Three-loop corrections to the quark and gluon decomposition of the QCD trace anomaly and their applications

Kazuhiro Tanaka (Juntendo U)

KT, JHEP1901, 120 & work in progress

(Belinfante-improved) energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{2}\overline{q}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)} q \; + \; F^{\mu\rho} F_{\rho}{}^{\nu} + \frac{g^{\mu\nu}}{4} F^2 \; + \; (\text{ghost}) \; + \; (\text{gauge fix})$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{q}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)} q} + \boxed{F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{g^{\mu\nu}}{4}F^2} \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

trace anomaly

Nielsen, NPB120, 212 ('77)
Collins, Duncan, Joglekar, PRD16, 438 ('77)

$$g_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1 + \gamma_m(g)\right)m\bar{q}q$$

$$\left(\beta(g) \equiv \mu\frac{dg}{d\mu},\ \gamma_m(g) = -\frac{\mu}{m}\frac{dm}{d\mu}\right)$$

classically, $g_{\mu\nu}T_q^{\mu\nu} = m\bar{q}q,\quad g_{\mu\nu}T_g^{\mu\nu} = 0$

does not coincide with quantum corr. to $m\bar{q}q$

quantum loop corr. & taking trace do not commute

$\partial_\mu T^{\mu\nu} = 0$ **implies total** $T^{\mu\nu}$ **is not renormalized**

each of $T_q^{\mu\nu}$, $T_g^{\mu\nu}$ **is renormalized**

➡ trace anomaly separately for q, g

1&2-loop Hatta, Rajan, KT, JHEP1812, 008

3-loop (& all orders) KT, JHEP1901, 120

trace anomaly separately for q, g

$\overline{\text{MS}}$ scheme

$$g_{\mu\nu} T_q^{\mu\nu} = m\bar{q}q + \frac{\alpha_s}{4\pi}\left(\frac{4}{3} C_F m\bar{q}q + \frac{1}{3} n_f F^2\right)$$

$$g_{\mu\nu} T_g^{\mu\nu} = \frac{\alpha_s}{4\pi}\left(\frac{14}{3} C_F m\bar{q}q - \frac{11}{6} C_A F^2\right)$$

$$g_{\mu\nu} T^{\mu\nu} = \frac{\beta(g)}{2g} F^2 + \left(1 + \gamma_m(g)\right) m\bar{q}q$$

$$C_A = N_c \ , \qquad C_F = \frac{N_c^2 - 1}{2N_c}$$

trace anomaly separately for q, g

$\overline{\text{MS}}$ scheme

$$g_{\mu\nu}T_q^{\mu\nu} = m\bar{q}q + \frac{\alpha_s}{4\pi}\Big(\frac{4}{3}C_F m\bar{q}q + \frac{1}{3}n_f F^2\Big) + \Big(\frac{\alpha_s}{4\pi}\Big)^2\Big[\Big(C_F\Big(\frac{61C_A}{27} - \frac{68n_f}{27}\Big) - \frac{4C_F^2}{27}\Big)m\bar{q}q + \Big(\frac{17C_A n_f}{27} + \frac{49C_F n_f}{54}\Big)F^2\Big]$$

$$+\Big(\frac{\alpha_s}{4\pi}\Big)^3\Big[\Big\{n_f\Big(\Big(\frac{64\zeta(3)}{9} - \frac{8305}{729}\Big)C_F^2 - \frac{2}{243}(864\zeta(3)+1079)C_A C_F\Big) - \frac{8}{729}(972\zeta(3)+143)C_A C_F^2$$

$$+\Big(\frac{32\zeta(3)}{9} + \frac{6611}{729}\Big)C_A^2 C_F - \frac{76}{243}C_F n_f^2 + \frac{8}{729}(648\zeta(3)-125)C_F^3\Big\}m\bar{q}q$$

$$+\Big\{n_f\Big(\Big(\frac{52\zeta(3)}{9} - \frac{401}{324}\Big)C_A C_F + \Big(\frac{134}{27} - 4\zeta(3)\Big)C_A^2 + \Big(\frac{2407}{1458} - \frac{16\zeta(3)}{9}\Big)C_F^2\Big) + n_f^2\Big(-\frac{697C_A}{729} - \frac{169C_F}{1458}\Big)\Big\}F^2\Big]$$

$$g_{\mu\nu}T_g^{\mu\nu} = \frac{\alpha_s}{4\pi}\Big(\frac{14}{3}C_F m\bar{q}q - \frac{11}{6}C_A F^2\Big) + \Big(\frac{\alpha_s}{4\pi}\Big)^2\Big[\Big(C_F\Big(\frac{812C_A}{27} - \frac{22n_f}{27}\Big) + \frac{85C_F^2}{27}\Big)m\bar{q}q + \Big(\frac{28C_A n_f}{27} - \frac{17C_A^2}{3} + \frac{5C_F n_f}{54}\Big)F^2\Big]$$

$$+\Big(\frac{\alpha_s}{4\pi}\Big)^3\Big[\Big\{n_f\Big(\Big(\frac{368\zeta(3)}{9} - \frac{25229}{729}\Big)C_F^2 - \frac{2}{243}(4968\zeta(3)+1423)C_A C_F\Big) + \Big(\frac{32\zeta(3)}{3} - \frac{91753}{1458}\Big)C_A C_F^2$$

$$+\Big(\frac{294929}{1458} - \frac{32\zeta(3)}{9}\Big)C_A^2 C_F - \frac{554}{243}C_F n_f^2 + \Big(\frac{95041}{729} - \frac{64\zeta(3)}{9}\Big)C_F^3\Big\}m\bar{q}q$$

$$+\Big\{n_f\Big(\Big(\frac{1123}{162} - \frac{52\zeta(3)}{9}\Big)C_A C_F + \Big(4\zeta(3) + \frac{293}{36}\Big)C_A^2 + \frac{16}{729}(81\zeta(3)-98)C_F^2\Big) + n_f^2\Big(\frac{655C_A}{2916} - \frac{361C_F}{729}\Big) - \frac{2857C_A^3}{108}\Big\}F^2\Big]$$

$$g_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \big(1+\gamma_m(g)\big)m\bar{q}q \qquad C_A = N_c\,, \quad C_F = \frac{N_c^2-1}{2N_c}$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{q}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)} q} + \boxed{F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{g^{\mu\nu}}{4}F^2} \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$O_q = i\bar{q}\gamma^{(\mu}\overleftrightarrow{D}^{\nu)}q\,, \qquad O_{q(4)} = g^{\mu\nu} m\bar{q}q\,, \qquad O_g = F^{\mu\rho}F_{\rho}{}^{\nu}\,, \qquad O_{g(4)} = g^{\mu\nu}F^2$$

$$T_q^{\mu\nu} = O_q \quad , \qquad T_g^{\mu\nu} = O_g + \frac{O_{g(4)}}{4}$$

$$O_q^R = Z_\psi O_q + Z_K O_{q(4)} + Z_Q O_g + Z_B O_{g(4)}$$

$$O_g^R = Z_L O_q + Z_S O_{q(4)} + Z_T O_g + Z_M O_{g(4)}$$

$$O_{g(4)}^R = Z_F O_{g(4)} + Z_C O_{q(4)}$$

$$O_{q(4)}^R = O_{q(4)}$$

$$T^{\mu\nu} = \frac{1}{2}\bar{q}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}q + F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{g^{\mu\nu}}{4}F^2 \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$O_q = i\bar{q}\gamma^{(\mu}\overleftrightarrow{D}^{\nu)}q\,, \quad O_{q(4)} = g^{\mu\nu}m\bar{q}q\,, \quad O_g = F^{\mu\rho}F_{\rho}{}^{\nu}\,, \quad O_{g(4)} = g^{\mu\nu}F^2$$

$$T_q^{\mu\nu} = O_q\,, \quad T_g^{\mu\nu} = O_g + \frac{O_{g(4)}}{4}$$

subtracting traces:

$$O^R_{q(2)} = Z_\psi O_{q(2)} + Z_Q O_{g(2)}$$

$$O^R_{g(2)} = Z_L O_{q(2)} + Z_T O_{g(2)}$$

renorm. constants can be determined by DGLAP splitting fn. up to 3-loop

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{q}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)} q} + \boxed{F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{g^{\mu\nu}}{4}F^2} \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$O_q = i\bar{q}\gamma^{(\mu}\overleftrightarrow{D}^{\nu)}q\,, \qquad O_{q(4)} = g^{\mu\nu}m\bar{q}q\,, \qquad O_g = F^{\mu\rho}F_{\rho}{}^{\nu}\,, \qquad O_{g(4)} = g^{\mu\nu}F^2$$

$$T_q^{\mu\nu} = O_q \quad , \qquad T_g^{\mu\nu} = O_g + \frac{O_{g(4)}}{4}$$

$$O_q^R = Z_\psi O_q + Z_K O_{q(4)} + Z_Q O_g + Z_B O_{g(4)}$$

$$O_g^R = Z_L O_q + Z_S O_{q(4)} + Z_T O_g + Z_M O_{g(4)}$$

$$O_{g(4)}^R = Z_F O_{g(4)} + Z_C O_{q(4)}$$

$$O_{q(4)}^R = O_{q(4)}$$

Z_F, Z_C are obtained by Feynman diagram calculation up to 2-loop

Tarrach, NPB196, 45 ('82)

