Comment on compact star mass formula

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References:

S. Aoki, T. Onogi and S. Yokoyama,
“Conserved charge in general relativity”,
Int. J. Mod. Phys. A36 (2021) 2150098 arXiv:2005.13233[gr-qc].

S. Aoki, T. Onogi and S. Yokoyama,
“Charge conservation, Entropy, and Gravitation”,
arXiv:2010.07660[gr-qc].

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0. Introduction
Energy in general relativity

(Conserved) energy in general relativity

Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G_d T_{\mu\nu}$$

gravity matter

$$T_{\mu\nu}(x) = \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}(x)}$$

energy momentum tensor
(EMT)

Bianchi identity

$$\nabla_\mu T^\mu{}_\nu = 0$$

covariant conservation

However what we need for a conservation is $\partial_\mu(\sqrt{-g}T^\mu{}_\nu) = 0$

but in general $\partial_\mu(\sqrt{-g}T^\mu{}_\nu) \neq 0$

(Historical) solutions

Pseudo-tensor **Traditional: give up covariance**

Einstein

$$\partial^\mu \left[\sqrt{|g|} \, (T_{\mu\nu} + \underline{t_{\mu\nu}}) \right] = 0$$

gravitational energy ?

$t_{\mu\nu}$ is not covariant under general coordinate transformation.

Noether's 2nd theorem

Aoki-Onogi (in preparation)

Quasi-local energy **Modern: give up a local definition of energy**

Komar, Bondi, Arnowitt-Deser-Misner, Gibbons-Hawking

$$E = \int dV \, (\text{local energy}) \quad \Longleftrightarrow \quad E = \int_{r\to\infty} dS \, (\text{quasi-local energy})$$

cf. Gauss's law in electromagnetism $Q = \int_V dV \, J_0 = \int_{\partial V} dS_\mu F^{0\mu}$

local energy can not be defined.

Plan of this talk

I. We propose a covariant and universal definition of (generalized) energy

by the volume integral of the EMT with some vector.

II. We evaluate the total energy of a compact star via our definition,

which is compared with the standard Misner-Sharp mass.

III. Implications.

I. Conserved (generalized) energy

S. Aoki, T. Onogi and S. Yokoyama, “Conserved charge in general relativity”,
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Our proposal

$$Q(\zeta) = \int_{\Sigma(x^0)} d\Sigma_0 \sqrt{-g} T^0{}_\nu \zeta^\nu$$

$\Sigma(x^0)$: a constant x^0 hypersurface

$d\Sigma_0$: a hypersurface element

where we require

$$T^\mu{}_\nu \nabla_\mu \zeta^\nu = 0$$

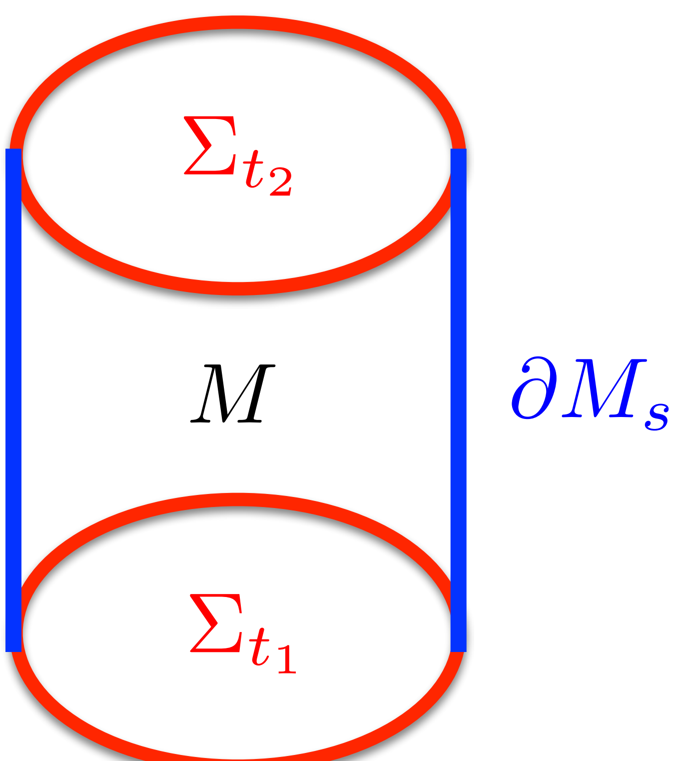
conserved current

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} J^\mu) = 0, \quad J^\mu := T^\mu{}_\nu \zeta^\nu$$

