Infrared renormalon in the $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$

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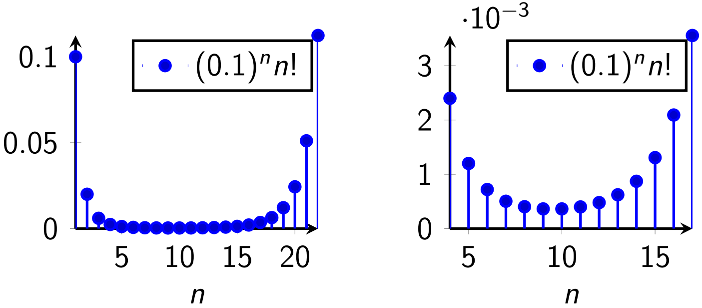
- K. Ishikawa, O.M., A. Nakayama, K. Shibata, H. Suzuki and H. Takaura, arXiv:1908.00373 [hep-th].
- K. Ishikawa, O.M., K. Shibata, H. Suzuki and H. Takaura, arXiv:1909.09579 [hep-th], to appear in PTEP.

Factorial growth of perturbation series

- Perturbation theory (PT) of QM/QFT is a quite successful tool.
- But, perturbative series are typically divergent as

$$F(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n, \qquad c_n \sim n! \text{ at large } n.$$



- Accuracy of perturbative predictions is limited. . .

Factorial growth of perturbation series

- E.g., ground state energy in QM (Rayleigh–Schrödinger PT):

	Perturbative coefficients
Zeeman effect	$\sim (-1)^n (2n)!$
Stark effect	$\sim (2n)!$
Anharmonic oscillator	
$V(\phi) \sim \phi^3$	$\sim \Gamma(n+1/2)$
$V(\phi) \sim \phi^4$	$\sim (-1)^n \Gamma(n+1/2)$
Double well	$\sim n!$
periodic cosine well	$\sim n!$
$\vdots$	$\vdots$

- These are due to the factorial growth of the number of Feynman diagrams.

Borel resummation

- The Borel (re)summation is useful for summing divergent asymptotic series.
- For the perturbative series of a quantity $f(\lambda)$,

$$f(\lambda) \sim \sum_{n=0}^{\infty} f_n \left(\frac{\lambda}{4\pi}\right)^{n+1},$$

we define the Borel transform by

$$B(u) \equiv \sum_{n=0}^{\infty} \frac{f_n}{n!} u^n.$$

- The Borel sum is given by

$$f(\lambda) \equiv \int_0^{\infty} du\, B(u) e^{-4\pi u/\lambda}.$$

Borel resummation

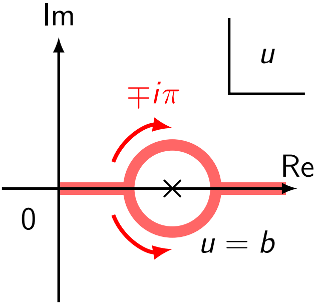
- If $f_n \sim b^{-n} n!$ as $n \to \infty$,

$$B(u) = \frac{1}{1 - u/b}.$$

This possesses a pole singularity at $u = b$.

- The integral is convergent for $b < 0$ (alternating series).

- If $b > 0$, the Borel sum becomes ill-defined. $\Rightarrow$ non-Borel summable
- Then, one should avoid the pole by contour deformation.
- This induces the imaginary ambiguity proportional to $\sim e^{-4\pi b/\lambda}$.



Resurgence theory and semi-classical picture

- $f(\lambda)$ is not analytic at $\lambda = 0$, but an asymptotic series.
- Asymptotic nature of perturbative series is related to
  1. possible instability of quantum theories,
     [Dyson '52, Hurst '52, Thirring '53, ...]
  2. nonperturbative effects such as quantum tunneling.
     [Vainshtein '64, Bender–Wu '73, Lipatov '77, ...]
- Ambiguity $\propto$ a nonperturbative factor $e^{-\text{const.}/\lambda}$

  $\updownarrow$ Cancellation (Resurgence structure)

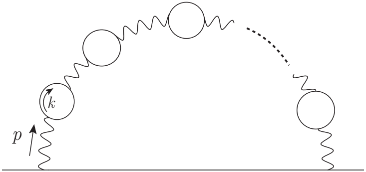
  Ambiguity associated with nonperturbative effects
  - Instanton calculus [Bogomolny 1980, Zinn-Justin 1981]

$$V(x) = \frac{1}{4}(x^2 - 1)^2 \quad \to \quad \text{Solution to EoM: } x(\tau) = \tanh \frac{\tau - \tau_0}{\sqrt{2}}$$