Z_F, Z_C can be determined up to 4-loop by RG-invariance of total anomaly

$$g_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{q}q$$

KT, JHEP1901, 120

$$T^{\mu\nu} = \frac{1}{2}\bar{q}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)} q + F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{g^{\mu\nu}}{4}F^2 \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$O_q = i\bar{q}\gamma^{(\mu}\overleftrightarrow{D}^{\nu)}q\,, \qquad O_{q(4)} = g^{\mu\nu}m\bar{q}q\,, \qquad O_g = F^{\mu\rho}F_{\rho}{}^{\nu}\,, \qquad O_{g(4)} = g^{\mu\nu}F^2$$

$$T_q^{\mu\nu} = O_q \quad , \quad T_g^{\mu\nu} = O_g + \frac{O_{g(4)}}{4}$$

taking trace parts:

$$O_q^R = Z_\psi O_q + Z_K O_{q(4)} + Z_Q O_g + Z_B O_{g(4)}$$

$$O_g^R = Z_L O_q + Z_S O_{q(4)} + Z_T O_g + Z_M O_{g(4)}$$

$$O_{g(4)}^R = Z_F O_{g(4)} + Z_C O_{q(4)}$$

$$O_{q(4)}^R = O_{q(4)}$$

$$Z_X = \frac{a_X}{\varepsilon} + \frac{b_X}{\varepsilon^2} + \frac{c_X}{\varepsilon^3} + \cdots$$

$$X = K, B, S, M \qquad d = 4 - 2\varepsilon$$

trace anomaly separately for q, g

$\overline{\text{MS}}$ scheme

$$g_{\mu\nu}T_q^{\mu\nu} = m\bar{q}q + \frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{q}q + \frac{1}{3}n_f F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{61C_A}{27} - \frac{68n_f}{27}\right) - \frac{4C_F^2}{27}\right)m\bar{q}q + \left(\frac{17C_A n_f}{27} + \frac{49C_F n_f}{54}\right)F^2\right]$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^3\left[\left\{n_f\left(\left(\frac{64\zeta(3)}{9} - \frac{8305}{729}\right)C_F^2 - \frac{2}{243}(864\zeta(3)+1079)C_A C_F\right) - \frac{8}{729}(972\zeta(3)+143)C_A C_F^2\right.\right.$$
$$\left.+\left(\frac{32\zeta(3)}{9} + \frac{6611}{729}\right)C_A^2 C_F - \frac{76}{243}C_F n_f^2 + \frac{8}{729}(648\zeta(3)-125)C_F^3\right\}m\bar{q}q$$
$$\left.+\left\{n_f\left(\left(\frac{52\zeta(3)}{9} - \frac{401}{324}\right)C_A C_F + \left(\frac{134}{27} - 4\zeta(3)\right)C_A^2 + \left(\frac{2407}{1458} - \frac{16\zeta(3)}{9}\right)C_F^2\right) + n_f^2\left(-\frac{697C_A}{729} - \frac{169C_F}{1458}\right)\right\}F^2\right]$$

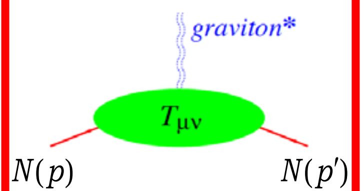
$$g_{\mu\nu}T_g^{\mu\nu} = \frac{\alpha_s}{4\pi}\left(\frac{14}{3}C_F m\bar{q}q - \frac{11}{6}C_A F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{812C_A}{27} - \frac{22n_f}{27}\right) + \frac{85C_F^2}{27}\right)m\bar{q}q + \left(\frac{28C_A n_f}{27} - \frac{17C_A^2}{3} + \frac{5C_F n_f}{54}\right)F^2\right]$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^3\left[\left\{n_f\left(\left(\frac{368\zeta(3)}{9} - \frac{25229}{729}\right)C_F^2 - \frac{2}{243}(4968\zeta(3)+1423)C_A C_F\right) + \left(\frac{32\zeta(3)}{3} - \frac{91753}{1458}\right)C_A C_F^2\right.\right.$$
$$\left.+\left(\frac{294929}{1458} - \frac{32\zeta(3)}{9}\right)C_A^2 C_F - \frac{554}{243}C_F n_f^2 + \left(\frac{95041}{729} - \frac{64\zeta(3)}{9}\right)C_F^3\right\}m\bar{q}q$$
$$\left.+\left\{n_f\left(\left(\frac{1123}{162} - \frac{52\zeta(3)}{9}\right)C_A C_F + \left(4\zeta(3) + \frac{293}{36}\right)C_A^2 + \frac{16}{729}(81\zeta(3)-98)C_F^2\right) + n_f^2\left(\frac{655C_A}{2916} - \frac{361C_F}{729}\right) - \frac{2857C_A^3}{108}\right\}F^2\right]$$

$$g_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{q}q$$

$$\langle p | T^{\mu\nu} | p\rangle = 2p^\mu p^\nu$$

1. hadron mass formula

$$2M^2 = g_{\mu\nu}\langle p \,|\, T_q^{\mu\nu} \,|\, p\rangle + g_{\mu\nu}\langle p \,|\, T_g^{\mu\nu} \,|\, p\rangle$$



$$T^{\mu\nu} = \frac{1}{2}\bar{q}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}q + F^{\mu\rho}F_{\rho}^{\ \nu} + \frac{g^{\mu\nu}}{4}F^2 \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$\langle N(p')|T_{q,g}^{\mu\nu}|N(p)\rangle = \bar{u}(p')\Big[A_{q,g}(t)\gamma^{(\mu}P^{\nu)} + B_{q,g}(t)\frac{P^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D_{q,g}(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} + \bar{C}_{q,g}(t)Mg^{\mu\nu}\Big]u(p)$$

$$P = \frac{p+p'}{2}$$

$$\Delta = p' - p$$

$$t = \Delta^2$$

$$A_q(0) + A_g(0) = 1$$

$$\langle N(p)|T^{\mu\nu}|N(p)\rangle = 2p^\mu p^\nu$$

$$\frac{\langle N(p)\,S|J^i|N(p)\,S\rangle}{\langle N(p)\,S|N(p)\,S\rangle} = \frac{1}{2}S^i$$

