Gradient flow, the renormalization group beta function and further applications

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Challenges and opportunities in Lattice QCD simulations and related fields
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QCD gradient flow [Narayanan, Neuberger JHEP 0603(2006)064] [Lüscher CMP 293(2010)899][JHEP 1008(2010)071]

$$\frac{\partial}{\partial t} B_\mu(t) = \mathcal{D}_\nu(t) G_{\nu\mu}(t) \qquad \text{with} \qquad B_\mu(t=0) = A_\mu$$

- Ordinary differential equation (ODE)
- Covariant derivative defined in terms of the flow field $B_\mu(t)$ and the Yang-Mills action $S_{YM}(B)$

Lattice gradient (gauge) flow

$$\frac{\partial}{\partial t} V_t(x,\mu) = -g_0^2 \left\{\partial_{x,\mu} S_{YM}\right\} V_t(x,\mu) \quad \text{with} \quad V_t(x,\mu)\Big|_{t=0} = U(x,\mu)$$

Lattice gradient (gauge) flow

- Wilson flow

$$S_W(U) = \frac{1}{g_0^2} \sum_P \mathrm{Re\,Tr}\,\{1 - U(P)\}$$

- Symanzik flow

$$S_S(U) = \frac{1}{g_0^2} \left[\frac{5}{3} \sum_P \mathrm{Re\,Tr}\,\{1 - U(P)\} - \frac{1}{12} \sum_R \mathrm{Re\,Tr}\,\{1 - U(R)\} \right]$$

- Zeuthen flow [Ramos, Sint PoS Lattice2014 (2015) 329][EPJC 76(2016)15]
  $\rightarrow$ Symanzik flow with two additional terms to build an $O(a^2)$ improved gradient flow

Gradient flow scale setting

- Define gradient flow renormalized coupling using the energy density E

$$g^2_{GF}(t) = \mathcal{N} t^2 \langle E(t) \rangle$$

→ $t^2\langle E(t)\rangle$ is dimensionless ⇒ t has mass dimension -2
→ Normalization to match tree-level $\overline{\text{MS}}$: $\mathcal{N} = 128\pi^2/(3N_c^2 - 3)$
→ Estimate $E(t)$ using plaquette, Symanzik, clover, . . . operator

- Determine gradient flow scale t_0 [Lüscher JHEP 1008(2010)071]

$$\left\{ t^2 \langle E(t) \rangle \right\}_{t=t_0} = 0.3$$

- Alternative gradient flow scale w_0 [Borsanyi et al. JHEP09 (2012)010]

$$\left\{ t \frac{d}{dt} \left[t^2 \langle E(t) \rangle \right] \right\}_{t=w_0} = 0.3$$

Zero-mode correction [Fodor et al. JHEP 11 (2012) 007]

- Numerical gradient flow calculations are performed in finite volume
  - → Non-Gaussian gauge zero modes are present and need to be treated exactly
  - → Integrate out using perturbation theory

$$\delta = -\frac{64t^2\pi^2}{3L^4} + \vartheta\left(\exp\left(-\frac{L^2}{8t}\right)\right)$$

with ϑ Jacobi elliptic function and L^4 volumes

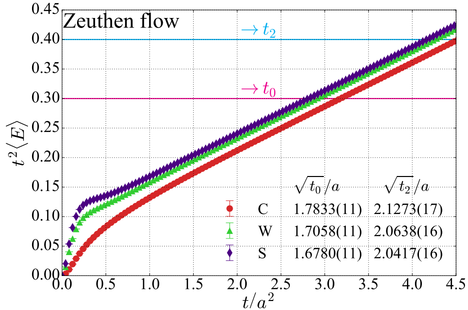
Tree-level normalization [Fodor et al. JHEP 09 (2014) 018]

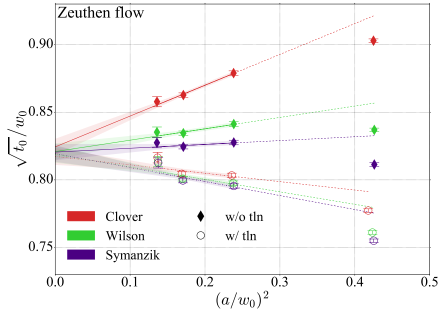
- Reduce discretization effects by calculating perturbative improvements of the gradient flow
  - → At tree-level no fermion effects are present
  - → Calculate coefficients in lattice perturbation theory for a given gauge action, gradient flow, operator to measure $E(t)$

$$C(a^2/t) = 1 + \sum_{m=1}^{\infty} C_{2m}\frac{a^{2m}}{t^m}$$

Gradient flow scales

[Schneider, Hasenfratz, OW arXiv:2211.12406]





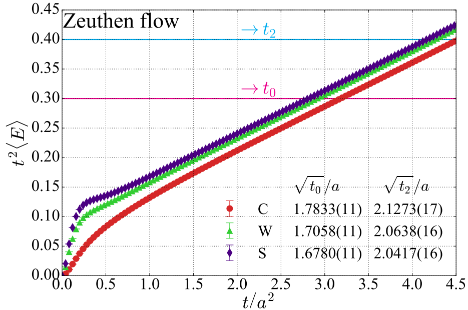
- $\sqrt{t_0}/a$ and w_0/a are lattice scale and have a continuum limit
  - $\to$ Use other scales/hadronic quantities to get continuum limit $\sqrt{t_0}$ or w_0 in physical units