$$0 = \int_M d^d x \sqrt{-g} \nabla_\mu J^\mu = \int_M d^d x \, \partial_\mu (\sqrt{-g} J^\mu) = \int_{\partial M} d\Sigma_\mu \sqrt{-g} J^\mu$$

boundaries $\partial M = \partial M_s \oplus \Sigma_{t_2} \ominus \Sigma_{t_1}$

assume $d\Sigma_k J^k = 0$ on ∂M_s



covariant conservation law

$$Q(\zeta)\Big|_{x^0 = t_2} = Q(\zeta)\Big|_{x^0 = t_1}$$

1. Energy conservation by symmetry

Killing vector $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \longrightarrow \zeta^\mu = \xi^\mu \longrightarrow T^\mu{}_\nu \nabla_\mu \zeta^\nu = 0$

ex. stationary space time a metric $g_{\mu\nu}$ does not contain x^0 $\longrightarrow \xi^\mu = -\delta^\mu_0$

conserved energy $E = -\int_{\Sigma(x_0)} d\Sigma_0 \sqrt{-g}\, T^0{}_0$

Covariant and universal definition of total energy

Ex.1 Black hole mass — **Black hole in NOT a vacuum !**

Ex.2 Compact star — **See II.**

This definition has been known, but rarely used.

1. V. Fock, *TheTheory of Space, Time and Gravitation* (Pergamon Press, New York 1959)

2. A. Tautman, Kings Collage lecture notes on general relativity, mimeographed note (unpublished), May-June 1958; Gen. Res. Grav. **34** (2002), 721-762, cited Fock.

3. A. Tautman's lecture notes was cited by Komar in PRD127(1962)1411.

These were forgotten in major textbooks (e.g. Landau-Lifshitz) except a few.

4. R. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984), p.286, footnote 3.

See also lecture notes by Blau; Shiromizu (Japanese); Sekiguchi (Japanese).