$$\frac{1}{2}\Big(A_q(0) + B_q(0) + A_g(0) + B_g(0)\Big) = \frac{1}{2}$$

$$B_q(0) + B_g(0) = 0$$

$$J^i = \frac{1}{2}\epsilon^{ijk}\int d^3x M^{+jk}$$

$$M^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

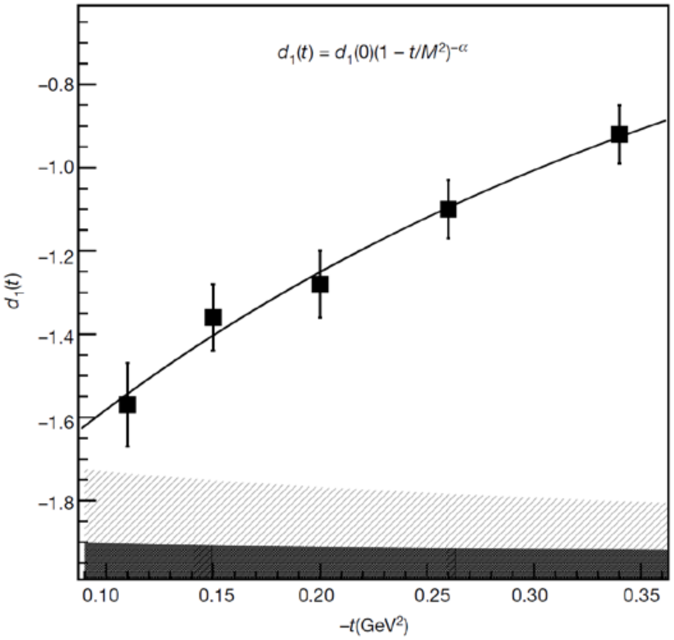
$$\bar{C}_q(t) + \bar{C}_g(t) = 0$$

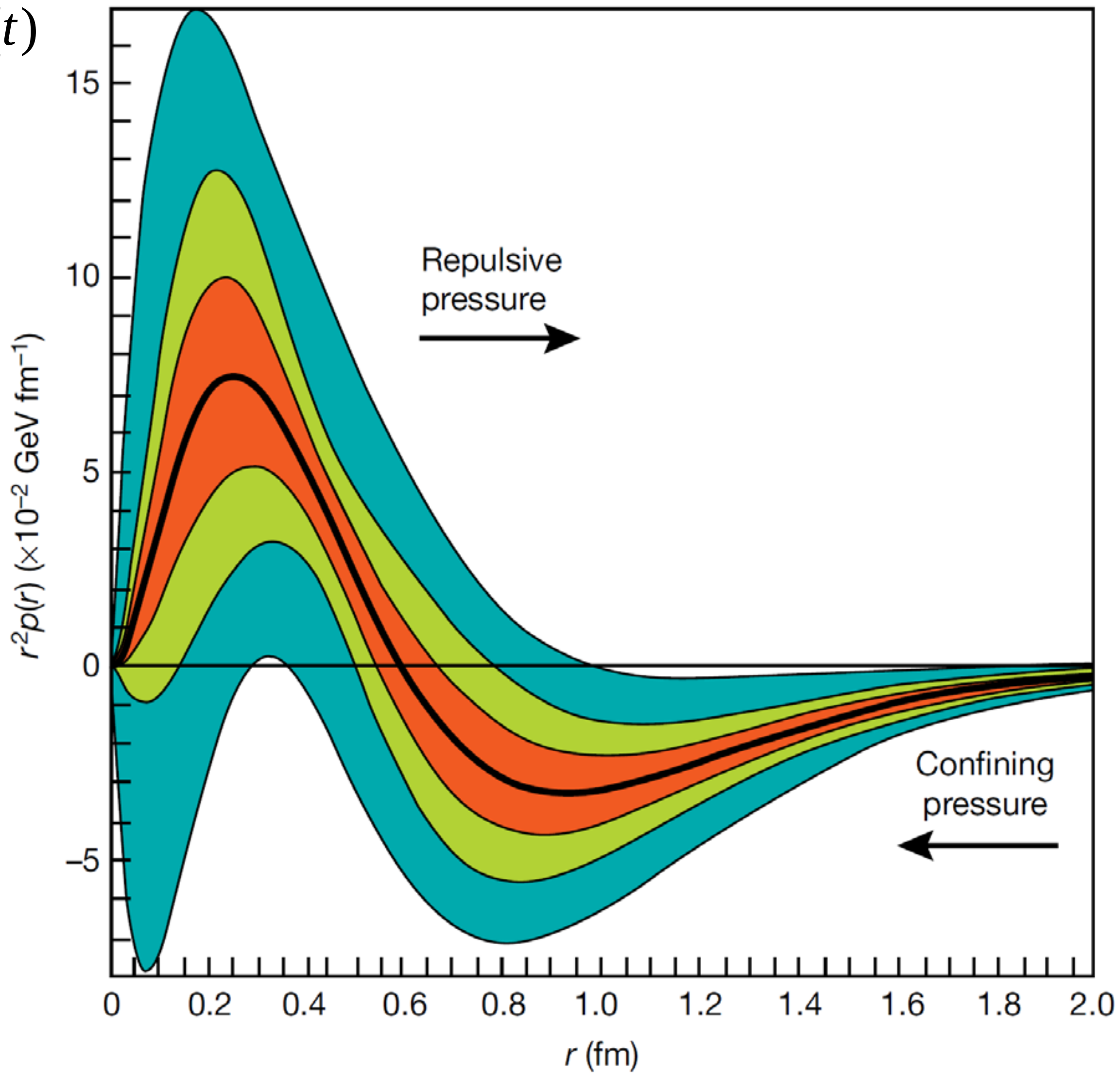
$$\partial_\mu T^{\mu\nu} = 0$$

V. D. Burkert et al, Nature 557 ('18) 396

$$\int_{-1}^{1} dx\, x H^q(x,\eta,t) = A_q(t) + 4\eta^2 D_q(t)$$

$$5\sum_q e_q^2 D_q(t) \sim d_1(t)$$



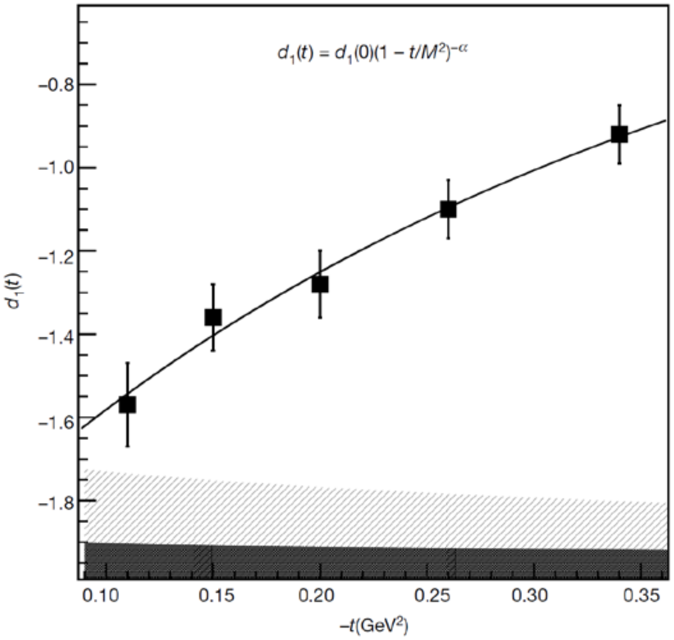


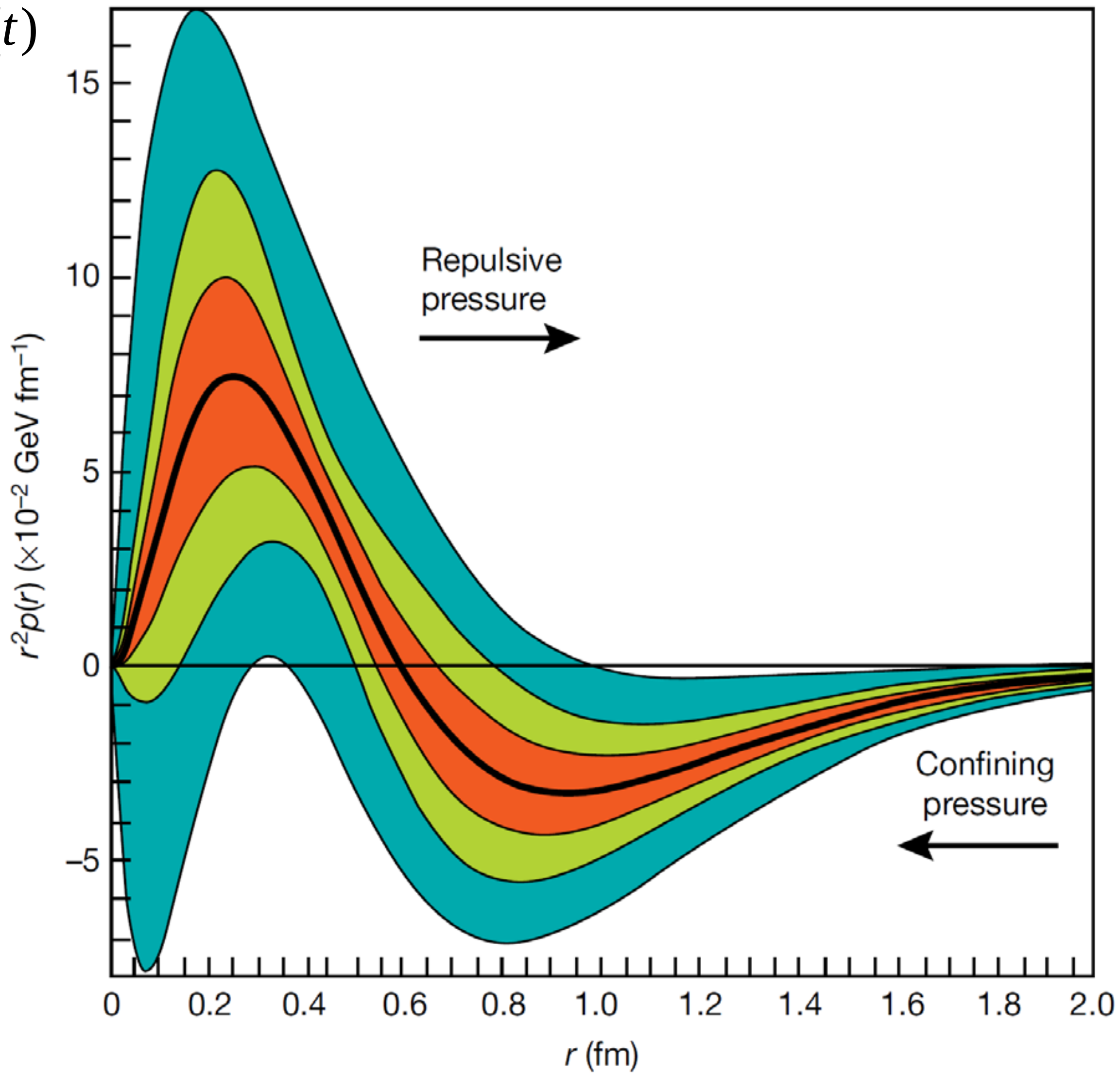
$$\langle N(p')| T^{ik}(0) | N(p)\rangle \sim \left(\Delta^i \Delta^k - \delta^{ik} \mathbf{\Delta}^2\right) D(t), \quad T^{ij}(r) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\right) s(r) + \delta^{ij}\, p(r)$$