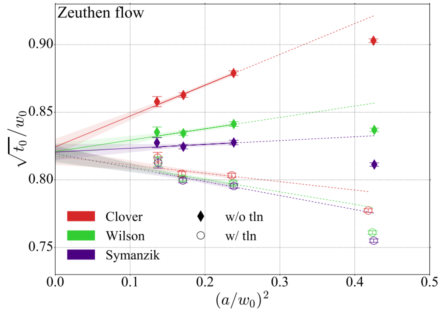
    [FLAG] (2+1) $\sqrt{t_0} = 0.14464(87)$ fm [RBC/UKQCD][BMWc][CLS]

    $w_0 = 0.17355(92)$ fm [RBC/UKQCD][BMWc][HotQCD]
  - $\to$ "Shortcut" to turn $\sqrt{t_0}/a$ or w_0 into determinations of a

Gradient flow scales

[Schneider, Hasenfratz, OW arXiv:2211.12406]





- $\sqrt{t_0}/a$ and w_0/a are lattice scale and have a continuum limit
- RBC/UKQCD's set of 2+1 flavor Shamir domain-wall and Iwasaki gauge field ensembles
  [PRD 78 (2008) 114509] [PRD 83 (2011) 074508] [JHEP 1712 (2017) 008] [PRD 93 (2016) 074505]
  → Coarse ensembles not described by a linear a^2 ansatz
  → Zeuthen flow with Symanzik operators has smallest cutoff effects

Renormalization Group β function

$$\beta(g^2) = \mu^2 \frac{dg^2}{d\mu^2}$$

- Encodes dependence of coupling g^2 on the energy scale μ^2
- β has no explicit dependence on μ^2, only implicit through $g^2(\mu)$
- Known perturbatively up to 5-loop order in the $\overline{\text{MS}}$ scheme (1- and 2-loop are universal)
  [Baikov, Chetyrkin, Kühn PRL118(2017)082002] [Ryttov and Shrock PRD94(2016)105015]
- Known perturbatively at 3-loop order in the GF scheme [Harlander, Neumann JHEP06(2016)161]
- Perturbative predictions reliable at weak coupling,
  nonperturbative methods needed for strong coupling

Step-Scaling β function

- Discretized β function determined using numerical lattice field theory calculations [Lüscher et al. NPB359(1991)221]
  - → Choose symmetric L^4 setup where the size L of the lattice is the only scale
  - → Determine β function by calculating scale change $L \to s \cdot L$
- Define gradient flow renormalized coupling for scheme c

$$g_c^2(L) = \frac{128\pi^2}{3(N_c^2 - 1)} \frac{1}{C(c,L)} t^2 \langle E(t) \rangle$$

  - → Relate flow time t to scale L: $\sqrt{8t} = c \cdot L$ [Fodor et al. JHEP11(2012)007]
  - → $C(c,L)$ tree-level improvement coefficient [Fodor et al. JHEP09(2014)018]
  - → Calculate scale difference

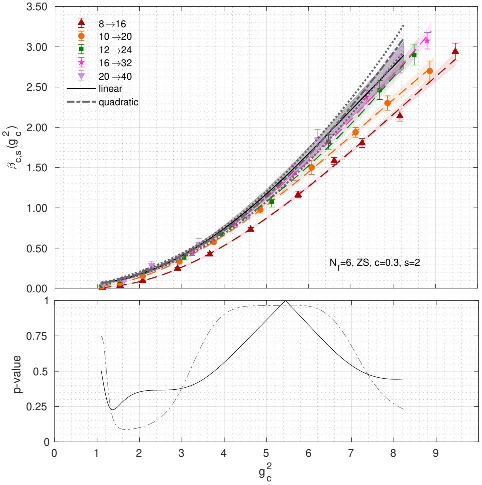
$$\beta_s^c(g_c^2; L) = \frac{g_c^2(sL) - g_c^2(L)}{\log(s^2)}$$

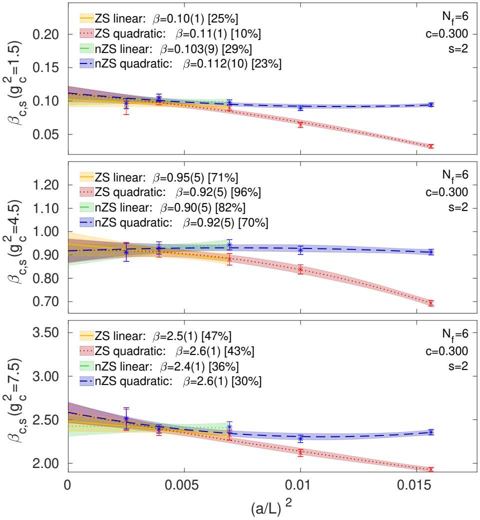
  - → Extrapolate $L \to \infty$ to remove discretization effects and take the continuum limit