2. Energy conservation without symmetry

$\xi^\mu = -\delta^\mu_0$ is not a Killing vector but the metric satisfies

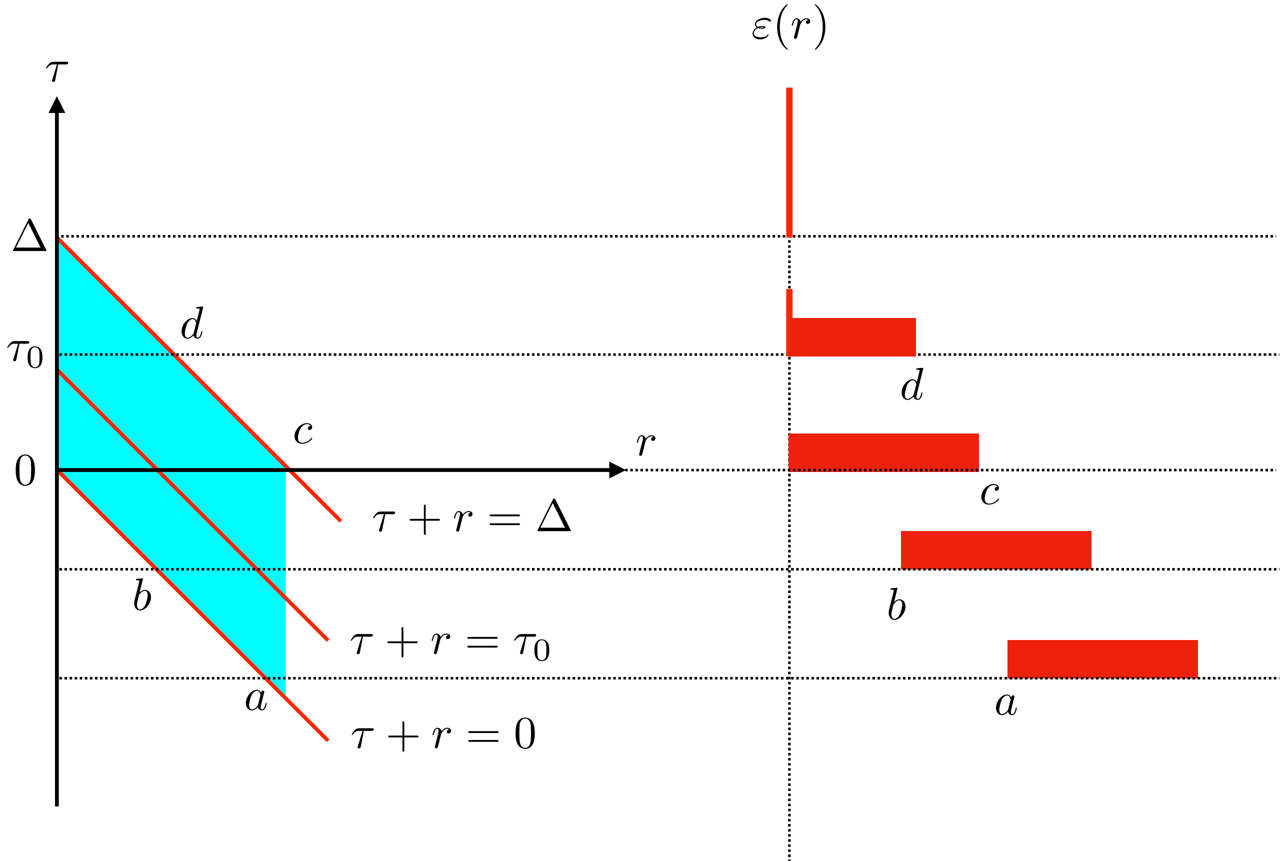
$$T^\mu{}_\nu \nabla_\mu \xi^\nu = -T^\mu{}_\nu \Gamma^\nu_{\mu 0} = 0$$

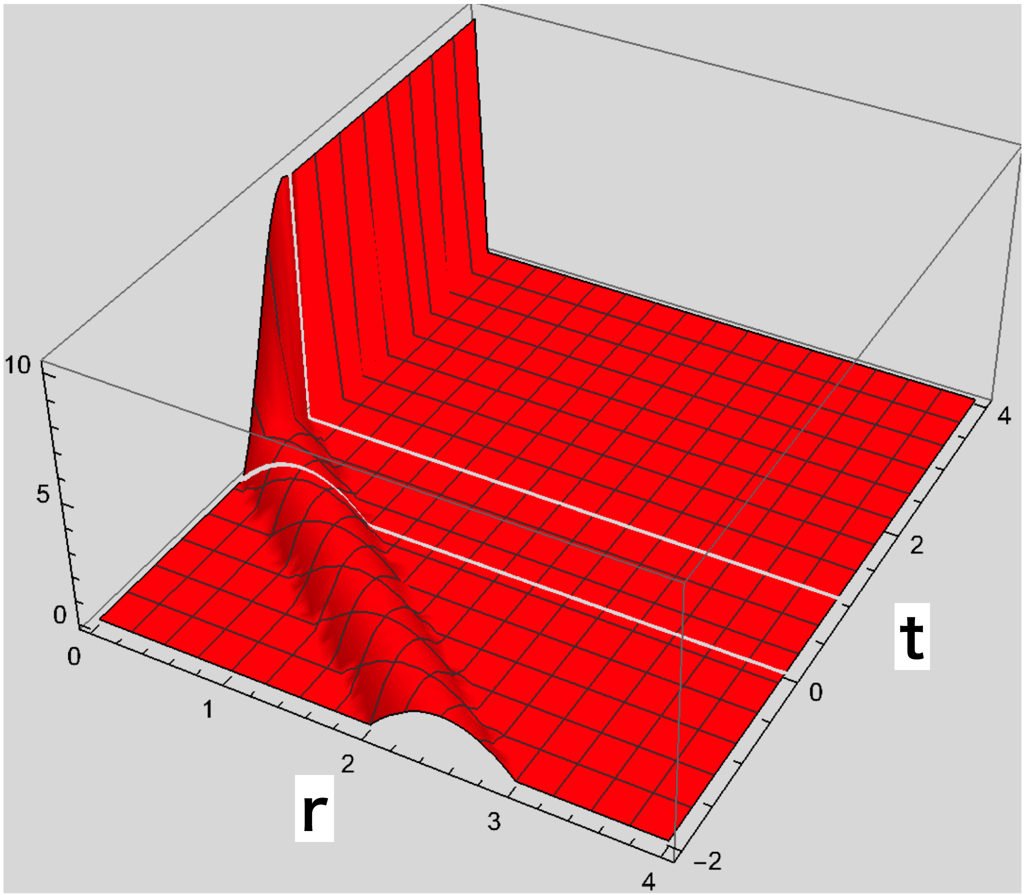
$$E = -\int_{\Sigma(x_0)} d\Sigma_0 \sqrt{-g}\, T^0{}_0$$

