Tensor renormalization group approach to higher-dimensional lattice field theories

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Introduction to TRG approach

Advantages of the TRG approach

✔ Tensor renormalization group (TRG) is a deterministic numerical method based on the idea of real-space renormalization group.

- **No sign problem**
- **The computational cost scales logarithmically w. r. t. system size**
- **Direct evaluation of the Grassmann integrals**
- **Direct evaluation of the path integral**

✔ **Applicability to the higher-dimensional systems**

TRG is a kind of tensor-network method and its application to higher-dimensional systems has recently made remarkable progress.

Lagrangian (TRG) approach: Meurice+, Rev. Mod. Phys. 94(2022)025005

Hamiltonian (TNS) approach: Bañuls-Cichy, Rep. Prog. Phys. 83(2020)024401

→ toward the 2d quantum systems: PEPS, Tree TN, isoTNS, Fermionic isoTNS…

Procedure of TRG approach

1) Write down the target function X defined on lattice as a tensor contraction (tensor network).

ex. Partition function, Path integral, …

2) Approximately perform the tensor contraction with TRG.

1) **TN representation for $\boldsymbol{X}$** : (# of tensors in TN) = (# of lattice sites)

$$X \rightarrow \Sigma_{abcd\cdots} T_{aiw\cdots} T_{bjx\cdots} T_{cky\cdots} T_{dlz\cdots} \cdots$$

2) **TRG** : Block-spin trans. for T to reduce # of tensors in TN

$$\approx \Sigma_{a'b'c'd'\cdots} T'_{a'i'w'\cdots} T'_{b'j'x'\cdots} T'_{c'k'y'\cdots} T'_{d'l'z'\cdots} \cdots$$

TN rep. for 2d Ising model w/ PBC

Decompose nearest-neighbor interactions

$$Z = \Sigma_{\{\sigma=\pm1\}}\Pi_{n,\mu}\exp\left[\beta J\sigma_n\sigma_{n+\hat{\mu}}\right] \quad \Rightarrow \quad Z = \mathrm{Tr}\left[\Pi_n T_{x_n y_n x'_n y'_n}\right]$$

$T_{x_n y_n x'_n y'_n}$ specifies the details of the model

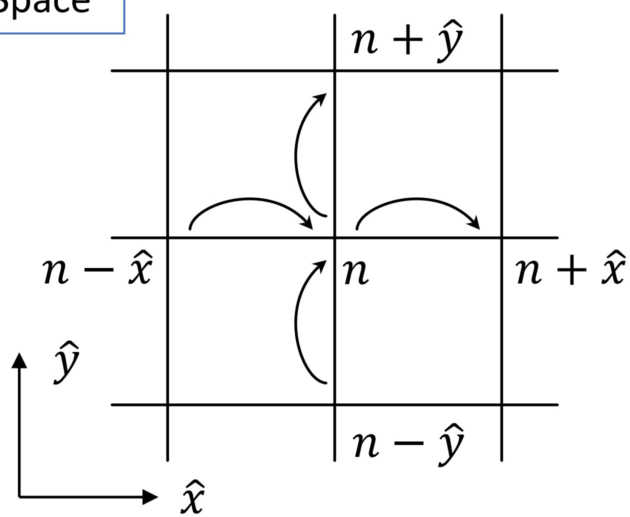
$$\exp\left[\beta J\sigma_n\sigma_{n+\hat{\mu}}\right] = \sum_{l_n}\sqrt{\lambda_{l_n}}U(\sigma_n,l_n)\sqrt{\lambda_{l_n}}U(\sigma_{n+\hat{\mu}},l_n) = \sum_{l_n}W(\sigma_n,l_n)W(\sigma_{n+\hat{\mu}},l_n)$$