Details SU(3) with $N_f = 6$ fundamental flavors

▸ Nf=4 ▸ Nf=8 ▸ Nf=10 ▸ Nf=12

[Hasenfratz, Rebbi, OW, PRD 106(2022)114509]





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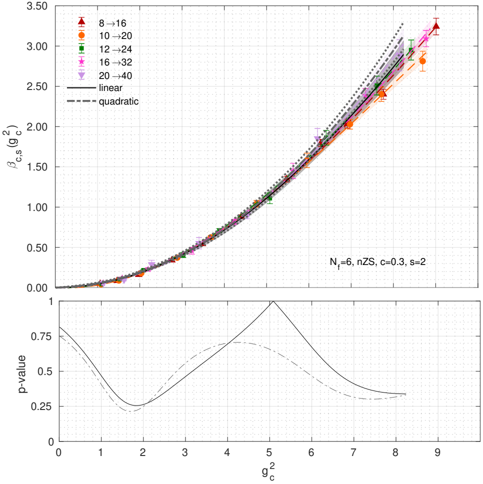
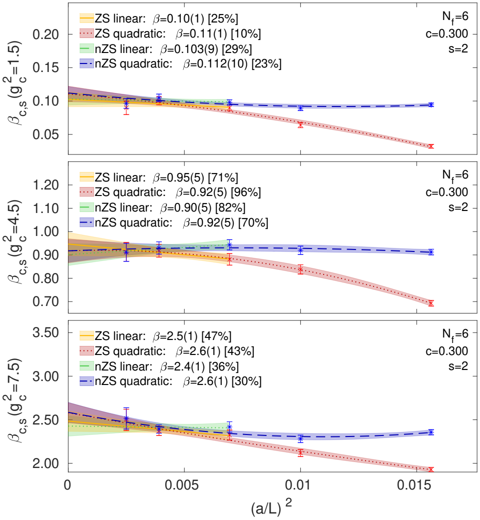
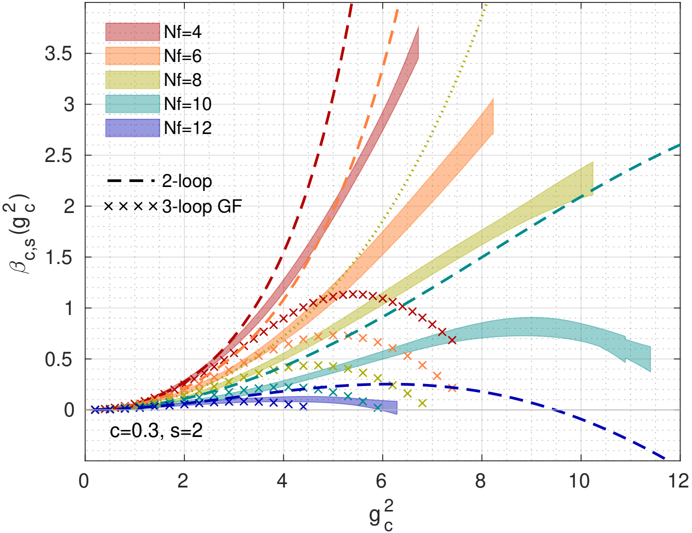


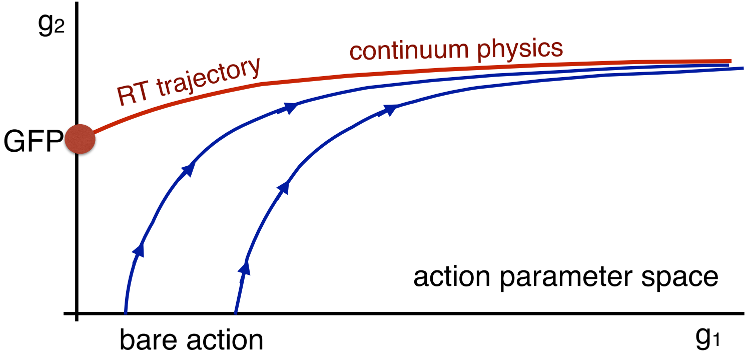


SU(3) with N_f fundamental flavors



[Hasenfratz, Rebbi, OW PLB 798(2019)134937][PRD 100(2019)114508][PRD 101(2020)114508][PRD 106(2022)114509][arXiv:2210.16760]

Beyond Step-Scaling: real-space Renormalization Group (RG) flow



Beyond Step-Scaling: real-space Renormalization Group (RG) flow

- RG flow: change of (bare) parameters and coarse graining (blocking)

- Gradient flow is a continuous transformation
  → Define real-space RG blocked quantities by incorporating coarse graining as part of calculating expectation values [Carosso, Hasenfratz, Neil PRL 121 (2018) 201601]

- Relate GF time t/a^2 to RG scale change $b \propto \sqrt{t/a^2}$
  → Quantities at flow time t/a^2 describe physical quantities at energy scale $\mu \propto 1/\sqrt{t}$
  → Local operator with non-vanishing expectation value can be used to define running coupling
  ⇝ Simplest choice: $t^2\langle E(t)\rangle$ [Lüscher JHEP 1008 (2010) 071]