is a conserved energy without symmetry.

Ex.1 Exact gravitational plane wave

Ex.2 Gravitational collapse





3. Conserved charge without energy conservation

$T^{\mu}{}_{\nu}\nabla_{\mu}\xi^{\nu} = -T^{\mu}{}_{\nu}\Gamma^{\nu}_{\mu 0} \neq 0$ but a solution to $T^{\mu}{}_{\nu}\nabla_{\mu}\zeta^{\nu} = 0$ always exists.

For a spherically symmetric system, a solution is a Kodama vector

Ex.1 Homogeneous and Isotropic expanding (FLRW) Universe

$$\zeta^{\mu} = -\beta(x^0)\delta\delta^{\mu}_{0} \longrightarrow Q(\zeta) = \int_{\Sigma(x^0)} d\Sigma_0 \sqrt{-g}\, T^{0}{}_{\nu}\zeta^{\nu}$$ is conserved.

What is a physical meaning ?

Our interpretation

$S := Q(\zeta)$ is an *entropy* with a (time-dependent) inverse temperature β.

The entropy of the Universe is conserved during its expansion.

An entropy is a source of gravitational interaction.

II. Compact star

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Oppenheimer-Volkoff equation

stationary spherically symmetric metric

Oppenheimer-Volkoff, PR55(1939)374.

$$ds^2 = -f(r)(dx^0)^2 + h(r)dr^2 + r^2 \tilde{g}_{ij} dx^i dx^j$$

with perfect fluid $T^0{}_0 = -\rho(r), \quad T^r{}_r = P(r), \quad T^i_j = \delta^i_j P(r)$

Einstein equation → **Oppenheimer-Volkoff equation**

$$-\frac{dP(r)}{dr} = \frac{G_4 M(r)}{r^2} \left(P(r) + \rho(r)\right) h(r) \left(1 + 4\pi r^3 \frac{P(r)}{M(r)}\right)$$