$$W(a,b) \coloneqq \sqrt{\lambda_b}U(a,b)$$

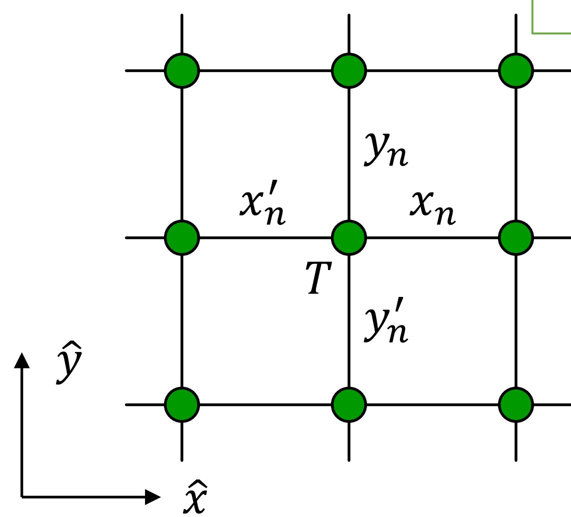
$$T_{x_n y_n x'_n y'_n} \coloneqq \sum_{\sigma_n=\pm1} W(\sigma_n,x_n)W(\sigma_n,y_n)W(\sigma_n,x'_n)W(\sigma_n,y'_n)$$

$$x'_n \coloneqq x_{n-\hat{x}},\ y'_n \coloneqq y_{n-\hat{y}}$$

Real Space

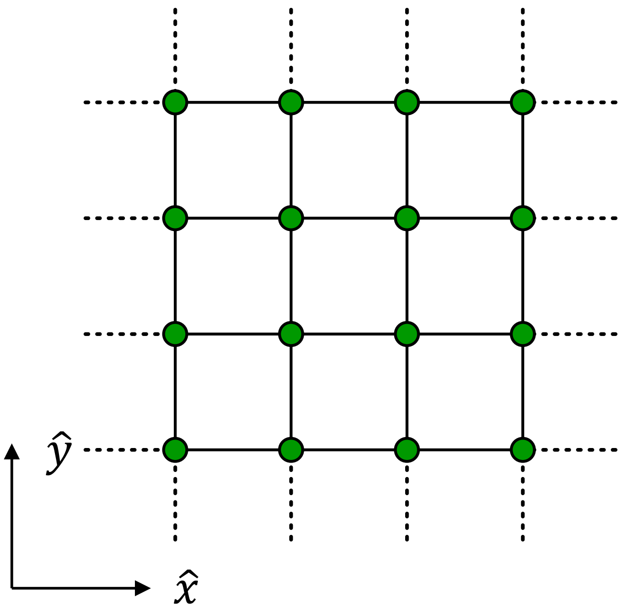


TN rep. for Z



Basic concept of TRG algorithm

We cannot perform the contractions in TN rep. exactly (too many d. o. f.)



Idea of real-space renormalization group

Iterate a simple transformation **w/ approximation** and we can investigate thermodynamic properties

+

Information compression w/ the Singular Value Decomposition (SVD)

$$A_{ij} = \Sigma_k U_{ik}\sigma_k V_{jk} \approx \Sigma_{k=1}^{D} U_{ik}\sigma_k V_{jk}$$

$$\text{w/}\quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)} \geq 0$$

(A: $m\times n$ matrix, U: $m\times m$ unitary, V: $n\times n$ unitary)