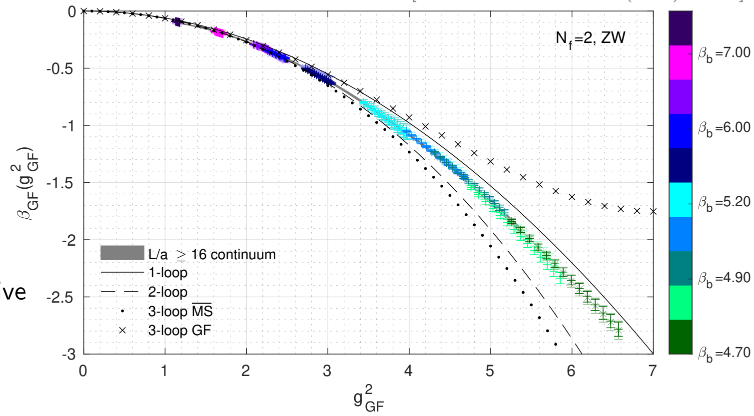
- Continuous RG β function

$$\beta(g^2_{GF}) = \mu^2 \frac{dg^2_{GF}}{d\mu^2} = -t\frac{dg^2_{GF}}{dt}$$

Example: QCD

- QCD: SU(3) gauge theory with two light flavors in the fundamental representation
- Fast "running" coupling
  ⇝ Confinement
- Plot: Comparison of nonperturbative and perturbative determinations

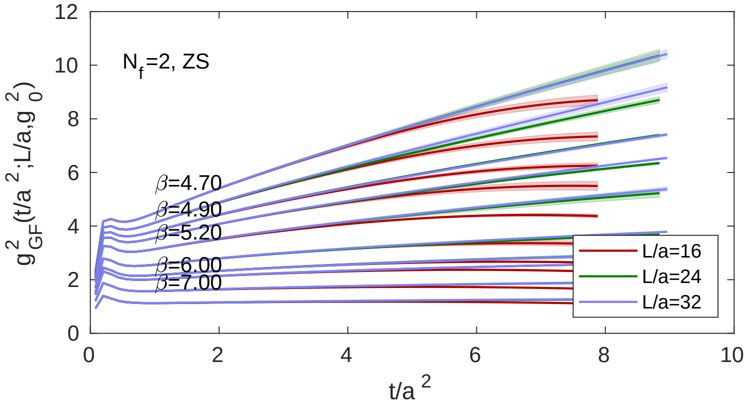
[Hasenfratz, OW PRD101(2020)034514]



$t^2\langle E(t)\rangle$ vs. t

- Determine g^2_{GF} for each ensemble and at all flow times t

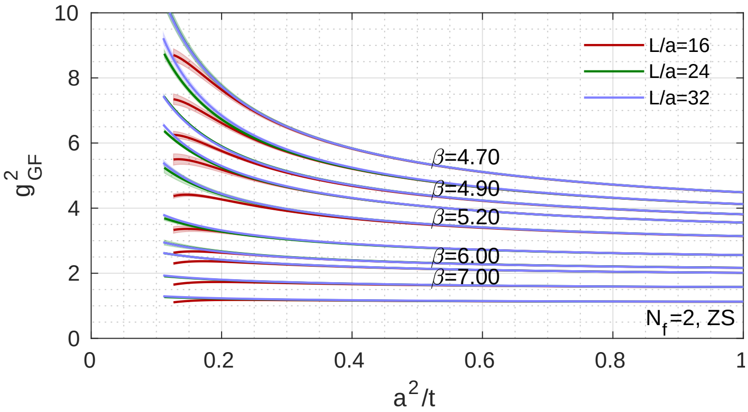
[Hasenfratz, OW PRD101(2020)034514]



$t^2\langle E(t)\rangle$ vs. $1/t$

- More instructive: plot g^2_{GF} vs. inverse flow times $1/t$

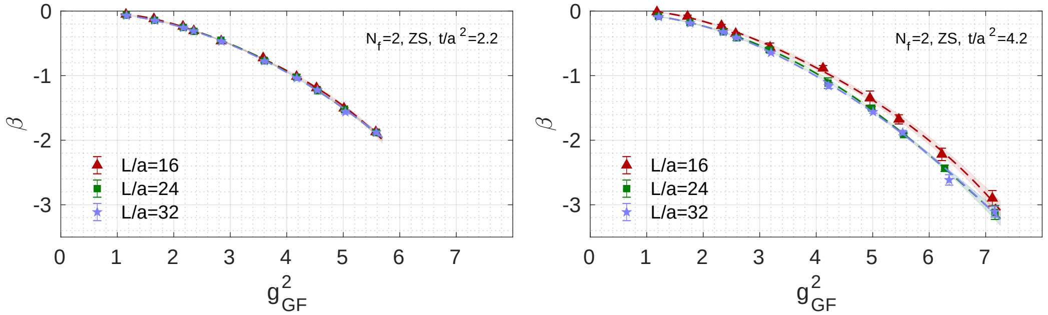
[Hasenfratz, OW PRD101(2020)034514]



β function at fixed flow time and volume

- Numerically calculate the derivative of g^2_{GF} in the flow time t to obtain $\beta(g^2_{GF}, L, t)$
- Polynomially interpolate $\beta(g^2_{GF}, L, t)$ in g^2_{GF} for each L and t

