = A(s) + \delta A(s)$, the earlier PDE for $\vec{\eta}$ can be written as

$$\partial_s \vec{\eta} = -\sum_{j=1}^{2d} \partial_{y_j} \left(([A(s) + \delta A(s)]y)_j \, \vec{\eta} \right) + \frac{1}{2} \sum_{j,k=1}^{2d} \left[B(s) B^{\mathrm{T}}(s) \right]_{j,k} \partial^2_{y_j y_k} \vec{\eta}(s,y) + \Omega(s,y)\vec{\eta}. \quad (1)$$

$$\vec{\eta}(0,y) = \vec{g}(y),$$

$$\lim_{y \to \infty} \vec{\eta}(s,y) e^{\alpha |y|^2} = 0, \quad \forall s \geq 0, \quad \text{for some } \alpha > 0.$$

- Components of $\vec{\eta}$ are coupled by Thomas–BMT precession matrix $\Omega(s,y)$. And that in fact causes the depolarization.
- If $d = 3$ (three degrees of freedom spin–orbit motion, 6D phase space), numerical solutions to the RBE hard to obtain (in general).

Radiation as perturbation

- To consider the synchrotron radiation as a perturbation the main assumption is that the effect of radiation is small.
- The unperturbed problem is the same as one that models the spin–orbit motion of heavy particles, like protons where orbital motion yields Hamiltonian dynamics.

Perturbed problem

$$Y' = [A(s) + \delta A(s)]Y + B(s)\xi(s),$$

$$\vec{S}' = [\Omega_0(s) + \sum_{j=1}^{2d} \Omega_j(s)Y_j]\vec{S}$$

Unperturbed problem

$$Y' = [A(s)]Y$$

$$\vec{S}' = [\Omega_0(s) + \sum_{j=1}^{2d} \Omega_j(s)Y_j]\vec{S}$$

ISF-approximation of polarization density

$$\partial_s \vec{\eta} = -\sum_{j=1}^{2d} \partial_{y_j} \left(([A(s) + \delta A(s)]y)_j \, \vec{\eta} \right) + \frac{1}{2} \sum_{j,k=1}^{2d} \left[B(s) B^{\mathrm{T}}(s) \right]_{j,k} \partial^2_{y_j y_k} \vec{\eta}(s,y) + \Omega(s,y)\vec{\eta}. \quad (2)$$

Invariant spin field (ISF)

In fact the ISF $\hat{n}(s,y)$ is the periodic solution of (2), without radiation damping and diffusion, such that $|\hat{n}| \equiv 1$.

$$\partial_s \hat{n} = -\sum_{j=1}^{2d} \left([A(s)y]_j \partial_{y_j} \hat{n} \right) + \left[\Omega_0(s) + \sum_{j=1}^{2d} \Omega_j(s) y_j \right] \hat{n}$$

- In spirit of DK, we look for approximate solution of RBE in the form

  $$\vec{\eta}(s,y) \approx \vec{\eta}_{\mathrm{DK}}(s,y) := P_{\mathrm{DK}}(s) \mathcal{P}_{\mathrm{eq}}(s,y) \hat{n}(s,y),$$

  where $\mathcal{P}_{\mathrm{eq}}$ is the equilibrium orbital probability density function.
- The residual $\vec{r}(s,y)$ of the approximation of $\vec{\eta}_{\mathrm{DK}}$ w.r.t. the Bloch equation is

  $$\vec{r}(s,y) := \partial_s \vec{\eta}_{\mathrm{DK}} - L_{\mathrm{Bloch}} \vec{\eta}_{\mathrm{DK}}.$$

- $\vec{\eta}_{\mathrm{DK}}$ points into the direction of $\hat{n}$, thus would hope for $\vec{r}(s,y) \cdot \hat{n}(s,y) = 0$, but this condition is too strong!