V. D. Burkert et al, Nature 557 ('18) 396

$$\int_{-1}^{1} dx\, x H^q(x,\eta,t) = A_q(t) + 4\eta^2 D_q(t)$$

$$5\sum_q e_q^2 D_q(t) \sim d_1(t)$$





$$\langle N(p') | T_q^{ik}(0) | N(p) \rangle \sim \left(\Delta^i \Delta^k - \delta^{ik} \boldsymbol{\Delta}^2\right) D_q(t) - 4M^2 \delta^{ik} \bar{C}_q(t)$$

$\left(i\not{D}-m\right)q=0$ $\quad D_\nu F^{\mu\nu}=g\bar{q}\gamma^\mu q$

KT, PRD98, 034009

$$\partial_\nu T_q^{\mu\nu}=-\bar{q}gF^{\mu\nu}\gamma_\nu q,\qquad \partial_\nu T_g^{\mu\nu}=-F_a^{\mu\nu}D_{ab}^{\rho}F_{\rho\nu}^{b}$$

$$\Delta^\mu\Lambda^2\bar{C}_q(t)=\langle h(p')|\bar{q}igF^{\mu\nu}\gamma_\nu q|h(p)\rangle$$

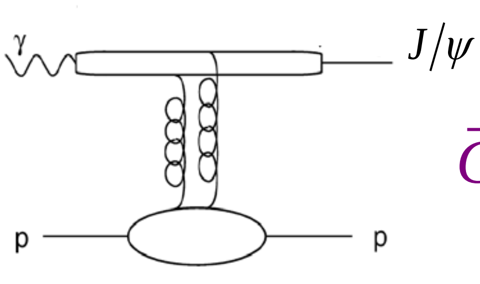
$$\Delta^\mu\Lambda^2\bar{C}_g(t)=\langle h(p')|F_a^{\mu\nu}iD_{ab}^{\rho}F_{\rho\nu}^{b}|h(p)\rangle$$

nucleon's transverse spin sum rule:

Hatta, KT, Yoshida, JHEP1302, 003

$$J_{q,g}=\frac{1}{2}(A_{q,g}+B_{q,g})+\frac{p^3}{2(p^0+M)}\bar{C}_{q,g}$$

$\gamma p\to J/\psi\ p$ **near threshold** JLab, EIC



$\bar{C}_g\ \left(=-\bar{C}_q\right)$

Y. Hatta, D. Yang, PR98, 074003
Y. Hatta, A. Rajan, D. Yang, PRD100, 014032

1. **hadron mass formula**

$$2M^2 = g_{\mu\nu}\langle p|T_q^{\mu\nu}|p\rangle + g_{\mu\nu}\langle p|T_g^{\mu\nu}|p\rangle$$

2. **gravitational form factor** **twist-four**

$$\bar{C}_{q/g}(t)$$

$$2M^2 = \langle N(p)|\left(\frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{q}q\right)|N(p)\rangle \simeq \langle N(p)|\frac{\beta(g)}{2g}F^2|N(p)\rangle$$

$$2M^2 = g_{\mu\nu}\langle N(p)|T_q^{\mu\nu}|N(p)\rangle + g_{\mu\nu}\langle N(p)|T_g^{\mu\nu}|N(p)\rangle$$

1-loop

$$\frac{\alpha_s}{4\pi}\frac{n_f}{3}F^2 \qquad \frac{\alpha_s}{4\pi}\left(-\frac{11C_A}{6}F^2\right)$$

2-loop

$$\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{17C_An_f}{27}+\frac{49C_Fn_f}{54}\right)F^2 \qquad \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{28C_An_f}{27}-\frac{17C_A^2}{3}+\frac{5C_Fn_f}{54}\right)F^2$$

3-loop

$$\left(\frac{\alpha_s}{4\pi}\right)^3\left[\left\{n_f\left(\frac{52\zeta(3)}{9}-\frac{401}{324}\right)C_AC_F + \left(\frac{134}{27}-4\zeta(3)\right)C_A^2 + \left(\frac{2407}{1458}-\frac{16\zeta(3)}{9}\right)C_F^2\right\} + n_f^2\left(-\frac{697C_A}{729}-\frac{169C_F}{1458}\right)\right]F^2$$

$$\left(\frac{\alpha_s}{4\pi}\right)^3\left[n_f\left(\left(\frac{1123}{162}-\frac{52\zeta(3)}{9}\right)C_AC_F + \left(4\zeta(3)+\frac{293}{36}\right)C_A^2 + \frac{16}{729}(81\zeta(3)-98)C_F^2\right) + n_f^2\left(\frac{655C_A}{2916}-\frac{361C_F}{729}\right) - \frac{2857C_A^3}{108}\right]F^2$$

$$2m_\pi^2 = \langle \pi | \left(\frac{\beta(g)}{2g} F^2 + \left(1+\gamma_m(g)\right) m\bar{q}q \right) | \pi \rangle$$

$$2 f_\pi^2 m_\pi^2 = -\left(m_u + m_d\right) \langle 0 | \left(\bar{u}u + \bar{d}d\right) | 0 \rangle$$

$$\left|\pi\right\rangle \to \left|\pi\right\rangle_0 + \left|\pi\right\rangle_1 + \cdots$$

$$m_\pi^2 = {}_0\left\langle \pi \left| m\bar{q}q \right| \pi \right\rangle_0$$

Gasser, Leutwyler ('82)

$${}_0\left\langle \pi \left| F^2 \right| \pi \right\rangle_0 = 0, \qquad \left(1-\gamma_m(g)\right) m_\pi^2 = {}_1\left\langle \pi \left| \frac{\beta(g)}{2g} F^2 \right| \pi \right\rangle_0 + {}_0\left\langle \pi \left| \frac{\beta(g)}{2g} F^2 \right| \pi \right\rangle_1$$

$$\gamma_m(g) = 0.63662\alpha_s + 0.768352\alpha_s^2 + 0.801141\alpha_s^3 \simeq 0.559$$

$$g_{\mu\nu}T_q^{\mu\nu} = m\bar{q}q + \frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{q}q + \frac{1}{3}n_f F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{61C_A}{27} - \frac{68n_f}{27}\right) - \frac{4C_F^2}{27}\right)m\bar{q}q + \left(\frac{17C_A n_f}{27} + \frac{49C_F n_f}{54}\right)F^2\right]$$

$$+\left(\frac{\alpha_s}{4\pi}\right)^3\left[\left\{n_f\left(\left(\frac{64\zeta(3)}{9} - \frac{8305}{729}\right)C_F^2 - \frac{2}{243}(864\zeta(3)+1079)C_AC_F\right) - \frac{8}{729}(972\zeta(3)+143)C_AC_F^2\right.\right.$$

$$\left.+\left(\frac{32\zeta(3)}{9} + \frac{6611}{729}\right)C_A^2C_F - \frac{76}{243}C_F n_f^2 + \frac{8}{729}(648\zeta(3)-125)C_F^3\right\}m\bar{q}q$$