with $\frac{1}{h(r)} = 1 - \frac{2G_4 M(r)}{r}$

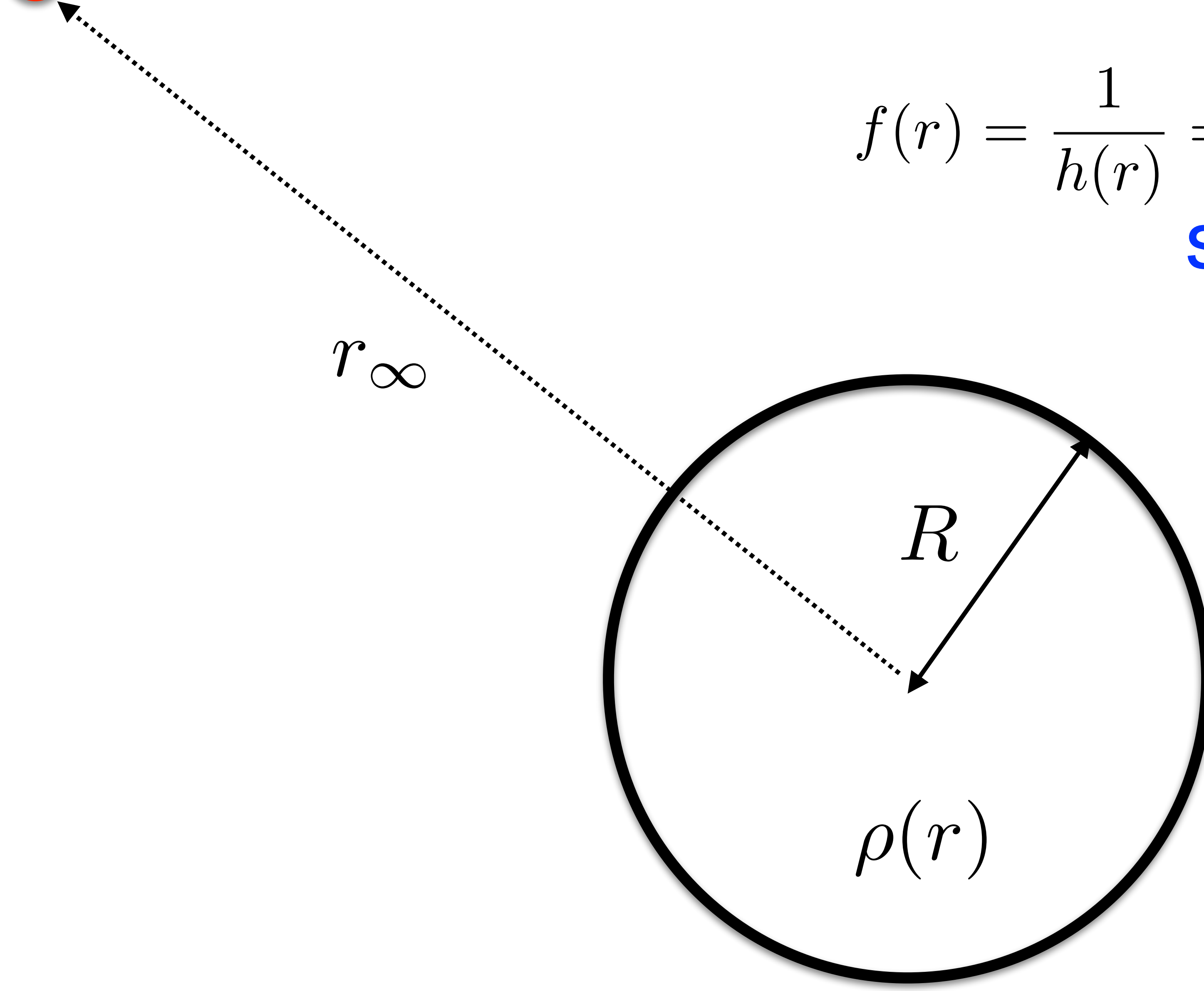
Misner-Sharp mass $M(r) = 4\pi \int_0^r ds\, s^2 \rho(s), \quad M(0) = 0$

EoS $P = P(\rho)$ → **solution to OV equation**

Schwarzschild metric with $m = M(R)$ **Misner-Sharp mass**

$$f(r) = \frac{1}{h(r)} = 1 - \frac{2G_4 M(R)}{r}$$

Schwarzschild metric



outside star

$$r > R, \quad \rho(r) = P(r) = 0$$

radius of compact star R from $P(r = R) = 0$

Energy of a compact star

conserved energy   Killing vector $\xi^\mu = -\delta^\mu_0$

$$E = -\int d^2x \int_0^\infty dr \sqrt{-g}\, T^0{}_0 = 4\pi \int_0^R dr \sqrt{f(r)h(r)} r^2 \rho(r)$$

Misner Sharp mass $M(R) = 4\pi \int_0^R dr\, r^2 \rho(r)$

$$E = M(R) - 4\pi G_4 \int_0^R dr \sqrt{f(r)h^3(r)}\, r M(r) \left(\rho(r) + P(r)\right)$$

observed mass   corrections due to a structure inside star $:= \Delta E$

Mass observed far from the star is NOT a total energy of the star.