- Reading out nonperturbative effects from PT: Resurgence theory

Renormalon and bion

- Another source of n-factorial: renormalon ['t Hooft 1979]
- Amplitude of a single Feynman diagram $\sim \beta_0^n n!$.
  (β_0: one-loop coefficient of the beta function)



- It is conjectured that renormalon ambiguities disappear thanks to the so-called bion [Argyres–Ünsal '12, Dunne–Ünsal '12, . . .].
- Bion: a pair of $\underbrace{\text{fractional instanton/anti-instanton}}_{\text{on } \mathbb{R}^{d-1}\times S^1 \text{ with twisted boundary conditions (BC)}}$
- It is important to clarify the renormalon structure on $\mathbb{R}^{d-1} \times S^1$.
- We study the 2D $\mathbb{C}P^{N-1}$ model in the large N limit.

2D (supersymmetric) $\mathbb{C}P^{N-1}$ model

- Complex projective space $\mathbb{C}P^{N-1} \cong S^{2N-1}/U(1)$: for $z^A \in \mathbb{C}$,

$$(z^1, z^2, \dots, z^N) \sim (cz^1, cz^2, \dots, cz^N) \qquad (c \in \mathbb{C}, c \neq 0)$$

  $\to \bar{z}^A z^A = 1$ ($|c| = 1$) and $U(1)$ gauge invariance ($\arg(c)$).
- Action of the 2D (supersymmetric) $\mathbb{C}P^{N-1}$ model:

$$S = \frac{N}{\lambda} \int d^2x \left[-\bar{z}^A D_\mu D_\mu z^A + \bar{\sigma}\sigma + \bar{\chi}^A (\not{D} + \bar{\sigma} P_+ + \sigma P_-) \chi^A \right]$$

  where $D_\mu = \partial_\mu + iA_\mu$, $\gamma_5 = -i\gamma_x\gamma_y$, $P_\pm = (1 \pm \gamma_5)/2$, and we impose $\bar{z}^A z^A = 1$ and $\bar{z}^A \chi^A = 0$.
- $U(1)$ gauge symmetry: $z^A \to g z^A$, $\chi^A \to g\chi^A$, and $A_\mu \to A_\mu + i g^{-1} \partial_\mu g$ with $g \in U(1)$.
- Lagrange multiplier fields f and $(\eta, \bar{\eta})$;

$$S' = S + \frac{N}{\lambda} \int d^2x \left[f(\bar{z}^A z^A - 1) + 2\bar{\eta}\bar{z}^A \chi^A + 2\bar{\chi}^A z^A \eta \right].$$

$\mathbb{Z}_N$ invariant twisted BC

- We assume the following BC

$$z^A(x, y + 2\pi R) = e^{2\pi i m_A R} z^A(x, y),$$
$$\chi^A(x, y + 2\pi R) = e^{2\pi i m_A R} \chi^A(x, y),$$

where

$$m_A = \begin{cases} A/NR & \text{for } A = 1, 2, \ldots, N-1 \\ 0 & \text{for } A = N. \end{cases}$$

- We impose periodic BC for all auxiliary fields A_μ, f, σ, η.
- Kaluza–Klein momentum along S^1 is given by $p_y = n/R$.

Gauge field propagator in $N \to \infty$

- We consider $NR\Lambda \to \infty$ where Λ is a dynamical scale.
- Effective action S_{eff} for fluctuations of the auxiliary fields

$$A_\mu \equiv A_{\mu 0} + \delta A_\mu, \qquad f \equiv f_0 + \delta f, \qquad \sigma \equiv \sigma_0 + \delta\sigma,$$

around the large N saddle point $f_0 = \bar{\sigma}_0 \sigma_0 = \Lambda^2$, $\eta_0 = 0$.

- $S_{\text{eff}}|_{\mathbb{R}\times S^1} \overset{N\to\infty}{\to} S_{\text{eff}}|_{\mathbb{R}^2}$ in [D'Adda–Di Vecchia–Lüscher '78&'79].
- E.g., A_μ propagator in the SUSY $\mathbb{C}P^{N-1}$ model is given by

$$\langle \delta A_\mu(p) \delta A_\nu(q) \rangle = \frac{4\pi}{N} \frac{\delta_{\mu\nu} + 4\Lambda^2 p_\mu p_\nu/(p^2)^2}{(p^2 + 4\Lambda^2)\mathcal{L}_\infty(p)} 2\pi\delta(p_x + q_x) 2\pi R \delta_{p_y+q_y,0}$$