$$\left.+\left\{n_f\left(\left(\frac{52\zeta(3)}{9} - \frac{401}{324}\right)C_AC_F + \left(\frac{134}{27} - 4\zeta(3)\right)C_A^2 + \left(\frac{2407}{1458} - \frac{16\zeta(3)}{9}\right)C_F^2\right) + n_f^2\left(-\frac{697C_A}{729} - \frac{169C_F}{1458}\right)\right\}F^2\right]$$

$$\frac{1}{2m_\pi^2}\langle\pi|g_{\mu\nu}T_q^{\mu\nu}|\pi\rangle = 0.388889 + 0.12215\alpha_s + 0.124659\alpha_s^2 + 0.0430357\alpha_s^3$$

$$g_{\mu\nu}T_g^{\mu\nu} = \frac{\alpha_s}{4\pi}\left(\frac{14}{3}C_F m\bar{q}q - \frac{11}{6}C_A F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{812C_A}{27} - \frac{22n_f}{27}\right) + \frac{85C_F^2}{27}\right)m\bar{q}q + \left(\frac{28C_A n_f}{27} - \frac{17C_A^2}{3} + \frac{5C_F n_f}{54}\right)F^2\right]$$

$$+\left(\frac{\alpha_s}{4\pi}\right)^3\left[\left\{n_f\left(\left(\frac{368\zeta(3)}{9} - \frac{25229}{729}\right)C_F^2 - \frac{2}{243}(4968\zeta(3)+1423)C_AC_F\right) + \left(\frac{32\zeta(3)}{3} - \frac{91753}{1458}\right)C_AC_F^2\right.\right.$$

$$\left.+\left(\frac{294929}{1458} - \frac{32\zeta(3)}{9}\right)C_A^2C_F - \frac{554}{243}C_F n_f^2 + \left(\frac{95041}{729} - \frac{64\zeta(3)}{9}\right)C_F^3\right\}m\bar{q}q$$

$$\left.+\left\{n_f\left(\left(\frac{1123}{162} - \frac{52\zeta(3)}{9}\right)C_AC_F + \left(4\zeta(3) + \frac{293}{36}\right)C_A^2 + \frac{16}{729}(81\zeta(3)-98)C_F^2\right) + n_f^2\left(\frac{655C_A}{2916} - \frac{361C_F}{729}\right) - \frac{2857C_A^3}{108}\right\}F^2\right]$$

$$\frac{1}{2m_\pi^2}\langle\pi|g_{\mu\nu}T_g^{\mu\nu}|\pi\rangle = 0.611111 - 0.12215\alpha_s - 0.124659\alpha_s^2 - 0.0430357\alpha_s^3$$

$$2m_\pi^2 = \langle \pi | \left(\frac{\beta(g)}{2g} F^2 + \left(1+\gamma_m(g)\right) m\bar{q}q \right) | \pi \rangle$$

$$2 f_\pi^2 m_\pi^2 = -\left(m_u + m_d\right) \langle 0 | \left(\bar{u}u + \bar{d}d\right) | 0 \rangle$$

$$\left| \pi \right\rangle \to \left| \pi \right\rangle_0 + \left| \pi \right\rangle_1 + \cdots$$

$$m_\pi^2 = {}_0\left\langle \pi \right| m\bar{q}q \left| \pi \right\rangle_0$$

Gasser, Leutwyler ('82)

$${}_0\left\langle \pi \right| F^2 \left| \pi \right\rangle_0 = 0, \qquad \left(1-\gamma_m(g)\right) m_\pi^2 = {}_1\left\langle \pi \right| \frac{\beta(g)}{2g} F^2 \left| \pi \right\rangle_0 + {}_0\left\langle \pi \right| \frac{\beta(g)}{2g} F^2 \left| \pi \right\rangle_1$$

$$\gamma_m(g) = 0.63662\alpha_s + 0.768352\alpha_s^2 + 0.801141\alpha_s^3 \simeq 0.559$$

$$2m_\pi^2 = g_{\mu\nu} \langle \pi | T_q^{\mu\nu} | \pi \rangle + g_{\mu\nu} \langle \pi | T_g^{\mu\nu} | \pi \rangle$$

$$= 2m_\pi^2 \times \left(0.479 + 0.521 \right)$$

KT, JHEP1901, 120

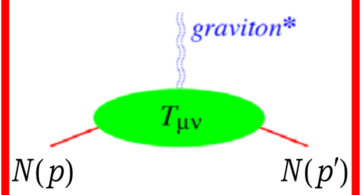
1. hadron mass formula

$$2M^2 = g_{\mu\nu}\langle p|T_q^{\mu\nu}|p\rangle + g_{\mu\nu}\langle p|T_g^{\mu\nu}|p\rangle$$

nucleon	-1	:	5
pion	1	:	1

2. gravitational form factor   twist-four

$$\bar{C}_{q/g}(t)$$



$$T^{\mu\nu} = \frac{1}{2}\bar{q}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}q + F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{g^{\mu\nu}}{4}F^2 \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$P = \frac{p+p'}{2}$$

$$\Delta = p' - p$$

$$t = \Delta^2$$

$$\langle N(p')|T_{q,g}^{\mu\nu}|N(p)\rangle = \bar{u}(p')\Big[A_{q,g}(t)\gamma^{(\mu}P^{\nu)} + B_{q,g}(t)\frac{P^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D_{q,g}(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} + \bar{C}_{q,g}(t)Mg^{\mu\nu}\Big]u(p)$$

$$A_q(0) + A_g(0) = 1$$

$$\langle N(p)|T^{\mu\nu}|N(p)\rangle = 2p^\mu p^\nu$$

$$\frac{\langle N(p)\,S|J^i|N(p)\,S\rangle}{\langle N(p)\,S|N(p)\,S\rangle} = \frac{1}{2}S^i$$

$$\frac{1}{2}\Big(A_q(0) + B_q(0) + A_g(0) + B_g(0)\Big) = \frac{1}{2}$$

$$B_q(0) + B_g(0) = 0$$

$$J^i = \frac{1}{2}\epsilon^{ijk}\int d^3x M^{+jk}$$

$$M^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

$$\bar{C}_q(t) + \bar{C}_g(t) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)Mg^{\mu\nu}\right]u(p)$$

$$g_{\mu\nu}\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = 2M^2\left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)$$

$$\bar{C}_q(0)\ \left(= -\bar{C}_g(0)\right) = -\frac{1}{4}A_q(0) + \frac{1}{8M^2}\langle N(p)|g_{\mu\nu}T^{\mu\nu}_q|N(p)\rangle$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{q}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}q} + \boxed{F^{\mu\rho}F_\rho{}^\nu + \frac{g^{\mu\nu}}{4}F^2} \equiv T^{\mu\nu}_q + T^{\mu\nu}_g$$

$$\int_{-1}^{1}dx\,xq(x,\mu) = A_q(t=0,\mu)$$

$$A_q(0,\mu) = \frac{n_f}{4C_F+n_f} + \frac{4C_F A_q(0,\mu_0) + n_f\left(A_q(0,\mu_0)-1\right)}{4C_F+n_f}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}} + \cdots$$

$$\bar{C}_q(0) \left(=-\bar{C}_g(0)\right) = -\frac{1}{4}A_q(0) + \frac{1}{8M^2}\langle N(p)|\, g_{\mu\nu}T_q^{\mu\nu}\,|N(p)\rangle$$

$$= -\frac{1}{4}A_q(0) + \frac{1}{2M^2}\langle N(p)|\, m\bar{q}q\,|N(p)\rangle$$

$$+\frac{1}{8M^2}\langle N(p)|\frac{\alpha_s}{4\pi}\frac{4}{3}C_F m\bar{q}q\,|N(p)\rangle + \frac{1}{8M^2}\langle N(p)|\frac{\alpha_s}{4\pi}\frac{1}{3}n_f F^2\,|N(p)\rangle + \cdots$$

$$2M^2 = \langle N(p)|\left(\frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{q}q\right)|N(p)\rangle$$

$$\langle N(p)|T^{\mu\nu}|N(p)\rangle = 2p^\mu p^\nu$$

$$= -\langle N(p)|\frac{\beta_0}{2}\frac{\alpha_s}{4\pi}F^2\,|N(p)\rangle + \left(1+\gamma_{m0}\frac{\alpha_s}{4\pi}\right)\langle N(p)|\, m\bar{q}q\,|N(p)\rangle + \cdots$$

$$\int_{-1}^{1} dx\, x q(x,\mu) = A_q(t=0,\mu)$$

$$A_q(0,\mu) = \frac{n_f}{4C_F+n_f} + \frac{4C_F A_q(0,\mu_0) + n_f\left(A_q(0,\mu_0)-1\right)}{4C_F+n_f}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}} + \cdots$$

$$\bar{C}_q(0,\mu) = -\frac{1}{4}\left(\frac{n_f}{4C_F + n_f} + \frac{2n_f}{3\beta_0}\right) + \frac{1}{4}\left(\frac{2n_f}{3\beta_0} + 1\right)\frac{\langle N(p)|\, m\bar{q}q \,|N(p)\rangle}{2M^2}$$

$$-\frac{4C_F A_q(\mu_0) + n_f\left(A_q(\mu_0) - 1\right)}{4(4C_F + n_f)}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F + 2n_f}{3\beta_0}}$$

