ABOUT TOROIDAL MODEL OF LEPTONS IN SPACE-TIME FILM THEORY

Alexander A. CHERNITSKII[1,2]

[1] A. Friedmann Laboratory for Theoretical Physics, St.-Petersburg

[2] Department of Mathematics, Saint-Petersburg State Chemical Pharmaceutical University

The 24th International Spin Symposium
18-22 October 2021
Matsue, Shimane Prefecture, Japan

Concept of unified field

Let us start from the concept of unified field.

This concept is known for a long time.

According to the concept of unified field we consider a fundamental nonlinear field model with the following two relations to the material world:

- The elementary particles must be represented as space-localized solutions of this model.
- All interactions between the particles must appear naturally as the manifestation of the nonlinearity of the model.

It is well known that the creator of relativity theory A. Einstein was a proponent of this concept.

Also many famous researchers worked in the framework of this concept.

But we see a relatively weak progress in this direction of investigation. This is connected with the appropriate extraordinary mathematical difficulties.

Extremal space-time film as unified field

A nonlinear space-time scalar field model considered here is known for a long time sufficiently.

This model is related to well known Born – Infeld nonlinear electrodynamics, and it is sometimes called Born – Infeld type scalar field model.

This model is attractive because it has relatively simple and geometrically clear form.

It can be considered as a relativistic generalization of the minimal surface or minimal thin film model in three-dimensional space.

In this generalization we have an extremal four-dimensional film in a five-dimensional space-time.

But the model equation appears as differential one for scalar field in four-dimensional space-time.

Realistic filed model

The model under consideration can provide the necessary effects which are required for a realistic filed model.

This model has a static spherically symmetric solution, which is identical to zero four-vector component of electromagnetic potential for the spherically symmetric solution of Born – Infeld electrodynamics (so-called Born's electron).

In nonlinear electrodynamics there are the conformity between long-range interaction of solitons and two known long-range interactions of physical particles, that is electromagnetic and gravitational ones.[1,2,3]

But the methods used in nonlinear electrodynamics applying to the scalar model under consideration give the results, which are similar to ones for nonlinear electrodynamics.[4,5]

Thus the model of extremal space-time film provides electromagnetic and gravitational interactions between solitons-particles.

---

[1] A. A. Chernitskii, Dyons and Interactions in Nonlinear (Born-Infeld) Electrodynamics, *J. High Energy Phys.* **1999**, No 12, Paper 10 (1999); `arXiv:hep-th/9911093`.

[2] A. A. Chernitskii, Born-Infeld equations, in *Encyclopedia of Nonlinear Science*, ed. A. Scott (Routledge, New York and London, 2005), p. 67; `arXiv:hep-th/0509087`.

[3] A. A. Chernitskii, *Nonlinear electrodynamics: singular solitons and their interactions*, 360 p. (in Russian). Saint-Petersburg, ENGECON, 2012; `chernitskii.ru/books/`.

[4] A. A. Chernitskii, About long-range interaction of spheroidal solitons in scalar field nonlinear model, *Journal of Physics: Conf. Series* **938** (2017) 012029; `arXiv:1804.09022`.

[5] A. A. Chernitskii, Induced gravitation in nonlinear field models, *Int. J. Mod. Phys. Conf. Ser.* **41** (2016) 1660119; `arXiv:1808.10266`.

Photon as twisted lightlike soliton of extremal space-time film

According to recent results we can consider a correspondence between twisted lightlike solitons of extremal space-time film and photons.

The soliton solutions propagating with the speed of light or lightlike solitons was obtained for the space-time film model.[6,7]

A subclass of these soliton solutions that are the first-order twisted lightlike solitons has a correspondence with photons.

In particular, the equilibrium energy spectral density for ideal gas of these solitons is obtained. It has the form of Planck distribution in some approximation.

A correspondence between higher-order twisted lightlike solitons and real elementary particles will be investigated later. In particular, we keep in mind the possible correspondence between these solitons and neutrinos.