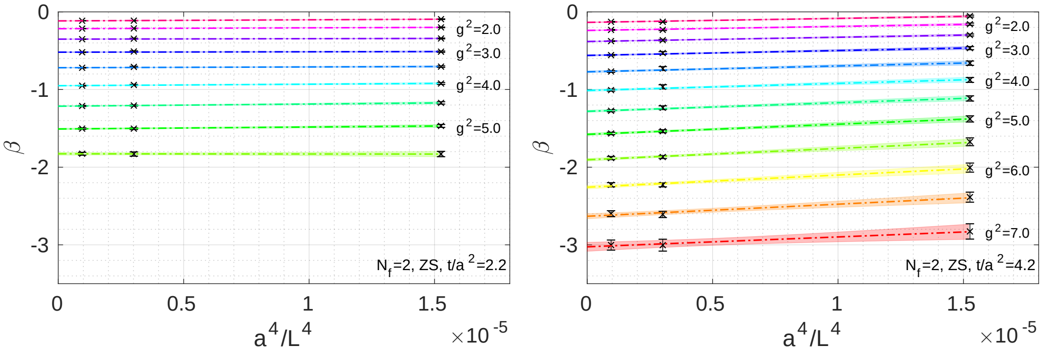
[Hasenfratz, OW PRD101(2020)034514]



Infinite volume extrapolation at fixed flow time

- Infinite volume extrapolation in (a^4/L^4) for fixed g^2_{GF} and flow time t
  - $\rightarrow$ Check for systematic effects, remove e.g. small volumes

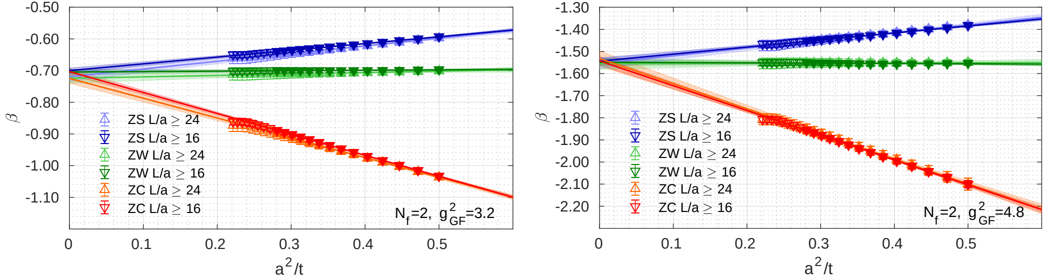
[Hasenfratz, OW PRD101(2020)034514]



$1/t$ continuum limit extrapolation at fixed coupling g^2_{GF}

- Fit range
  - → Small t (initial rise) polluted by UV effects
  - → Large t subject to finite volume effects
  - → Fit range depends on used volumes
  - → Compromise between range in g^2_{GF} and statistical uncertainties

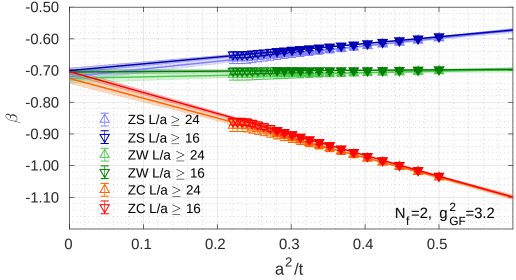
[Hasenfratz, OW PRD101(2020)034514]

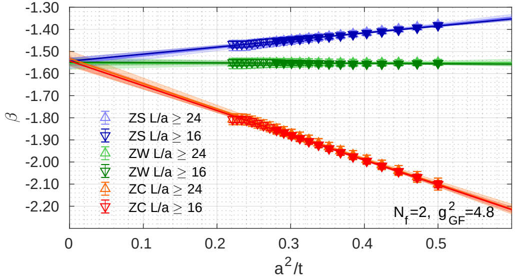


$1/t$ continuum limit extrapolation at fixed coupling g^2_{GF}

- Different operators (S, W, C) show different discretization effects but are expected to agree in the continuum
- Check for systematic effects
  - → Vary fit range (filled symbols)
  - → Use data from different continuum extrapolations ($L/a \geq 24$, $L/a \geq 16$)

[Hasenfratz, OW PRD101(2020)034514]

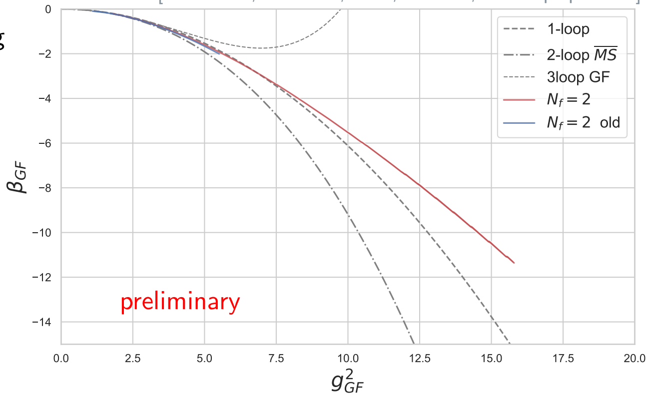




Example: QCD

- Extended data to stronger coupling
  → Confining region, $g^2_{\text{GF}} \sim 16$
- At strong coupling RG β function is highly linear
  → Nonperturbative phenomenon
- Integrate β function to obtain Λ_{GF}
  → g^2_m GF renormalized coupling at energy scale $\mu = 1/\sqrt{8t_0}$