↓

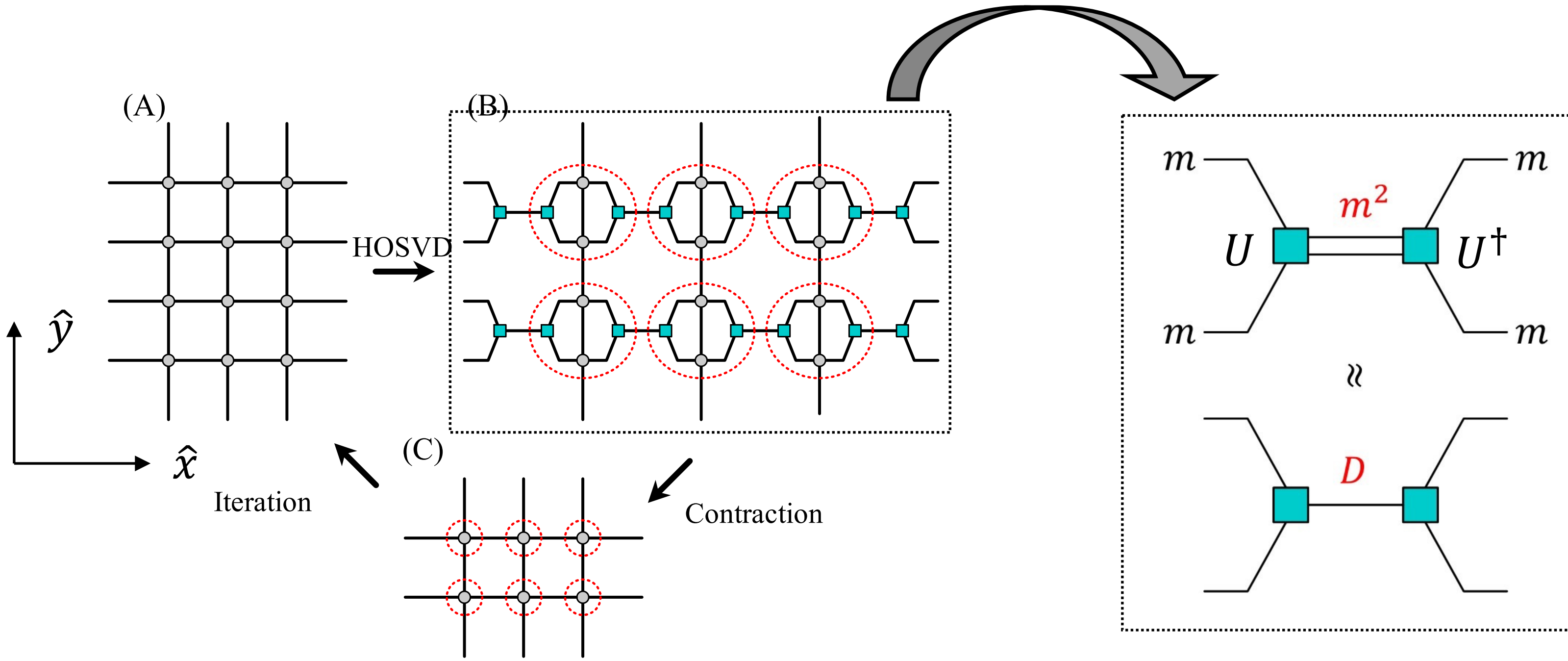
TRG employs the SVD to reduce d. o. f. and perform the tensor contraction approximately

Higher-order TRG (= TRG w/ isometry insertion)

Xie+, PRB86(2012)045139

Cf. See poster by Xiao Luo

✔ Applicable to any d-dimensional lattice



Sequential coarse-graining along with each direction

$\boldsymbol{D}$: **bond dimension**

✔ # of tensors are reduced to half.
Iterating this CG $\boldsymbol{n}$ times, we can approximately contract $\boldsymbol{2^n}$ tensors.

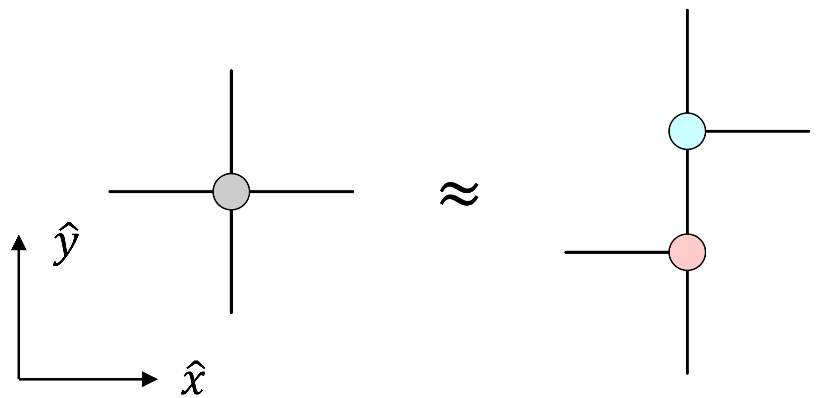
Anisotropic TRG (= TRG w/ indirect SVD)