---

[6] A. A. Chernitskii, Lightlike shell solitons of extremal space-time film, *J. Phys. Commun.* **2** (2018) 105013; `arXiv:1506.09137`.

[7] A. A. Chernitskii, Lightlike solitons with spin, *J. Phys. Conf. Ser.* **678** (2016) 012016

Lepton as ringed soliton of extremal space-time film

Now we continue the investigation of the time-periodic solitons with the general toroidal symmetry as massive elementary particles.

Such field configurations in toroidal coordinates was considered in the framework of Born – Infeld nonlinear electrodynamics.[8,9,10,11,12]

In the present talk we consider the problem formulation and some results concerning the representation of leptons by ringed solitons of extremal space-time film.

[8] A. A. Chernitskii, Electromagnetic wave-particle with spin and magnetic moment, *Proc. of XII Adv. Res. Workshop on High Energy Spin Physics, DSPIN-07*, JINR (2008) 433–436; `arXiv:0711.2499`.

[9] A. A. Chernitskii, About spin particle solution in Born-Infeld nonlinear electrodynamics, *Proc. of XII Adv. Res. Workshop on High Energy Spin Physics, DSPIN-09*, JINR (2010) 443–446; `arXiv:0911.3230`.

[10] A. A. Chernitskii, About spin electromagnetic wave-particle with ring singularity, *Proc. of XIII Adv. Res. Workshop on High Energy Spin Physics, DSPIN-11*, JINR (2012) 395–398; `arXiv:1112.4437`.

[11] A. A. Chernitskii, *Nonlinear electrodynamics: singular solitons and their interactions*, 360 p. (in Russian). Saint-Petersburg, ENGECON, 2012; `chernitskii.ru/books/`.

[12] A. A. Chernitskii, Electromagnetic singular soliton as particle with spin and magnetic moment, *Proc. of XIV Adv. Res. Workshop on High Energy Spin Physics, DSPIN-13*, JINR (2014) 395–398.

Extremal world volume action

Let us consider the variational principle and the action which has the world volume form:

$$\delta\mathcal{A} = 0\ , \qquad \mathcal{A} = \int\limits_{\overline{V}} \sqrt{|\mathfrak{M}|}\ (\mathrm{d}x)^4 = \int\limits_{\overline{V}} \pounds\ \mathrm{d}\overline{V}\ , \tag{1a}$$

where $\mathfrak{M} \doteqdot \det(\mathfrak{M}_{\mu\nu})$, $(\mathrm{d}x)^4 \doteqdot \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$, $\overline{V}$ is space-time volume, $\mathrm{d}\overline{V} \doteqdot \sqrt{|\mathfrak{m}|}\ (\mathrm{d}x)^4$ is four-dimensional volume element,

$$\mathfrak{M}_{\mu\nu} = \mathfrak{m}_{\mu\nu} + \chi^2\,\frac{\partial\Phi}{\partial x^\mu}\,\frac{\partial\Phi}{\partial x^\nu}\ , \qquad \pounds \doteqdot \sqrt{\left|1 + \chi^2\,\mathfrak{m}^{\mu\nu}\,\frac{\partial\Phi}{\partial x^\mu}\,\frac{\partial\Phi}{\partial x^\nu}\right|} \tag{1b}$$

$\mathfrak{m}_{\mu\nu}$ are components of metric tensor for flat four-dimensional space-time, Φ is scalar real field function, χ is dimensional constant. The Greek indices take values $\{0, 1, 2, 3\}$. The tensor $\mathfrak{M}_{\mu\nu}$ used here can be called also as world tensor.

The model (1) can be considered as a relativistic generalization of the appropriate expression for the mathematical model of two-dimensional minimal thin film in the tree-dimensional space of our everyday experience.