ISF-approximation. Residual condition

Lets try a weaker condition:

Problem

For $\vec{\eta}_{\mathrm{DK}}(s,y) = P_{\mathrm{DK}}(s)\mathcal{P}_{\mathrm{eq}}(s,y)\hat{n}(s,y)$, find $P_{\mathrm{DK}}(s)$ such that

$$\int_{\mathbb{R}^{2d}} \vec{r}(s,y)\cdot\hat{n}(s,y)\;dy = \int_{\mathbb{R}^{2d}} (\partial_s\vec{\eta}_{\mathrm{DK}} - L_{\mathrm{Bloch}}\vec{\eta}_{\mathrm{DK}})\cdot\hat{n}(s,y)\;dy = 0.$$

Theorem (generalization of the DK formula)

P_{DK} satisfies the first-order ODE

$$P'_{\mathrm{DK}} = -\frac{\tau^{-1}_{\mathrm{dep}}(s)}{c}P_{\mathrm{DK}},\;\; \tau^{-1}_{\mathrm{dep}}(s) = \frac{c}{2}\sum_{j,k=1}^{2d}\int_{\mathbb{R}^{2d}}\left[B(s)B^{\mathrm{T}}(s)\right]_{j,k}\mathcal{P}_{\mathrm{eq}}(s,y)[\partial_{y_j}\hat{n}(s,y)]\cdot[\partial_{y_k}\hat{n}(s,y)]\;dy,$$

if and only if the residual $\vec{r}$ satisfies the condition

$$\int_{\mathbb{R}^n} \vec{r}(s,y)\cdot\hat{n}(s,y)\;dy = 0.$$

Reproducing the DK formula for the depolarization time

Assume that the stochastic radiation only contributes in the sixth component of the orbital vector, $y_6 = \delta$ and it has a strength*

$$B^{\mathrm{T}}(s)B(s) = \frac{55}{24\sqrt{3}} r_e \lambda\!\!\!^{-}_e \gamma_0^5 \frac{1}{\rho(s)^3}$$

then we obtain a simple formula for a beam at equilibrium

$$P_{\mathrm{DK}}(s) = P_0 e^{-\frac{1}{c}\int_0^s \tau_{\mathrm{dep}}^{-1}(s')\ ds'},$$

$$\tau_{\mathrm{dep}}^{-1}(s) = \frac{55}{48\sqrt{3}} r_e \lambda\!\!\!^{-} \gamma_0^5 c \int_{\mathbb{R}^6} \frac{1}{\rho^3(s)} \mathcal{P}_{\mathrm{eq}}(s,y) \left| \frac{\partial \hat{n}}{\partial \delta}(s,y) \right|^2$$

$$\implies \overline{\tau_{\mathrm{dep}}^{-1}(s)} \approx \frac{5\sqrt{3}}{8} \frac{r_e \hbar \gamma_0^5}{m_e} \frac{1}{C} \oint_s \left\langle \frac{\frac{11}{18}\left| \frac{\partial \hat{n}}{\partial \delta}(s,y) \right|^2}{\rho^3(s)} \right\rangle_s ds$$

∗ H. Mais, G. Ripken, Theory spin-orbit motion in electron-positron storage rings. Summary of results, Tech. Rep. DESY 83-062, DESY

Single resonance model (SRM)

Consider the single harmonic vertical orbital motion.
We write the spin motion for the vertical dipole and the radial quadruple field with strength g.
The BMT equation takes a simplest form

$$\vec{S}' = (\vec{\Omega}_0 + \vec{\Omega}_1 \sqrt{J}\cos(\sqrt{g}s + \chi_0)) \times \vec{S},$$

where

$$\vec{\Omega}_0 = a\gamma_0 \hat{e}_y, \quad \vec{\Omega}_1 = (1 + a\gamma_0) g \hat{e}_x$$

Invoke the rotating wave approximation

$$\begin{aligned}(1 + a\gamma_0) g \sqrt{J}\cos(\sqrt{g}s + \chi_0)\hat{e}_x =& \frac{(1 + a\gamma_0) g}{2}\left(\sqrt{J}\cos(\sqrt{g}s + \chi_0)\hat{e}_x + \sqrt{J}\sin(\sqrt{g}s + \chi_0)\hat{e}_s\right) \\ &+ \frac{(1 + a\gamma_0) g}{2}\left(\sqrt{J}\cos(\sqrt{g}s + \chi_0)\hat{e}_x - \sqrt{J}\sin(\sqrt{g}s + \chi_0)\hat{e}_s\right)\end{aligned}$$