Adachi-Okubo-Todo, PRB102(2020)054432

Cf. See poster by Katsumasa Nakayama

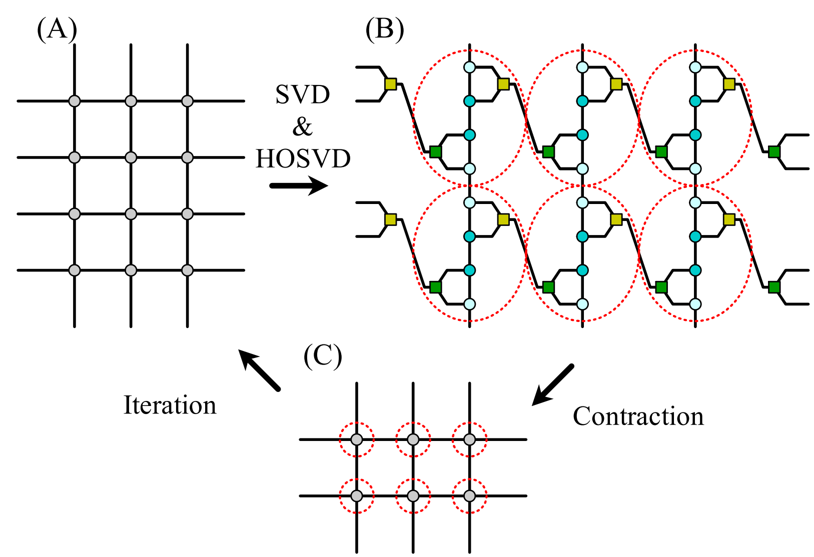
✔ Applicable to any d-dimensional lattice

✔ Accuracy with the fixed computational time is improved compared with the HOTRG, which is a conventional algorithm to the higher-dimensional systems



$$T_{xyx'y'} \approx \sum_{i=1}^{D} A_{xyi} B_{x'y'i}$$

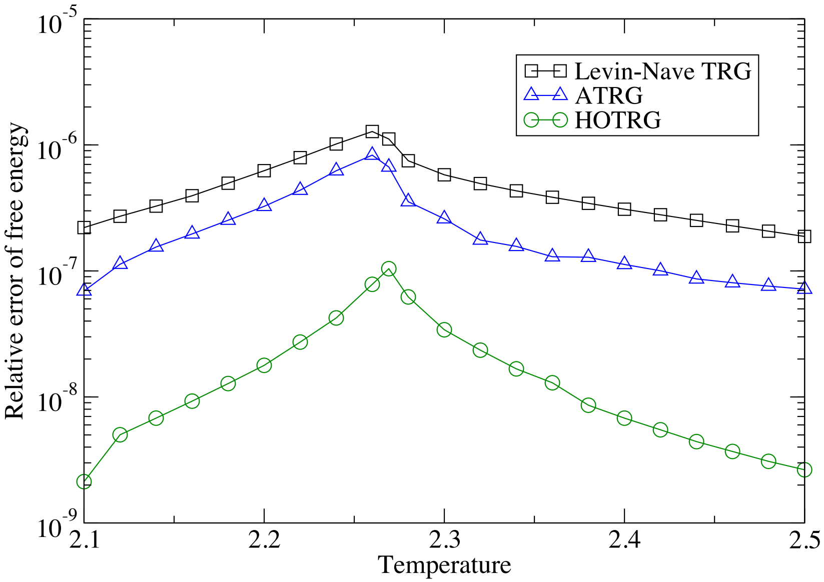
ATRG considers the block-spin transformation within lower-rank tensors (Memory: $O(D^{2d}) \to O(D^{d+1})$)