Physical meaning of ΔE

Newtonian limit

$$P(r) \ll \rho(r)$$

$$\Delta E \simeq -4\pi G_4 \int_0^\infty dr\, r\, M(r)\rho(r) + \cdots$$

gravitational potential energy

$$U_4 := -\frac{G_4}{2}\int d^3x\, d^3y\, \frac{\rho(\mathbf{x})\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} = -4\pi G_4 \int_0^R dr\, r\, M(r)\rho(r)$$

correction term represents the gravitational interaction (potential) energy !

$$\longrightarrow \Delta E \leq 0$$

total energy $\quad E = M(R) + \Delta E \leq M(R) \quad$ Misner Sharp mass

Interpretations in standard textbooks

Total energy = Misner Sharp mass

$$M(R) = \epsilon(R) + \underbrace{M(R) - \epsilon(R)}_{:=\Delta M(R)}$$

$\epsilon(R)$: static mass + internal energy

$\Delta M(R)$: gravitational potential energy

Def. 1 Weinberg, Schutz

$$\epsilon_1(R) = \int d^3x \sqrt{-g_4(x)}\rho(x) = 4\pi \int_0^R dr\, r^2 \sqrt{f(r)h(r)}\rho(r) = M(R) + \Delta E$$

$$\longrightarrow \quad \Delta M(R) = -\Delta E \geq 0$$

positive, thus incorrect !

Def. 2 Misner-Thorne-Wheeler, Hawking-Ellis, Wald

$$\epsilon_2(R) = \int d^3x \sqrt{g_3(x)}\rho(x) = 4\pi \int_0^R dr\, r^2 \sqrt{h(r)}\rho(r)$$

Newtonian limit $\Delta M(R) \simeq -4\pi G_4 \int_0^R dr\, r M(r)\rho(r) \simeq \Delta E \leq 0$ negative, OK

but $M(R) = 4\pi \int_0^R dr\, r^2 \rho(r)$ **even in Newtonian limit**

Interpretations in standard textbooks are physically inadequate.

Size of gravitational interaction energy

constant density $\rho(r) = \rho_0$ $M_{\rm BH} = \frac{4\pi R^3 \rho_0}{3}$ fixed ⇒ $R \geq R_{\rm min} = \frac{9GM_{\rm BH}}{4}$

⇒ $$E = M_{\rm BH} - \pi\rho_0 \left[R(3r_0^2 - R^2) - 3r_0^2 \sqrt{r_0^2 - R^2} \sin^{-1}\left(\frac{R}{r_0}\right) \right]$$ $r_0^2 := \frac{3}{8\pi G_4 \rho_0}$

interaction energy $\simeq -68\%$ of $M_{\rm BH}$ at $R = R_{\rm min}$

Gravitational interaction energy could be large !

A neutron star may have a much smaller total energy than the observed mass.