[Hasenfratz, Monahan, Rizik, Shindler, OW in preparation]



$$\Lambda_{GF} = \mu\left(b_0 g_m^2\right)^{-\frac{b_1}{2b_0^2}} \exp\left(-\frac{1}{2b_0 g_m^2}\right) \exp\left[-\int_0^{g_m^2} \mathrm{d}g^2 \left(\frac{1}{\beta(g^2)} + \frac{1}{b_0 g^4} - \frac{b_1}{b_0^2 g^2}\right)\right]$$

⇒ $\Lambda^{\text{prelim}}_{\overline{\text{MS}}} = 326(13)$ MeV (stat. error only) compare to ALPHA: f_K: 310(20) MeV

[Fritzsch et al. NPB865(2012)397]

Summary

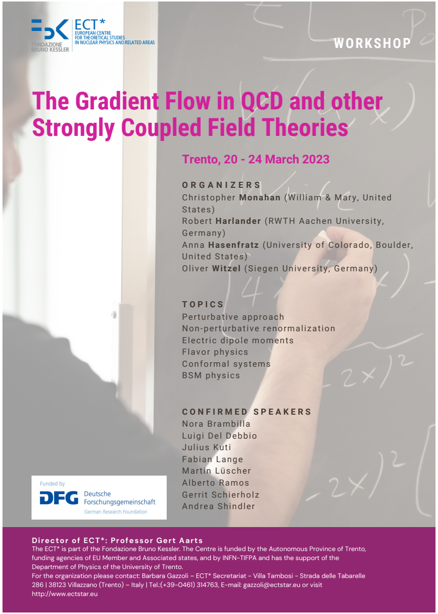
- Gradient flow default method for scale setting
  - → Useful to study systematic effects
    for specific choices of gauge action, gradient flow, and operator $(E(t))$
- Gradient flow can be used to determine the RG β function
  - → Step-scaling function
  - → Continuous β function
- Further applications
  - → Obtain Λ parameter by integrating β function from hadronic t_0 scale
    [Wong et al. 2301.06611] [Hasenfratz, Peterson, van Sickle, OW in preparation]
  - → Anomalous dimensions [Hasenfratz, Monahan, Rizik, Shindler, OW, PoS Lattice2021 (2022) 155]
  - → Gradient flow as renormalization scheme [Suzuki et al. PRD 102 (2020) 034508], ...
  - → Short-flow-time expansion [Lange, Harlander PRD 105 (2022) L071504], ...

Workshop

ECT*, Trento, Italy

March 20-24, 2023

https://indico.ectstar.eu/event/164/



Resources

XSEED: stampede2 (TACC)
NERSC: cori
USQCD: qcd16p/18p (Jlab), sdcc (BNL)
lq (Fermilab)
U Colorado: summit
BU: engaging and scc (MGHPCC)
Siegen U: omni

Software

GRID [Boyle et al. PoS Lattice2015 023]
Qlua [Pochinsky PoS Lattice2008 040]

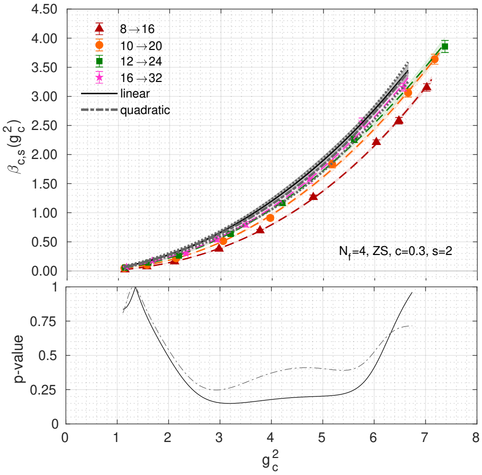
Thank you

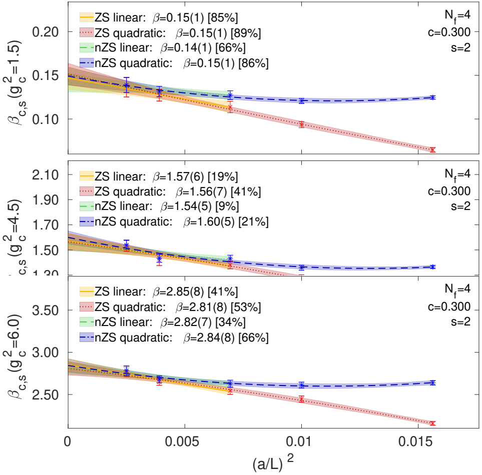
Anna Hasenfratz, Christian Schneider, Claudio Rebbi
Chris Monahan, Matthew Rizik, Andrea Shindler

Details SU(3) with $N_f = 4$ fundamental flavors

▸ Nf=6 ▸ Nf=8 ▸ Nf=10 ▸ Nf=12

[Hasenfratz, Rebbi, OW, PRD 106(2022)114509]