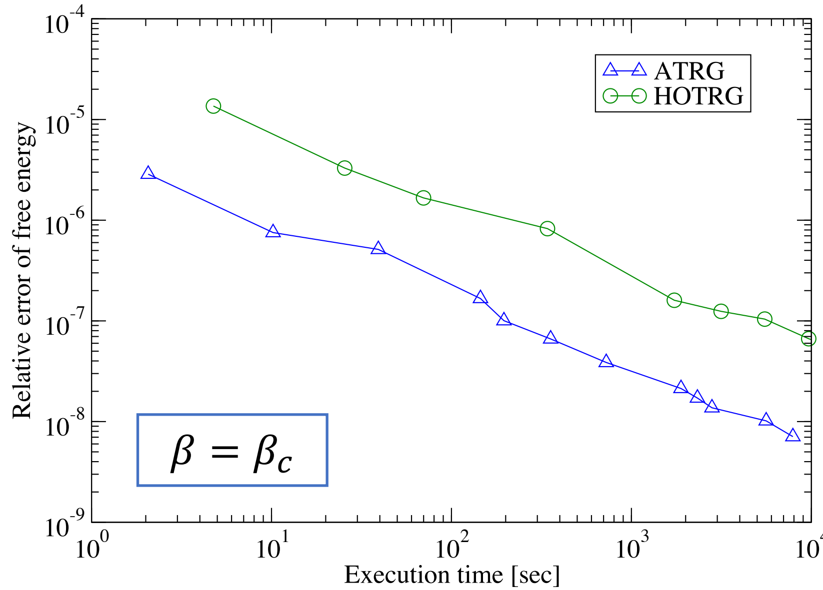
# of tensors are reduced to half

Benchmarking w/ 2d Ising model

Comparison of three types of TRG
w/ $D = 24$



Relative error vs execution time



- ✔ HOTRG & ATRG improve the accuracy of the original (LN-)TRG at the same D. The exact solution is well reproduced.
- ✔ **ATRG shows better performance than the HOTRG at the same execution time.**

Current status of TRG in the higher-dimensional systems

Algorithm	Cost	Applications to 3d	Applications to 4d
HOTRG Xie+, PRB86(2012)045139	$D^{4d-1}\ln L$	Ising Xie+, Potts model Wang+, free Wilson fermion Sakai+, $\mathbb{Z}_2$ gauge theory Dittirich+, Kuramashi-Yoshimura	Ising model SA+, Staggered fermion w/strongly coupled U(N) Milde+
Anisotropic TRG (ATRG) Adachi-Okubo-Todo, PRB102(2020)054432	$D^{2d+1}\ln L$	Ising model Adachi+, SU(2) gauge Kuwahara-Tsuchiya, Real ϕ^4 theory SA+, Hubbard model SA-Kuramashi $\mathbb{Z}_2$ gauge-Higgs SA-Kuramashi	Complex ϕ^4 theory SA+, NJL model SA+, Real ϕ^4 theory SA+ $\mathbb{Z}_2$ gauge-Higgs SA-Kuramashi
Triad RG Kadoh-Nakayama, arXiv:1912.02414	$D^{d+3}\ln L$	Ising model Kadoh-Nakayama, O(2) model Bloch+, $\mathbb{Z}_3$ (extended) clock model Bloch+ Potts models Raghav G. Jha	-

D: bond dimension, L: linear system size, d: spacetime dimension

TRG approach for fermions on a lattice

Auxiliary fermion fields to derive the TN rep.

SA-Kadoh, JHEP10(2021)188

Decompose hopping structures via

$$\mathrm{e}^{A\bar{\boldsymbol{\psi}}_n\boldsymbol{\psi}_{n+\mu}} = \left(\int\int \mathrm{d}\bar{\eta}_n \mathrm{d}\eta_n \mathrm{e}^{-\bar{\eta}_n\eta_n}\right)\exp\left[-\sqrt{A}\bar{\boldsymbol{\psi}}_{\boldsymbol{n}}\eta_n + \sqrt{A}\bar{\eta}_n\boldsymbol{\psi}_{\boldsymbol{n}+\boldsymbol{\mu}}\right]$$

Original Z	**TN rep for Z**
$\int$ over $\{\bar{\psi},\psi\}$	$\int$ with the weight $\mathrm{e}^{-\bar{\eta}\eta}$ over $\{\bar{\eta},\eta\}$

Integrating out $\{\bar{\psi},\psi\}$, we can find a tensor defined on each site

Original Grassmann numbers are manifestly converted into the Grassmann numbers (= auxiliary fermion fields)

Structure of the Grassmann tensor

✔ The G tensor is defined as a multi-linear combination of the G numbers.

$$\boxed{\mathcal{T}_{\eta_1\eta_2\eta_3\cdots} = \sum_{i_1,i_2,i_3,\cdots} T_{i_1 i_2 i_3\cdots}\eta_1^{i_1}\eta_2^{i_2}\eta_3^{i_3}\cdots}$$

✔ A clear correspondence btw usual tensors and G tensors.