Model equation and its symmetry

The field equation in Cartesian coordinates has the following form:

$$\left(\underline{\mathfrak{m}}^{\mu\nu}\left(1+\chi^2\,\underline{\mathfrak{m}}_{\sigma\rho}\,\Phi^\sigma\,\Phi^\rho\right)-\chi^2\,\Phi^\mu\,\Phi^\nu\right)\frac{\partial^2\,\Phi}{\partial x^\mu\,\partial x^\nu}=0\;, \tag{2a}$$

where $\underline{\mathfrak{m}}^{\mu\nu}$ is Minkowski metric with signature $\{+,-,-,-\}$ or $\{-,+,+,+\}$,

$$\Phi^\alpha \doteqdot \underline{\mathfrak{m}}^{\alpha\beta}\,\frac{\partial\Phi}{\partial x^\beta}\;. \tag{2b}$$

As we see, obtained equation does not include radicals.

It is evident that the model under consideration keep invariance for space-time rotation and scale transformation.

Thus any solution give birth to the appropriate class of solutions with the following transform:

$$\Phi(\{x^\mu\})\;\rightarrow\;a\,\Phi(\{L^\mu_{.\nu}\,x^\nu/a\})\;, \tag{3}$$

where $L^\mu_{.\nu}$ are components of space-time rotation matrix, a is scale parameter.

Energy-momentum density tensor

Customary method gives the following canonical energy-momentum density tensor of the model in Cartesian coordinates

$$\mathcal{T}^{\mu\nu} = \frac{1}{4\pi}\left(\frac{\Phi^{\mu}\,\Phi^{\nu}}{\mathcal{L}} - \frac{\underline{\mathfrak{m}}^{\mu\nu}}{\chi^2}\,\mathcal{L}\right) . \tag{4}$$

As we see, the canonical tensor is symmetrical.

To use finite integral characteristics of solutions in infinite space-time we introduce regularized energy-momentum density tensor with the following formula:

$$\mathcal{T}^{\mu\nu} = \mathcal{T}^{\mu\nu} - \overset{\infty}{\mathcal{T}}{}^{\mu\nu} . \tag{5}$$

where $\overset{\infty}{\mathcal{T}}{}^{\mu\nu}$ is regularizing symmetrical energy-momentum density tensor. Here we will use constant regularizing tensor

$$\overset{\infty}{\mathcal{T}}{}^{\mu\nu} = -\frac{1}{4\pi\,\chi^2}\,\underline{\mathfrak{m}}^{\mu\nu} . \tag{6}$$

Differential conservation laws

We have conservation law for regularized energy-momentum density tensor in Cartesian coordinates

$$\frac{\partial \not{T}^{\mu\nu}}{\partial x^{\nu}} = 0 \; . \tag{7}$$

Let us define angular momentum density tensor by customary way.
We have the following appropriate conservation law:

$$\frac{\partial \not{M}^{\mu\nu\rho}}{\partial x^{\rho}} = 0 \; , \tag{8}$$

where

$$\not{M}^{\mu\nu\rho} \doteqdot x^{\mu}\, \not{T}^{\nu\rho} - x^{\nu}\, \not{T}^{\mu\rho} \; . \tag{9}$$

Energy, momentum, and angular momentum

We have the following energy ($\mathcal{E}$), momentum vector ($\boldsymbol{\mathcal{P}}$), and angular momentum vector ($\boldsymbol{\mathcal{J}}$) densities of the model:

$$\mathcal{E} \doteqdot \mathcal{T}^{00} = \frac{1}{4\pi}\left(\frac{\Phi^0\,\Phi^0}{\pounds} - \frac{\pounds - 1}{\chi^2}\right) , \tag{10a}$$

$$\mathcal{P}^i \doteqdot \mathcal{T}^{0i} = \mathcal{T}^{i0} = \frac{1}{4\pi}\,\frac{\Phi^0\,\Phi^i}{\pounds} , \tag{10b}$$

$$\mathcal{J}_i \doteqdot \epsilon_{ijk}\, x^j\, \mathcal{P}^k . \tag{10c}$$

Let us define energy, momentum, and angular momentum of field in a three-dimensional volume V:

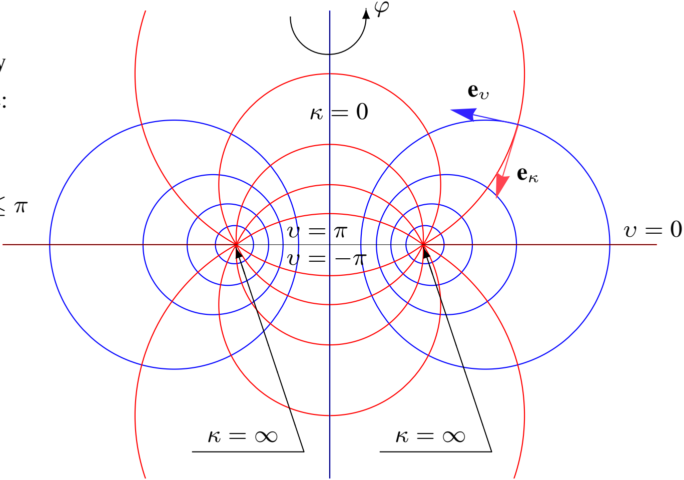
$$\mathbb{E}_V \doteqdot \int\limits_V \mathcal{E}\,\mathrm{d}V , \quad \mathbb{P}_V \doteqdot \int\limits_V \boldsymbol{\mathcal{P}}\,\mathrm{d}V , \quad \mathbb{J}_V \doteqdot \int\limits_V \boldsymbol{\mathcal{J}}\,\mathrm{d}V . \tag{11}$$

Toroidal coordinates $\{\kappa, \upsilon, \varphi\}$

It is obtained with rotation of bipolar coordinate system around the standing axis.

Space infinity is described by the closed line:

$$\begin{cases} \kappa = 0 \\ \upsilon = 0 \\ -\pi < \varphi \leq \pi \end{cases}$$



Linear wave equation in toroidal coordinates

The linear wave equation (D'Alembert equation) in toroidal coordinates has the following form:

$$
\begin{aligned}
&(\cos \upsilon - \cosh \kappa)^2 \left(\sinh^2\!\kappa \left(\frac{\partial^2 \Phi}{\partial \kappa^2} + \frac{\partial^2 \Phi}{\partial \upsilon^2} \right) + \frac{\partial^2 \Phi}{\partial \varphi^2} \right) - \rho_\circ^2 \, \sinh^2\!\kappa \, \frac{\partial^2 \Phi}{\partial (x^0)^2} \\
&+ \sinh \kappa \, (\cos \upsilon - \cosh \kappa) \left((\cosh \kappa \, \cos \upsilon - 1) \frac{\partial \Phi}{\partial \kappa} + \sinh \kappa \, \sin \upsilon \, \frac{\partial \Phi}{\partial \upsilon} \right) = 0 \,,
\end{aligned}
\tag{12}
$$

where $\rho_\circ$ is the radius of the coordinate system singular ring.

Helmholtz equation in toroidal coordinates

For the harmonic time dependence with the angular frequency ω
$\Phi = \bar{\Phi}(\kappa, \upsilon, \varphi)\ \exp(\mathrm{i}\,\omega\, x^0)$,
we have the following Helmholtz equation for the space-dependent function $\bar{\Phi}$:

$$
\begin{aligned}
&(\cos\upsilon - \cosh\kappa)^2 \left(\sinh^2\!\kappa \left(\frac{\partial^2\bar{\Phi}}{\partial\kappa^2} + \frac{\partial^2\bar{\Phi}}{\partial\upsilon^2}\right) + \frac{\partial^2\bar{\Phi}}{\partial\varphi^2}\right) + \rho_\circ^2\,\omega^2\ \sinh^2\!\kappa\ \bar{\Phi} \\
&+ \sinh\kappa\,(\cos\upsilon - \cosh\kappa)\left((\cosh\kappa\ \cos\upsilon - 1)\,\frac{\partial\bar{\Phi}}{\partial\kappa} + \sinh\kappa\ \sin\upsilon\,\frac{\partial\bar{\Phi}}{\partial\upsilon}\right) = 0\,,
\end{aligned}
\tag{13}
$$

It is well known that the variables are not separated completely in this equation for the general case $\omega \neq 0$.

At the present time we do not know a solution for this linear equation.