We now pick the component rotating in the same dirrection as a “tip” of a vector precessing around the vertical axis at rate via Ω_0.

rSRM: adding the radiation terms into SRM

Recognising $\sqrt{J}\cos(\sqrt{g}s+\chi_0)$ as an orbit we generalize a new $\vec{\Omega}$ from SRM as a field in y

$$\vec{\Omega}_r(s,y) = a\gamma_0 e_y + \frac{(1+a\gamma_0)g}{2}\left(y_1\hat{e}_x + y_2\hat{e}_s\right).$$

Next couple the spin motion to an orbital motion governed by an SDE derived using dispesion formalism

Radiative SRM

$$Y' = \begin{pmatrix} 0 & -\sqrt{g} \\ \sqrt{g} & -2\alpha \end{pmatrix} Y + D_3\sqrt{\frac{55}{24\sqrt{3}} r_e \bar{\lambda}\gamma_0^5 K_x^3}\,\xi(s),$$

$$\vec{S}' = \vec{\Omega}_r(s,Y)\times\vec{S}.$$

Y_1 – vertical position, $Y_2 = -P_y/\sqrt{g}$, α – damping constant, D_3 – average dispersion in the ring, $\xi(s)$ – scalar white noise.

Roughly speaking, we have reused the SRM “optics” designed for a non-radiative beam to guide the spin motion of radiative particles.
rSRM carries the useful properties of SRM to the electrons: it has known ISF!

rSRM: ISF and ISF- approximation

ISF for rSRM is known analytically

$$\hat{n}(y) = \frac{1}{\sqrt{\zeta^2 + |y|^2}}(y_1, y_2, \zeta)^{\mathrm{T}}, \quad \zeta = \frac{2(a\gamma_0 - \sqrt{g})}{(1 + a\gamma_0)g}$$

Here we use the generalized DK formula(noise is in Y_1). The depolarization time can be calculated

$$\tau_{\mathrm{dep}}^{-1}(s) = \frac{55}{48\sqrt{3}} r_e \bar{\lambda} \gamma_0^5 K_x^3 \frac{c}{C} \int_{\mathbb{R}^2} \mathcal{P}_{\mathrm{eq}}(y) \frac{y_2^2 + \zeta^2}{(y_1^2 + y_2^2 + \zeta^2)^2}$$

where equilibrium orbital density is

$$\mathcal{P}_{\mathrm{eq}}(y) = \frac{1}{2\pi\varepsilon} e^{-\frac{1}{2\varepsilon}((y_1 - \frac{2\alpha}{\sqrt{g}} y)^2 + y_2^2)}$$

Write polarization density $\vec{\eta}$ as

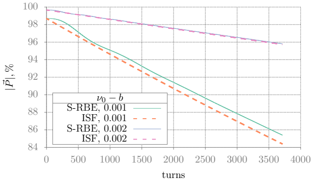
$$\vec{\eta}(s, y) = P_{\mathrm{DK}}(s) \mathcal{P}_{\mathrm{eq}}(y) \hat{n}(y) + \Delta\vec{\eta}(s, y),$$

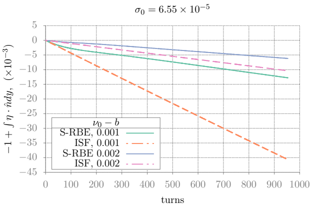
Simulation of the polarization density via the reduced Bloch equation

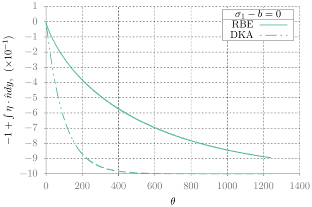
rSRM, as well as more general models can be averaged to an effective model with the Bloch equation that can be simulated numerically.
Here we have two simulations demonstrating the devastating effect of spin-orbit resonance.

Verifying ISF-approximation

Spin–orbit resonance occurs when the rates of spin precession and orbital revolution are commensurable It causes a massive depolarization.
For high energy machine (in this example we use parameters from HERA), we verify ISF-approximation using the numerical simulations of the RBE. Three test cases: when the system is far from resonance, close to resonance and exactly at resonance.