	Tensor	**Grassmann tensor**
index	integer	Grassmann number
contraction	$\Sigma_i \cdots$	$\int\int \mathrm{d}\bar{\eta}\mathrm{d}\eta \mathrm{e}^{-\bar{\eta}\eta}\cdots$

✔ **Any TRG algorithm can be easily applied to evaluate path integrals including fermions.**

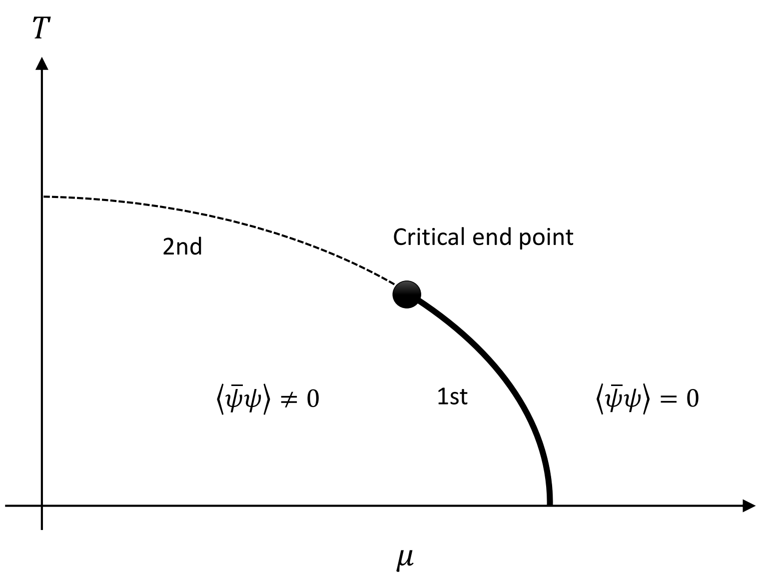
Restoration of chiral symmetry in the cold & dense (3+1)d NJL model

SA+, JHEP01(2021)121

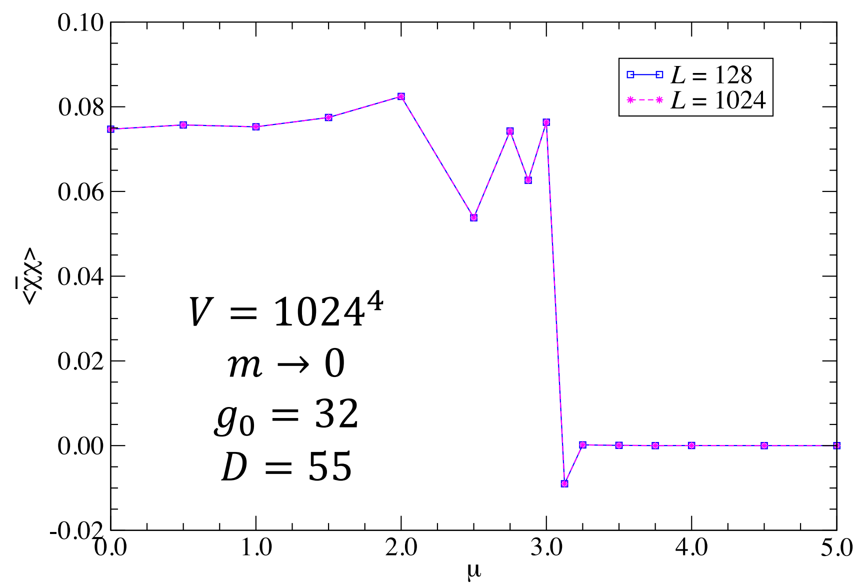
✔ **ATRG w/ parallel computation** allows us to investigate the cold & dense regime, where the MC suffers from the severe sign problem.

✔ **The resulting chiral condensate shows the first-order transition** as expected by several analytic methods.