Nonlinear equation of space-time film in toroidal coordinates

Nonlinear equation for the model under investigation in toroidal coordinates is obtained from the variational principle (1) with the following nonzero components of the diagonal metric tensor:

$$\mathfrak{m}_{\kappa\kappa} = \mathfrak{m}_{\upsilon\upsilon} = \frac{\rho_\circ^2}{(\cosh\kappa - \cos\upsilon)^2} \;, \tag{14a}$$

$$\mathfrak{m}_{\varphi\varphi} = \frac{\rho_\circ^2\,\sinh^2\kappa}{(\cosh\kappa - \cos\upsilon)^2} \;, \tag{14b}$$

$$\mathfrak{m}_{\kappa\kappa}\,\mathfrak{m}^{\kappa\kappa} = \mathfrak{m}_{\upsilon\upsilon}\,\mathfrak{m}^{\upsilon\upsilon} = \mathfrak{m}_{\varphi\varphi}\,\mathfrak{m}^{\varphi\varphi} = 1 \;. \tag{14c}$$

The obtained equation is very complicated for the representation of it on the slide.

This equation can be analyzed with the help of the computer mathematical programs shush that *Wolfram Mathematica*.

Rational toroidal coordinates

We consider also the coordinates which can be called rational toroidal ones. In these coordinates, new independent variable is introduced by the formula:

$$\bar{\kappa} = \mathrm{e}^{\kappa} - 1 \; . \tag{15}$$

This coordinate system is helpful because the hyperbolic functions are represented in it by the rational functions of variable $\bar{\kappa}$:

$$\cosh\kappa = \frac{2 + \bar{\kappa}\,(\bar{\kappa} + 2)}{2\,(\bar{\kappa} + 1)} \; , \quad \sinh\kappa = \frac{\bar{\kappa}\,(\bar{\kappa} + 2)}{2\,(\bar{\kappa} + 1)} \; . \tag{16}$$

As result the model equation can be represented in the form with polynomial coefficients by the variable $\bar{\kappa}$.

Rational toroidal coordinates with rotation

Let us introduce also the rational toroidal coordinates with rotation $\{\bar{\theta}, \bar{\bar{\theta}}, \bar{\kappa}, \upsilon\}$ by the following transformation of variables:

$$\bar{\theta} = \varphi - \tilde{\omega}\, x^0 \,, \quad \bar{\bar{\theta}} = \varphi + \tilde{\omega}\, x^0 \,. \tag{17}$$

The coordinates $\bar{\theta}$ and $\bar{\bar{\theta}}$ can be called the right and left phases accordingly. The parameter $\tilde{\omega}$ is an angular velocity. Also we take the condition

$$\tilde{\omega}\, \rho_\circ = 1 \tag{18}$$

such that the phase velocity of the circular wave on the singular ring equals the speed of light.

Energy and angular momentum densities in the coordinates with rotation

We have the following expressions for energy and angular momentum densities in the coordinates with rotation:

$$\mathcal{E} = \frac{1}{4\pi}\left(\frac{1}{\rho_\circ^2\,\pounds}\left(\frac{\partial\Phi}{\partial\bar{\theta}} - \frac{\partial\Phi}{\partial\bar{\bar{\theta}}}\right)^2 \pm \frac{1}{\chi^2}\left(1-\pounds\right)\right) , \tag{19}$$

$$\mathcal{J}_z = \frac{1}{4\pi\,\rho_\circ\,\pounds}\left(\left(\frac{\partial\Phi}{\partial\bar{\theta}}\right)^2 - \left(\frac{\partial\Phi}{\partial\bar{\bar{\theta}}}\right)^2\right) . \tag{20}$$

Thus if we consider a solution with one side rotation $\Phi = \Phi(\bar{\theta}, \bar{\kappa}, \upsilon)$ or $\Phi = \Phi(\bar{\bar{\theta}}, \bar{\kappa}, \upsilon)$ we have the following notable connection between energy and angular momentum densities:

$$\mathcal{E} = \tilde{\omega}\,\mathcal{J}_z \pm \frac{1}{4\pi\,\chi^2}\left(1-\pounds\right) . \tag{21}$$

Circinate lightlike soliton

As noted above we consider the first-order twisted lightlike soliton of extremal space-time film as photon.