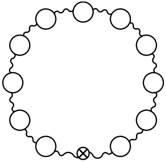
where

$$\mathcal{L}_\infty(p) \equiv \frac{2}{\sqrt{p^2(p^2+4\Lambda^2)}} \ln\left(\frac{\sqrt{p^2+4\Lambda^2}+\sqrt{p^2}}{\sqrt{p^2+4\Lambda^2}-\sqrt{p^2}}\right).$$

IR renormalon in gluon condensate

- We compute the gluon (photon) condensate in $N \to \infty$, and study Borel singularities associated with it.
- Gluon condensate in the large N limit

$$\langle F_{\mu\nu}(x) F_{\mu\nu}(x) \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \frac{2p^2}{(p^2 + 4\Lambda^2)\mathcal{L}_\infty(p)}$$



- Positive powers of $\Lambda^2 = \mu^2 e^{-4\pi/\lambda_R(\mu^2)}$ are regarded as the non-perturbative part; $\langle FF \rangle$ in PT is given by

$$\langle F_{\mu\nu}(x) F_{\mu\nu}(x) \rangle_{\mathrm{PT}} = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \left. \frac{p^2}{\ln(p^2/\Lambda^2)} \right|_{\text{expansion in } \lambda_R(\mu^2)} .$$

Note that $\mathcal{L}_\infty(p) = \frac{2}{p^2} \ln(p^2/\Lambda^2) + \mathcal{O}(\Lambda^2)$.

Renormalon ambiguity in $\mathbb{R}^d$

- Noting $\lambda_R(p^2) = 4\pi/\ln(p^2/\Lambda^2)$, $\langle FF\rangle_{\text{PT}}$ is a typical form from which a renormalon appears. Generally, on $\mathbb{R}^d$, we have

$$\int \frac{d^d p}{(2\pi)^d}(p^2)^\alpha \frac{\lambda_R(p^2)}{(4\pi)^{d/2}} = \int \frac{d^d p}{(2\pi)^d}(p^2)^\alpha \sum_{n=0}^{\infty} \ln^n\left(\frac{\mu^2}{p^2}\right)\left[\frac{\lambda_R(\mu^2)}{(4\pi)^{d/2}}\right]^{n+1}.$$

  Note that $\ln(p^2/\Lambda^2) = \ln(p^2/\mu^2) + 4\pi/\lambda_R(\mu^2)$.

- Focusing on the IR region by introducing a cutoff q $(p^2 \leq q^2)$,

$$B(u) = \int_{p^2\leq q^2} \frac{d^d p}{(2\pi)^d}(p^2)^\alpha \left(\frac{\mu^2}{p^2}\right)^u = \frac{\mu^{2u}}{(4\pi)^{d/2}\Gamma(d/2)} \frac{q^{2\alpha+d-2u}}{\alpha + d/2 - u}.$$

- The Borel singularity at $u = \alpha + d/2 = 2$ gives rise to

$$\langle F_{\mu\nu}(x)F_{\mu\nu}(x)\rangle_{\text{renormalon on } \mathbb{R}^2 \text{ (at } u=2)} = \pm\pi i\Lambda^4/N.$$

Renormalon ambiguity in $\mathbb{R}^{d-1} \times S^1$

- On the other hand, on $\mathbb{R}^{d-1} \times S^1$,

$$\int \frac{d^d p}{(2\pi)^d} \to \int \frac{d^{d-1} p}{(2\pi)^{d-1}} \frac{1}{2\pi R} \sum_{p_d = n/R,\, n \in \mathbb{Z}} .$$

- Now, only the $p_d = 0$ term can be singular; the dimension of the momentum integration is effectively reduced:

$$u = \alpha + \frac{d}{2} \to \alpha + \frac{d-1}{2}$$

- The Borel singularity at $u = 3/2$ gives rise to the renormalon:

$$\langle F_{\mu\nu}(x) F_{\mu\nu}(x) \rangle_{\text{renormalon}} = \pm \pi i \frac{\Lambda^3}{\pi R N} .$$

Peculiar to the compactified space $\mathbb{R} \times S^1$!

Discussion and conclusion

- The Borel singularity is generally shifted by $-1/2$ under the S^1 compactification and the following assumptions:
  1. volume independence of a loop integrand of a renormalon diagram
  2. loop momentum variable along S^1 associated with the periodic BC (not twisted!)
- Then, in the large-N (SUSY) $\mathbb{C}P^{N-1}$ model, we find an unfamiliar renormalon singularity at $u = 3/2$.
- But bion calculus $\to$ $u = 2$ [Fujimori *et al.*].

  Thus, no obvious semi-classical interpretation so far.
- 4D $SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$ $\Rightarrow$ Takaura-san's talk