$E_{\rm NS} \simeq \frac{2}{3} M_{\rm sun}$ $M_{\rm NS} \simeq 2M_{\rm sun}$

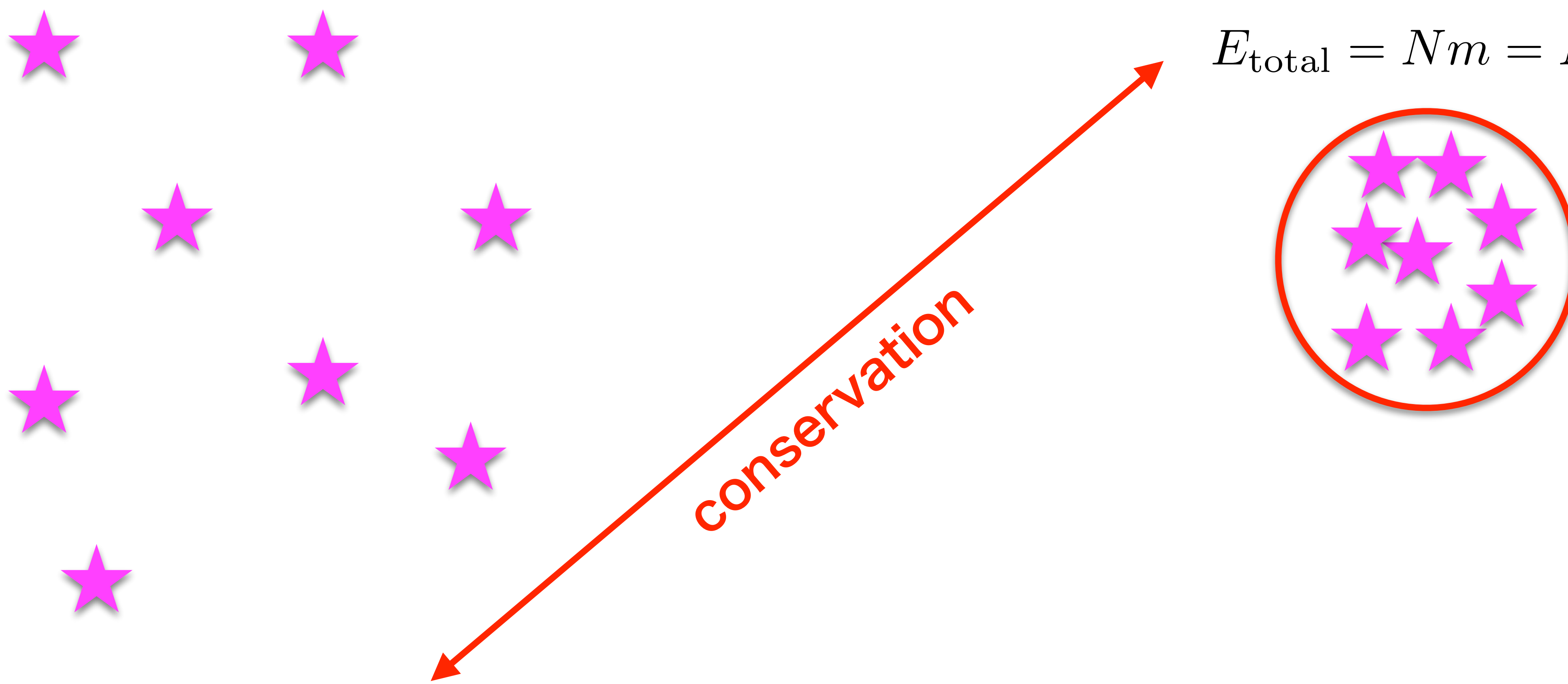
Please calculate $E(R)$ and $M(R)$ with your EOS.

III. Implications

Compact star **Total energy** $E \ll M$ **Misner Sharp mass**

Dark matters may change the current estimate for an amount of dark matter.

$$E_{\rm total} = Nm = \tilde{M}_{\rm total} + \Delta E$$



$$E_{\rm total} = M_{\rm total} = Nm$$ **Misner Sharp mass** $$\tilde{M}_{\rm total} = Nm - \Delta E \gg Nm$$

Quantitative analysis will be called for.

correct understanding of general relativity

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G_d T_{\mu\nu}$$

after 106 years from Einstein

Energy/entropy(?) is a source of the gravitational field.

Matters carry it but gravitational fields do not.

cf. Photon/gluon fields have no electric/color charges.

Aoki-Onogi (in preparation)

A total energy/entropy in the whole system is always conserved, as nothing can escape from a censorship of gravity.

天網恢恢疎にして漏らさず。

Gravity may provide a new tool to define entropy and temperature in an arbitrary system.

Future

binary stars How do they loose “energy” ?

Aoki-Onogi (work in progress)

Quantum gravity

Is it necessary to quantize gravity ?

Gravitational fields classically have no energy/entropy.

No exchange of energy/entropy between matters and gravitational fields.

If necessary, how can we quantize gravitons with no observed energy/entropy ?

若い学生や研究者の方へ

今、一般相対性理論に関して歴史的な仕事をするチャンスが目の前にあります。

過去のしがらみに囚われず、どんどん挑戦してください。

興味のある方は連絡をください。一緒に仕事をしましょう。