Also we have the higher-order twisted lightlike solitons which may represent neutrinos.

The appropriate circinate solitons with a static part can be considered as a massive charged particle with spin and magnetic moment. Such field configurations was considered in the framework of nonlinear electrodynamics.

The idea is that the appropriate solutions of extremal space-time film may represent leptons.

Rotating toroidal solution

We consider a rotating subclass of time-periodic toroidal solutions with dependence on three variables $\{\bar{\theta}, \bar{\kappa}, \varphi\}$:

$$\Phi = \frac{\bar{e}}{\rho_\circ} \sqrt{\bar{\mathcal{K}}}\ \bar{\Phi}\ , \tag{22}$$

where $\bar{\Phi} = \bar{\Phi}(\bar{\theta}, \bar{\kappa}, \upsilon)$ is a dimensionless function, the factor $\bar{\mathcal{K}}$ is defined as

$$\bar{\mathcal{K}} \doteqdot (\bar{\kappa} + 1)\ \sin^2\!\left(\frac{\upsilon}{2}\right) + \frac{\bar{\kappa}^2}{4}\ , \tag{23}$$

$$\frac{\bar{e}}{\rho_\circ} \sqrt{\bar{\mathcal{K}}} = \frac{\bar{e}}{r} + \frac{\bar{e}\,\rho_\circ\ \sin\vartheta}{r^2} + \frac{\bar{e}\,\rho_\circ^2\ (1 - 3\ \cos(2\,\vartheta))}{4\,r^3} + \mathcal{O}\Big(r^{-4}\Big)_{r\to\infty}\ . \tag{24}$$

After substitution the function (22) into the model equation we have an equation for the function $\bar{\Phi}$ with one dimensionless parameter

$$\varepsilon \doteqdot \pm \frac{\bar{e}^2\,\chi^2}{\rho_\circ^4}\ , \tag{25}$$

where χ is the constant of nonlinearity of the model, top and bottom signs correspond to two signatures of metric of the model.

Solution in a form of power series in $\bar{\kappa}$

We search an approximate solution for the function $\bar{\Phi}$ in the following partial sum form of a formal power series in $\bar{\kappa}$:

$$\bar{\Phi} = \sum_{i=0}^{N} \bar{\Phi}_i \, \bar{\kappa}^i \;, \tag{26}$$

where the coefficients $\bar{\Phi}_i$ are functions $\bar{\Phi}_i = \bar{\Phi}_i(\bar{\theta}, \upsilon)$.

Each iteration gives the equation for next coefficient of the power series $\bar{\Phi}_i$. These equations with the exception of the first one are two-order ordinary linear differential equations with respect to variable $\bar{\theta}$. These equations do not include the derivatives of the coefficient $\bar{\Phi}_i$ with respect to variable υ.

We can find the solution of equation for each iteration.

Thus we can build a solution in the form (29) for any order of approximation N.

But the solution for i-th coefficient $\bar{\Phi}_i$ contains an arbitrary function of the variable υ. These functions can be defined by using boundary conditions for this solution and solutions for higher orders.

Static part of the solution

The first equation which appear in the iteration method under consideration has the following form:

$$\varepsilon\ \left(\bar{\Phi}_0 + 2\,\bar{\Phi}_1\right)\left(\frac{\partial\bar{\Phi}_0}{\partial\bar{\theta}}\right)^2 = 0\ . \tag{27}$$

This equation is satisfied for the linear case $\varepsilon = 0$ but we take the interest in nonlinear one $\varepsilon \neq 0$.

We take $\bar{\Phi}_0$ to be constant which provides the right behavior of the solution at space infinity as charged elementary particle. We consider the following values for this coefficient:

$$\bar{\Phi}_0 = \pm 1\ , \tag{28}$$

where the sign $(+)$ or $(-)$ is appropriated to the particle with positive or negative elementary charge.