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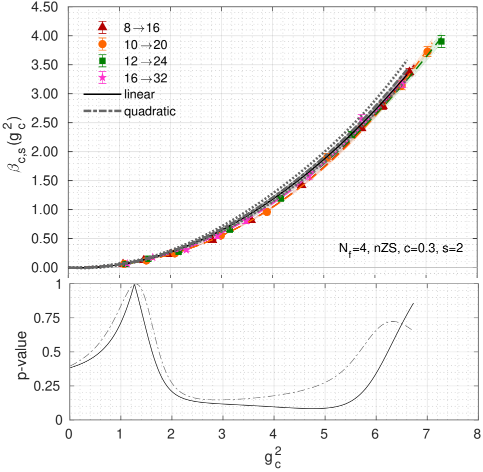
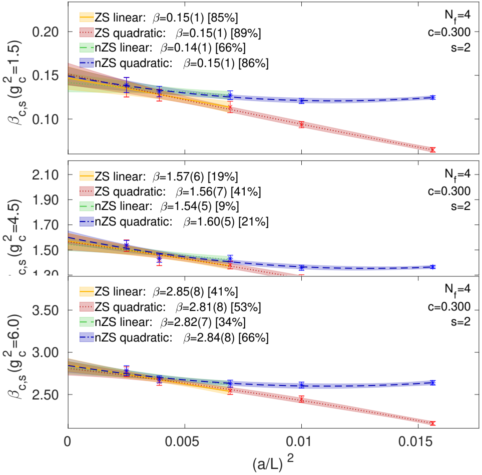
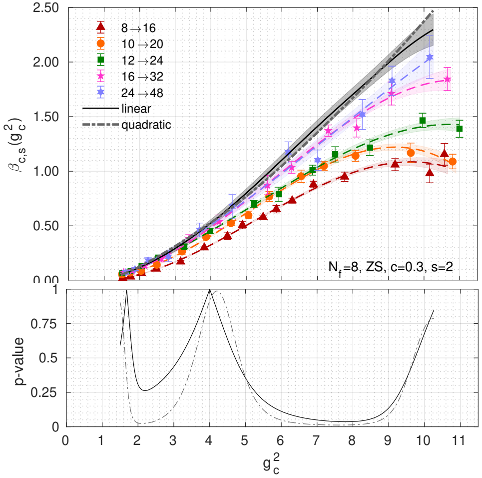


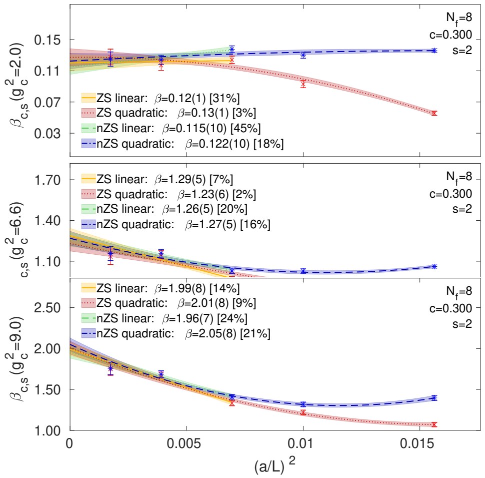


Details SU(3) with $N_f = 8$ fundamental flavors

▸ Nf=4 ▸ Nf=6 ▸ Nf=10 ▸ Nf=12

[arXiv:2210.16760]





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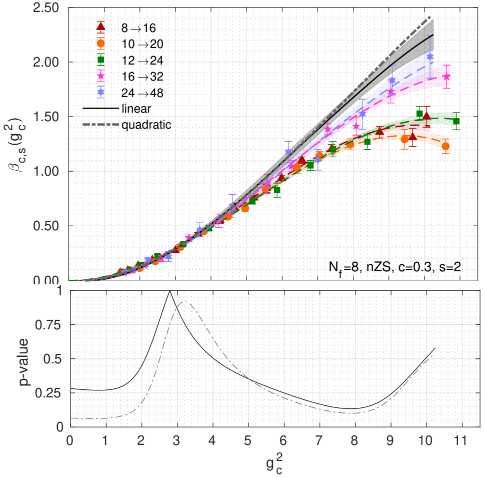
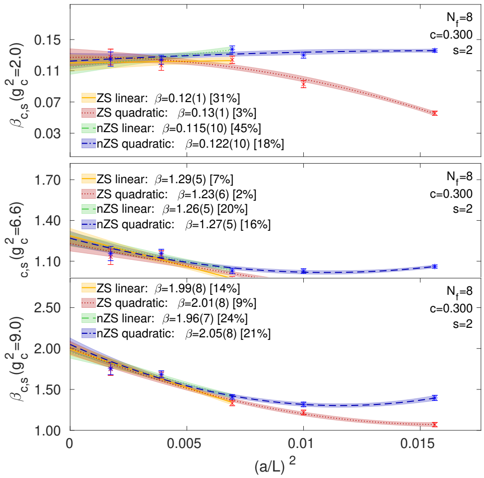
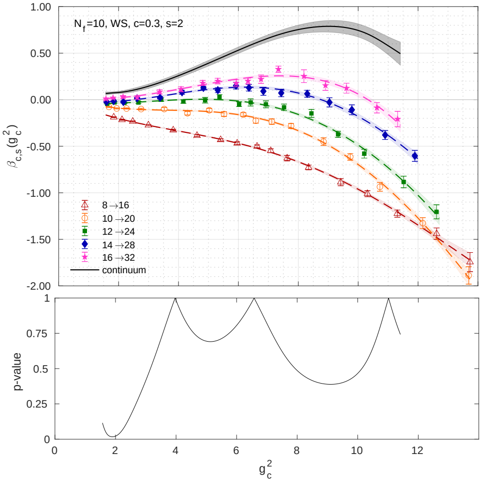