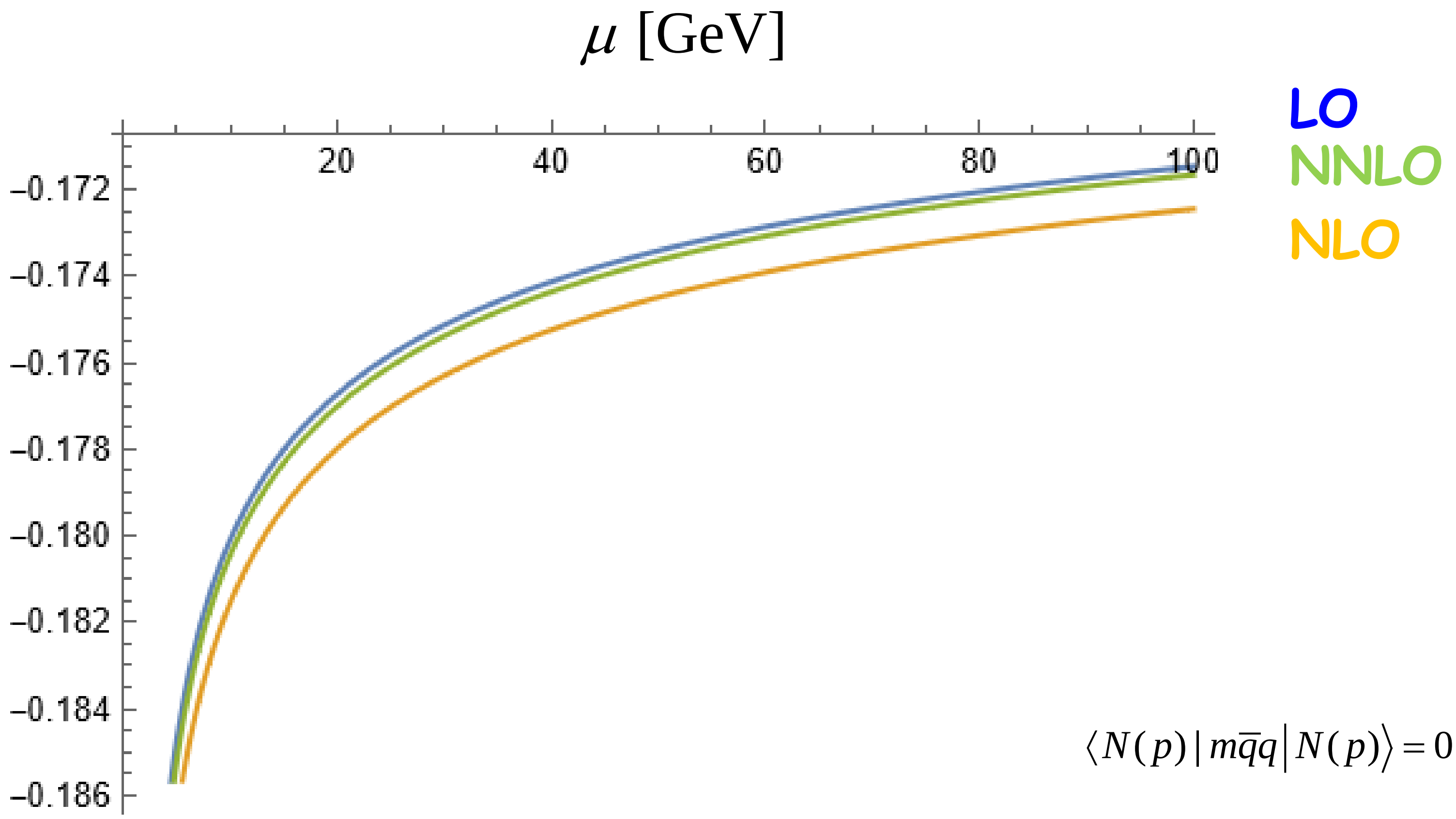
$$+\frac{\alpha_s(\mu)}{4\pi}\left[\frac{n_f}{4\beta_0}\left(-\frac{34C_A}{27} - \frac{49C_F}{27}\right) + \frac{\beta_1 n_f}{6\beta_0^2} + \cdots\right]$$

$$+\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\left[\frac{2C_F^3 n_f\left(10873C_A - 2632C_F\right)}{81\left(4C_F + n_f\right)\left(-3\beta_0 + 4C_F + n_f\right)\left(-3\beta_0 + 8C_F + 2n_f\right)} + \cdots\right]$$

$$\bar{C}_q(0,\mu)\Big|_{n_f=3} = -0.146 + 0.306\frac{\langle N(p)|m\bar{q}q|N(p)\rangle}{2M^2} - 0.25\left(A_q(\mu_0) - 0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$ LO

$$+\alpha_s(\mu)\left(0.006 + 0.08\frac{\langle N(p)|m\bar{q}q|N(p)\rangle}{2M^2} - 0.035\left(A_q(\mu_0) - 0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} + \cdots\right)$$ NLO

$$+\left[\alpha_s(\mu)\right]^2\left(0.002 + 0.03\frac{\langle N(p)|m\bar{q}q|N(p)\rangle}{2M^2} + 0.017\left(A_q(\mu_0) - 0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} + \cdots\right)$$ NNLO