Two integer-valued parameters $\{m, n\}$ of the solution

The calculations gives that the series of the solution can have the following structure for the positive particle:

$$\bar{\Phi} = 1 + \bar{\kappa}\left(-\frac{1}{2} + a_1 \,\sin^4\!\left(\frac{\upsilon}{2}\right)\sin\left(\bar{\theta} - m\,\upsilon\right)\right) + \ldots \\ + \bar{\kappa}^j\left(\underline{\bar{\bar{\Phi}}}_j + \bar{\bar{\Phi}}_{s,j}\,\sin\left(j\,\bar{\theta} - m\,\upsilon\right) + \bar{\bar{\Phi}}_{c,j}\,\cos\left(j\,\bar{\theta} - m\,\upsilon\right)\right) + \ldots\,, \qquad (29)$$

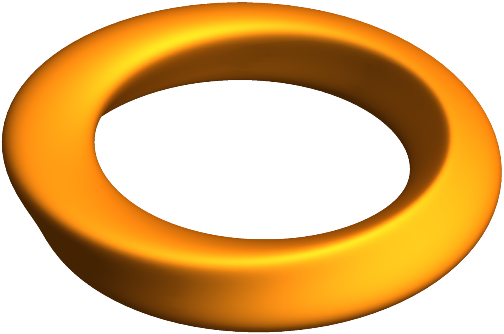
where a_1 is a constant which must be defined, $\underline{\bar{\bar{\Phi}}}_j = \underline{\bar{\bar{\Phi}}}_j(\upsilon)$, $\bar{\bar{\Phi}}_{s,j} = \bar{\bar{\Phi}}_{s,j}(\upsilon)$, $\bar{\bar{\Phi}}_{c,j} = \bar{\bar{\Phi}}_{c,j}(\upsilon)$, m is the twist parameter of the appropriate circinate lightlike soliton.

The function $\bar{\bar{\Phi}}_j(\upsilon)$ is arbitrary for the iterative solution of the j-th order. But this function can be concretized in higher orders.

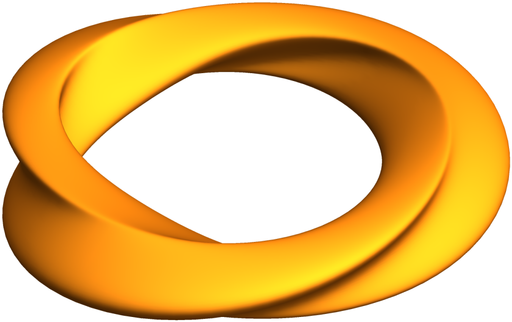
Let us designate the number of minimal harmonics of phase $\bar{\theta}$ as n. That is, for example, if $n = 2$ we must take $a_1 = 0$. Thus the parameter n defines the number of wavelength which be consisted on the singular ring.

We can suppose that the different number pairs $\{m, n\}$ represents the different particles.

Visualization for time evolution of an appropriate surface for $m = 1, n = 1$



Visualization for time evolution of an appropriate surface for $m = 2$, $n = 3$



Visualization for time evolution of an appropriate surface for $m = 4$, $n = 5$

Three first terms of the solution for $n = 1$

For example, let us write the following three first terms of the positive particle for $n = 1$:

$$\bar{\Phi} = 1 + \bar{\kappa}\left(-\frac{1}{2} + a_1\,\sin^4\!\left(\frac{\upsilon}{2}\right)\sin\left(\bar{\theta} - m\,\upsilon\right)\right) + \bar{\kappa}^2\left(\underline{\bar{\bar{\Phi}}}_2\right.$$
$$-a_1\,\sin^4\!\left(\frac{\upsilon}{2}\right)\sin\left(\bar{\theta} - m\,\upsilon\right) + \bar{\bar{\Phi}}_{s,2}\,\sin\left(2\,\bar{\theta} - m\,\upsilon\right) + \bar{\bar{\Phi}}_{c,2}\,\cos\left(2\,\bar{\theta} - m\,\upsilon\right)\Big)$$
$$+ \mathcal{O}\left(\kappa^3\right)_{\kappa\to\infty}\,, \qquad (30)$$