Far from the resonance the ISF-approximation is accurate.

Close to the resonance the ISF-approximation overestimates the polarization decay.

At the resonance the ISF-approximation shows much stronger polarization decay compared to the RBE simulation.

Numerical results

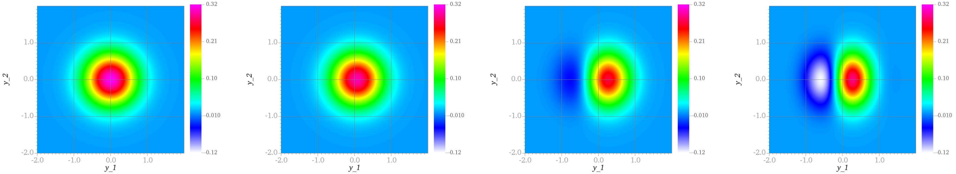
Effective Bloch equation in 1 DOF

$$\partial_s \eta = \varepsilon\left(\partial_{y_1}(y_1\eta) + \partial_{y_2}(y_2\eta)\right) + \frac{\varepsilon}{4}\nabla^2\eta - \varepsilon g y_1 J_2 \eta - \frac{\varepsilon}{2} g J_2 \partial_{y_1}\eta - \frac{\varepsilon}{4} g^2 \eta, \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

Here we consider 2–dimensional spin and 2–dimensional phase space, thus $\eta \in \mathbb{R}^2$, $y \in \mathbb{R}^2$. The exact solution is known. Polarization density approaches nonzero equilibrium.

$$\eta(0,y) = \frac{2}{\pi}\begin{pmatrix} \cos(\psi_0) \\ \sin(\psi_0) \end{pmatrix} e^{-2(y_1^2+y_2^2)}, \quad \eta_{\mathrm{eq}}(y) = \frac{2}{\pi} e^{-\frac{g^2}{8}} \begin{pmatrix} \cos(\psi_0 - g y_1) \\ \sin(\psi_0 - g y_1) \end{pmatrix} e^{-2(y_1^2+y_2^2)}.$$

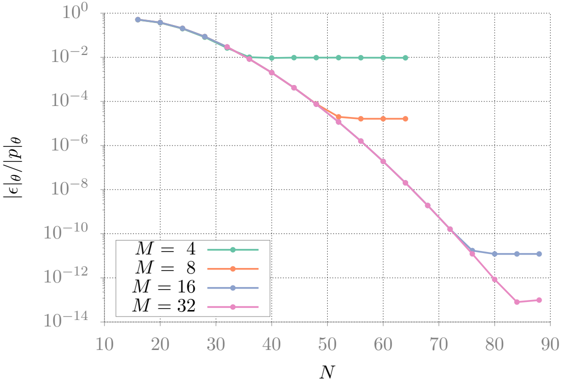
Simulation results for η_1, $s = 0, 10, 100, 600$



Convergence test

The error for the numerical solution $s = t_\nu$ is (recall, p approximates $u = r^2\eta$)

$$\epsilon(t_\nu, r, \phi) := \frac{1}{r^2} p(t_\nu, r, \phi) - \eta(t_\nu, r\cos\phi, r\sin\phi),$$



- N is the order of radial discretization, M is the order of Fourier discretization.
- Measure the error size on a fine $N_0 \times M_0$ grid

$$|\epsilon|_s := \left(\sum_{i=1}^{N_0} \sum_{j=1}^{M_0} |\epsilon(s, r_i, \phi_j)|^2 \right)^{\frac{1}{2}}$$

Summary

- DK formulas can be generalized and put on the solid ground via SDE and Bloch equation analysis.
- Using the SDE framework we formalized the known approaches and developed new approaches to study depolarizaiton in electron storage rings.
- We developed the highly efficient numerical method for solving the equations posed in up to 6 spatial dimensions + time.
- Using this method we studied the resonance behavior of the simple models.
- Our study can be extended to realistic polarized particle accelerator via the method of averaging.

References

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Thank you!