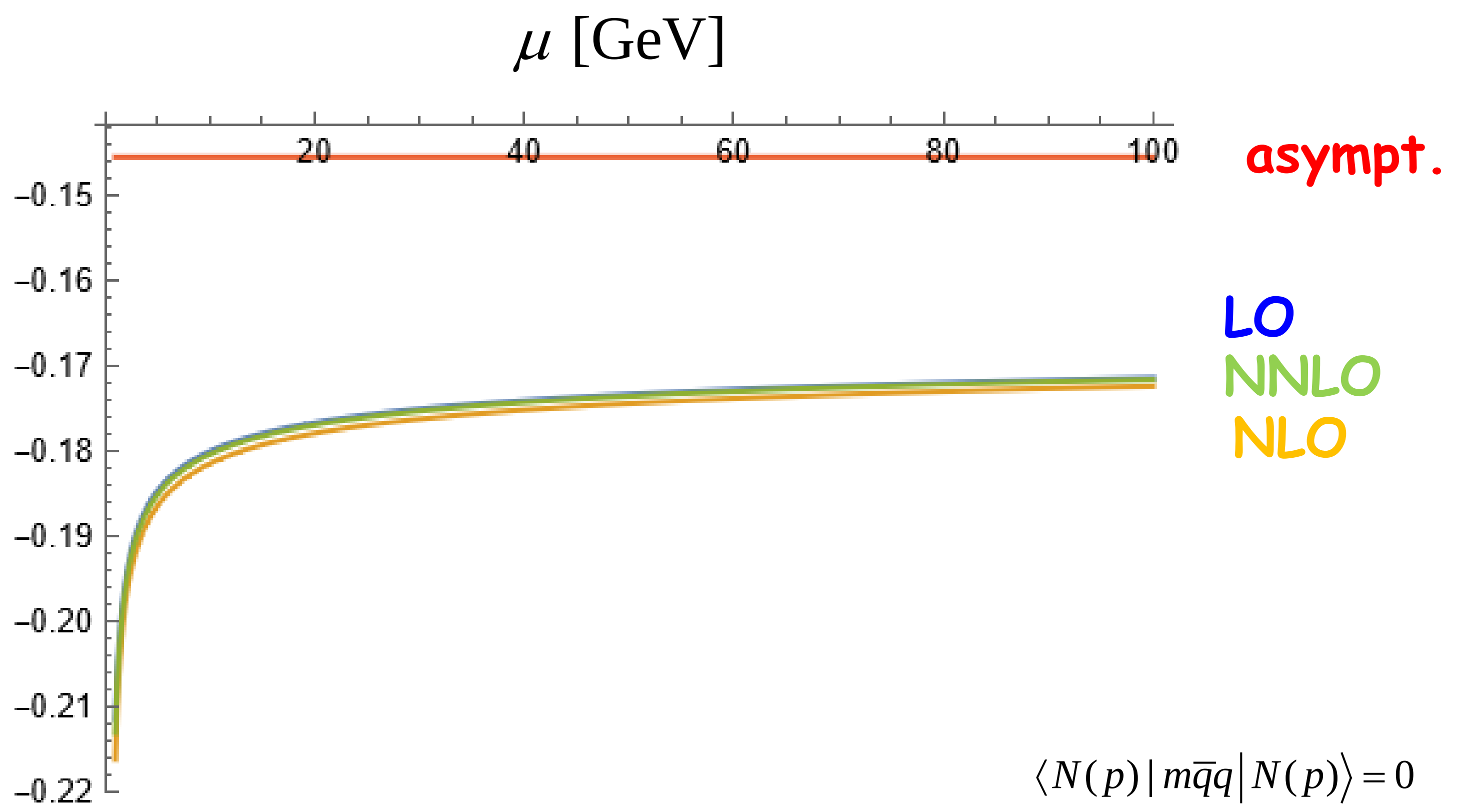


$\langle N(p)|m\bar{q}q|N(p)\rangle = 0$

$$\bar{C}_q(0,\mu)\Big|_{n_f=3} = -0.146 + 0.306\frac{\langle N(p)|m\bar{q}q|N(p)\rangle}{2M^2} - 0.25\left(A_q(\mu_0)-0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$ LO

$$+\alpha_s(\mu)\left(0.006 + 0.08\frac{\langle N(p)|m\bar{q}q|N(p)\rangle}{2M^2} - 0.035\left(A_q(\mu_0)-0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} + \cdots\right)$$ NLO

$$+\left[\alpha_s(\mu)\right]^2\left(0.002 + 0.03\frac{\langle N(p)|m\bar{q}q|N(p)\rangle}{2M^2} + 0.017\left(A_q(\mu_0)-0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} + \cdots\right)$$ NNLO

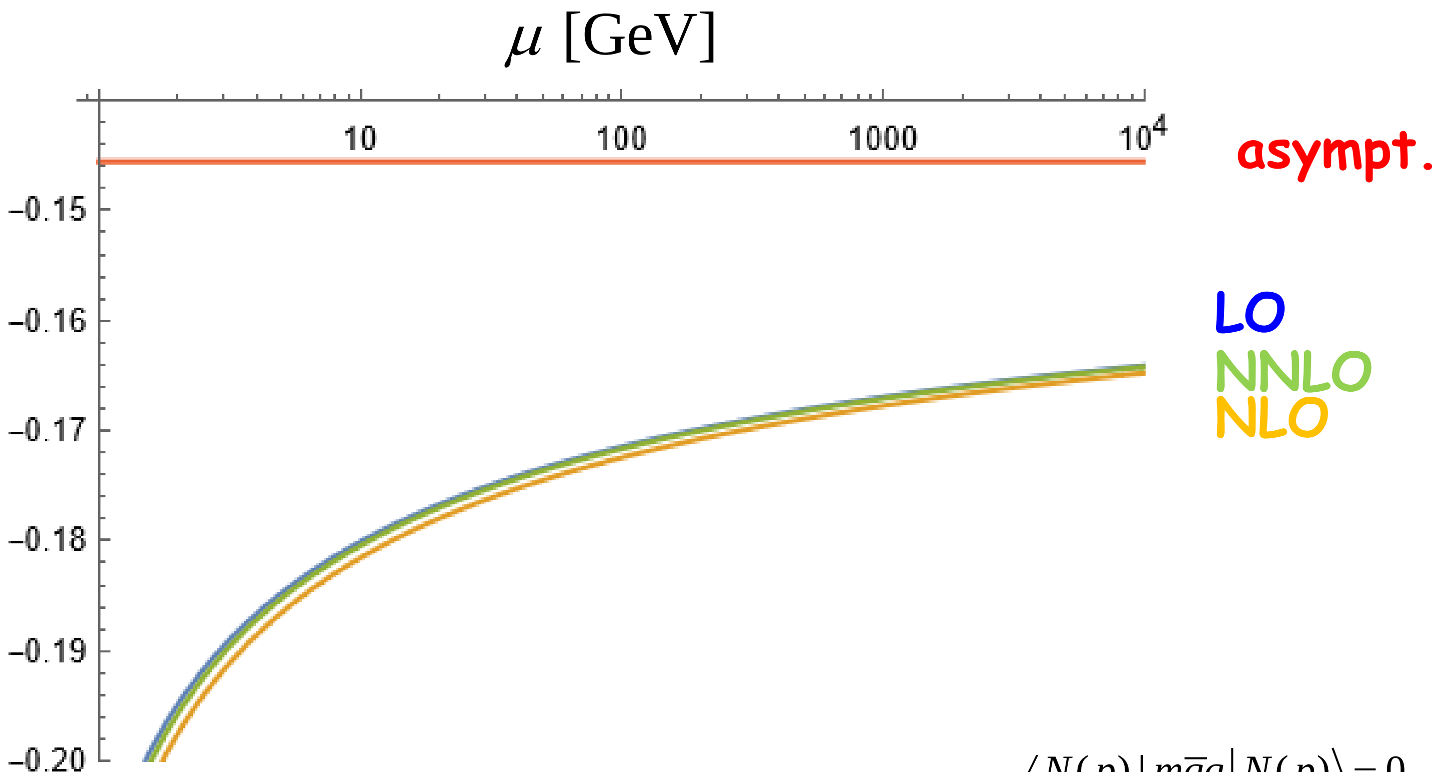


$\langle N(p)|m\bar{q}q|N(p)\rangle = 0$

$$\bar{C}_q(0,\mu)\Big|_{n_f=3} = -0.146 + 0.306\frac{\langle N(p)|m\bar{q}q|N(p)\rangle}{2M^2} - 0.25\left(A_q(\mu_0) - 0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$ LO

$$+\alpha_s(\mu)\left(0.006 + 0.08\frac{\langle N(p)|m\bar{q}q|N(p)\rangle}{2M^2} - 0.035\left(A_q(\mu_0) - 0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} + \cdots\right)$$ NLO

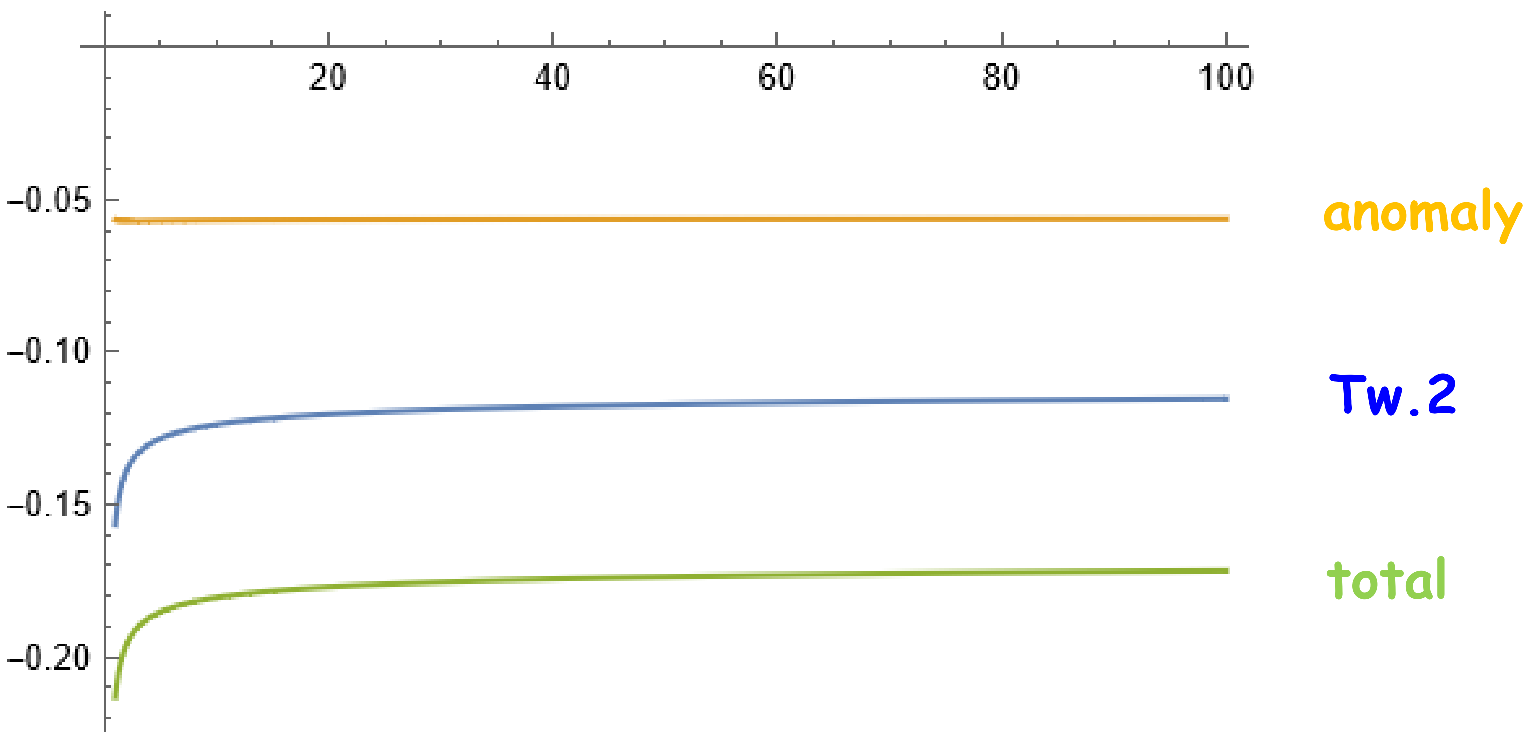
$$+\left[\alpha_s(\mu)\right]^2\left(0.002 + 0.03\frac{\langle N(p)|m\bar{q}q|N(p)\rangle}{2M^2} + 0.017\left(A_q(\mu_0) - 0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} + \cdots\right)$$ NNLO