where the function $\underline{\bar{\bar{\Phi}}}_2$ is defined as

$$\underline{\bar{\bar{\Phi}}}_2 = \frac{5 - \varepsilon\left(1 - \cos\upsilon + 4\,a_1^2\,\sin^{12}(\upsilon/2)\,(5\cos\upsilon + 8) - 2\,\sin^6(\upsilon/2)\right)}{8\left(2 - \varepsilon\,\sin^4(\upsilon/2)\left(1 + \cos\upsilon + 4\,a_1^2\,\sin^{10}(\upsilon/2)\right)\right)}\,, \qquad (31)$$

$\bar{\bar{\Phi}}_{s,2} = \bar{\bar{\Phi}}_2(\upsilon)$ and $\bar{\bar{\Phi}}_{c,2} = \bar{\bar{\Phi}}_2(\upsilon)$ are arbitrary functions in this order of iteration, a value of the parameter a_1 is also not defined in this step of the iterative process.

It should be noted that for the substitution $\varepsilon = 0$, $a_1 = 0$, $\bar{\bar{\Phi}}_{s,2} = 0$, $\bar{\bar{\Phi}}_{c,2} = 0$ we obtain the right series expansion for well known static solution for the appropriate linear equation.

Behavior of the solution at space infinity

Let us transform the expression of the previous slide with $m = 1$, $\bar{\bar{\Phi}}_{s,2} = 0$, $\bar{\bar{\Phi}}_{c,2} = 0$ to spherical coordinates $\{r, \vartheta, \varphi\}$:

$$\bar{\bar{\Phi}} \approx \frac{e}{r} - \frac{e\,\rho_\circ^2 (1 + 3\cos(2\vartheta))}{8\,r^3} + \frac{7\,e\,\rho_\circ^3\,\sin^3\vartheta}{r^4} + \frac{e\,\rho_\circ^4\,f_1(\varepsilon,\vartheta)}{r^5} + \frac{e\,\rho_\circ^5\,f_2(\varepsilon,\vartheta)}{r^6} + \frac{2\,a_1\,e\,\rho_\circ^5\,\sin(\vartheta)\,\cos^4(\vartheta)}{r^6}\,\sin\theta + \frac{e\,\rho_\circ^6\,f_3(\varepsilon,\vartheta)}{r^7} - \frac{4\,a_1\,e\,\rho_\circ^6\,\sin\vartheta\,\cos^5\vartheta}{r^7}\,\cos\theta\,. \qquad (32)$$

As we see the dependence on the phase $\theta \doteqdot \varphi - \tilde{\omega}\,x^0$ appears in the sixth and seventh order by inverse radius only.

Thus we can talk that the time dependence is deeply embedded in the solution. This provides the integral convergence of energy and angular momentum at infinity.

As well known the elementary time-periodic solution of wave equation in spherical coordinates has the factor $\dfrac{1}{r}\,\exp(\mathrm{i}\,\omega\,x^0)$ which gives the integral divergence for energy and angular momentum at infinity.

Conclusions

- We have considered the field model of extremal space-time film.
- We have obtained the model equation in the rational toroidal coordinates.
- We have considered the time-periodic solution dependent on three variables. This solution contain the circular wave with the phase velocity on the coordinate singular ring equal the speed of light. Also the solution contain a static part.
- We have proposed the iterative algorithm for obtaining this solution in the form of formal power series in the rational toroidal variable $\bar{\kappa}$.
- Using this algorithm we can obtain any term of the power series. But the appropriate calculations is very complicated and can be realized only with the help of the program systems for symbolic computing such that *Wolfram Mathematica*.
- We discover that the time dependence in the toroidal solution has a deeply embedded character. Such that the full energy and angular momentum of the solution converge at space infinity.
- Thus the soliton solution under consideration can represent a massive charged wave-particle with spin, in particular, lepton.

Thanks for your attention.