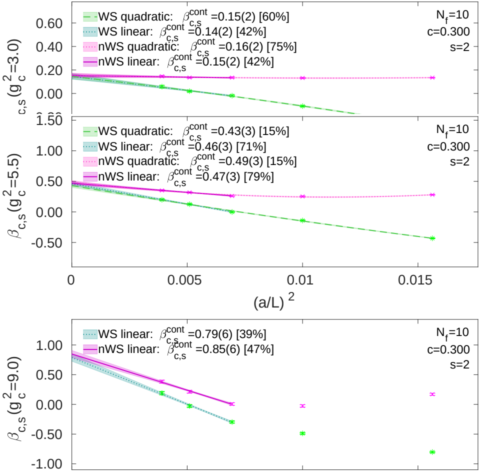


Details SU(3) with $N_f = 10$ fundamental flavors

▸ Nf=4 ▸ Nf=6 ▸ Nf=8 ▸ Nf=12

[Hasenfratz, Rebbi, OW PLB 798(2019)134937][PRD 101(2020)114508]





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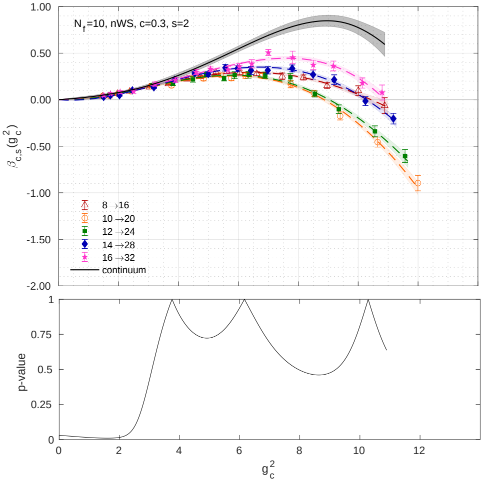
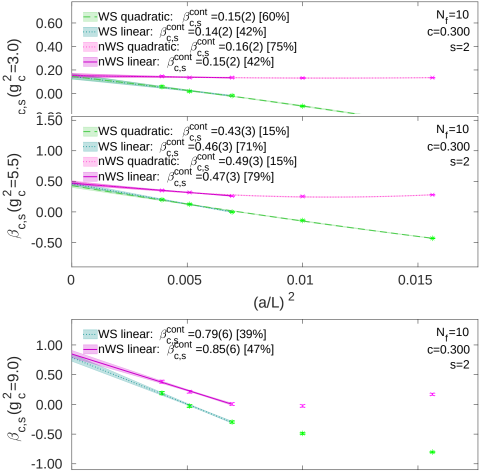
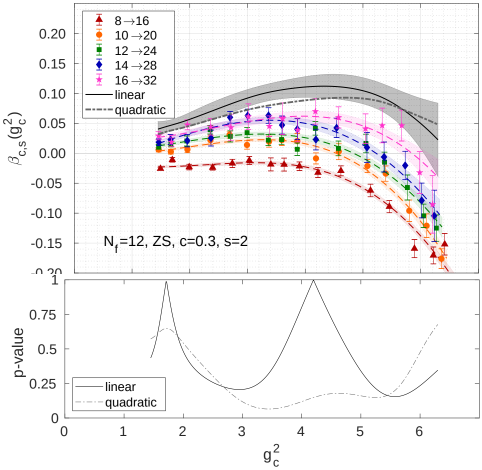


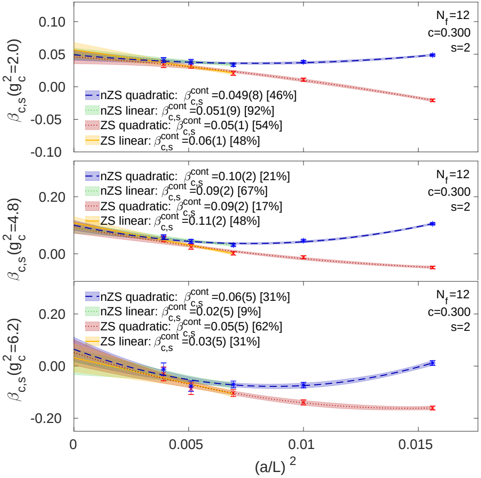


Details SU(3) with $N_f = 12$ fundamental flavors

▸ Nf=4 ▸ Nf=6 ▸ Nf=8 ▸ Nf=10

[Hasenfratz, Rebbi, OW PLB 798(2019)134937][PRD 100(2019)114508]





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