$\langle N(p)|m\bar{q}q|N(p)\rangle = 0$

$$\bar{C}_q(0)\ \left(=-\bar{C}_g(0)\right)=-\frac{1}{4}A_q(0)+\frac{1}{8M^2}\langle N(p)\,|\,g_{\mu\nu}T_q^{\mu\nu}\,|\,N(p)\rangle$$

μ [GeV]



$$\langle N(p)\,|\,m\bar{q}q\,|\,N(p)\rangle=0$$

$$\bar{C}_q(0,\mu)\Big|_{n_f=3} = -0.146 + 0.306\frac{\langle N(p)|m\bar{q}q|N(p)\rangle}{2M^2} - 0.25\left(A_q(\mu_0) - 0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$

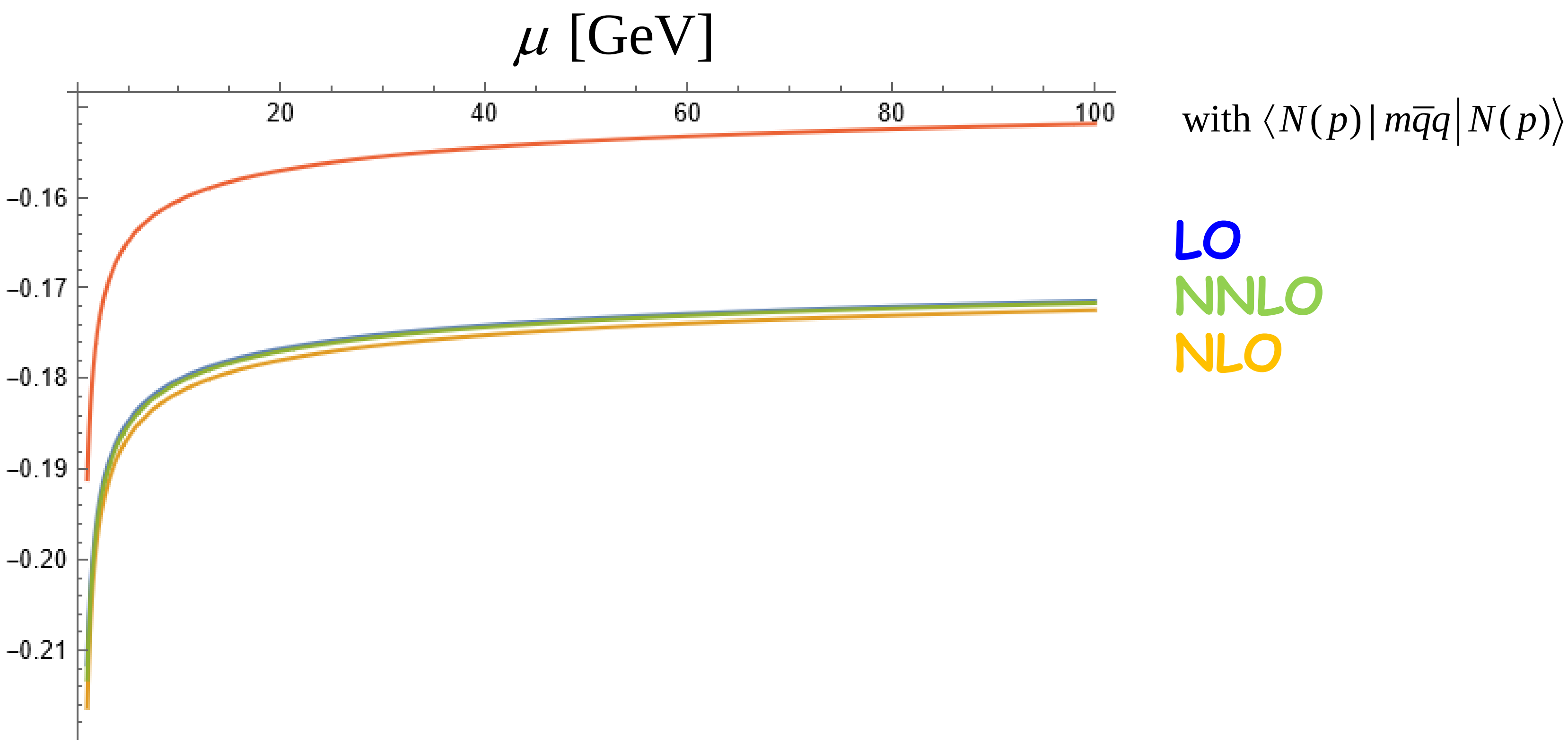
LO

$$+\alpha_s(\mu)\left(0.006 + 0.08\frac{\langle N(p)|m\bar{q}q|N(p)\rangle}{2M^2} - 0.035\left(A_q(\mu_0) - 0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} + \cdots\right)$$

NLO

$$+\left[\alpha_s(\mu)\right]^2\left(0.002 + 0.03\frac{\langle N(p)|m\bar{q}q|N(p)\rangle}{2M^2} + 0.017\left(A_q(\mu_0) - 0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} + \cdots\right)$$

NNLO



Summary

explicit quark/gluon separation of QCD trace anomaly is available at 3-loop in the MSbar scheme

$$g_{\mu\nu}T_q^{\mu\nu} = m\bar{q}q + \frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{q}q + \frac{1}{3}n_f F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{61C_A}{27} - \frac{68n_f}{27}\right) - \frac{4C_F^2}{27}\right)m\bar{q}q + \left(\frac{17C_A n_f}{27} + \frac{49C_F n_f}{54}\right)F^2\right] + \left(\frac{\alpha_s}{4\pi}\right)^3[\cdots]$$

$$g_{\mu\nu}T_g^{\mu\nu} = \frac{\alpha_s}{4\pi}\left(\frac{14}{3}C_F m\bar{q}q - \frac{11}{6}C_A F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{812C_A}{27} - \frac{22n_f}{27}\right) + \frac{85C_F^2}{27}\right)m\bar{q}q + \left(\frac{28C_A n_f}{27} - \frac{17C_A^2}{3} + \frac{5C_F n_f}{54}\right)F^2\right] + \left(\frac{\alpha_s}{4\pi}\right)^3[\cdots]$$

$$g_{\mu\nu}T_q^{\mu\nu} + g_{\mu\nu}T_g^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + (1+\gamma_m(g))m\bar{q}q$$

1. hadron mass formula $2M_h^2 = g_{\mu\nu}\langle h|T_q^{\mu\nu}|h\rangle + g_{\mu\nu}\langle h|T_g^{\mu\nu}|h\rangle$

nucleon	**-1**	**:**	**5**
pion	**1**	**:**	**1**

2. gravitational form factor $\bar{C}_{q/g}(t)$

$$\bar{C}_q(0,\mu) = -\bar{C}_g(0,\mu) = \mathrm{LO} + \mathrm{NLO} + \mathrm{NNLO}$$

the approach to the asymptotic value is quite slow

PT corrections ~% level **σ-term important**