Phase diagram on T-μ plane



Chiral condensate



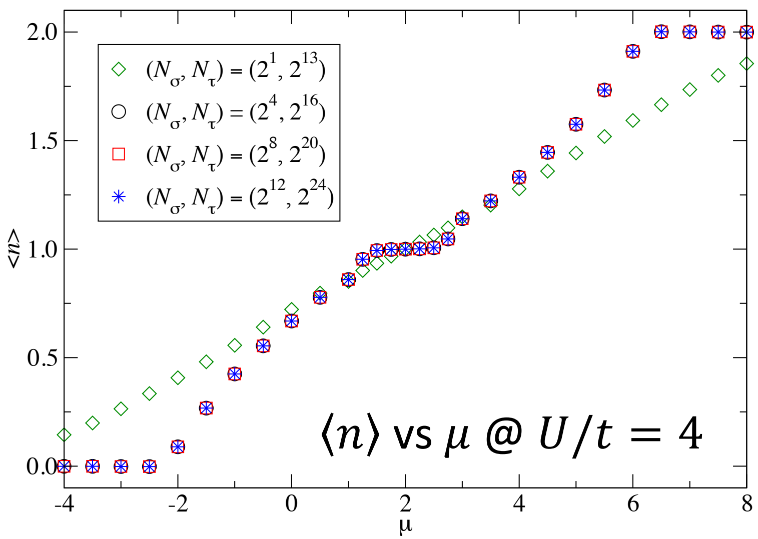
Metal-insulator transition in the Hubbard model at finite density

SA-Kuramashi, PRD104(2021)014504,
SA-Kuramashi-Yamashita, PTEP2022(2022)023I01

✔ Non-relativistic lattice fermions can also be dealt w/ the TRG approach.

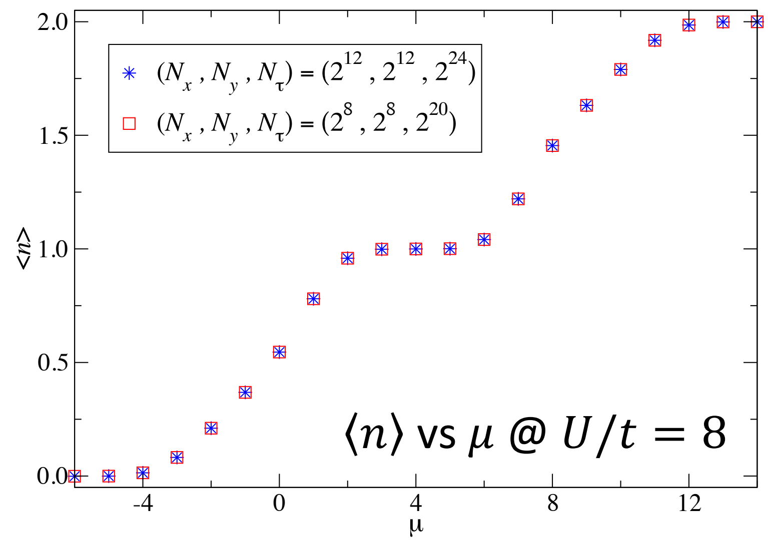
✔ In (1+1)d, the TRG provides us with the transition point as $\mu_c = 2.642(05)(13)$, which is consistent with the exact one ($\mu_c = 2.643 \ldots$).

Number density in (1+1)d



Chemical potential

Number density in (2+1)d



Chemical potential

Bond-weighting method for the Grassmann TRG

✔ Bond-weighting method is **a novel way to improve the accuracy of LN-TRG algorithms without increasing their computational costs.**

Adachi-Okubo-Todo, PRB105(2022)L060402

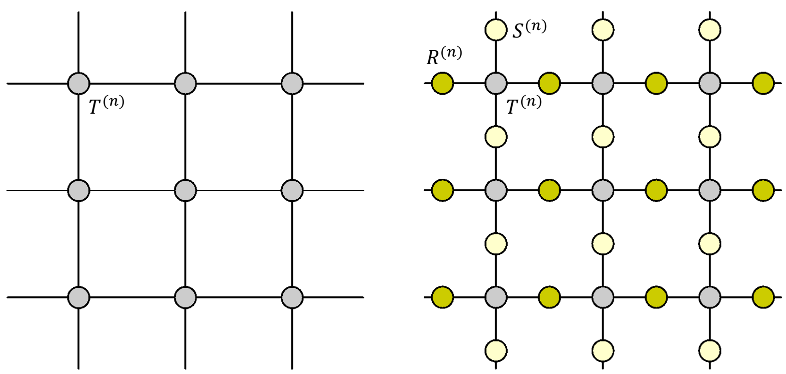
✔ Bond-weighting method works well also for lattice fermions.

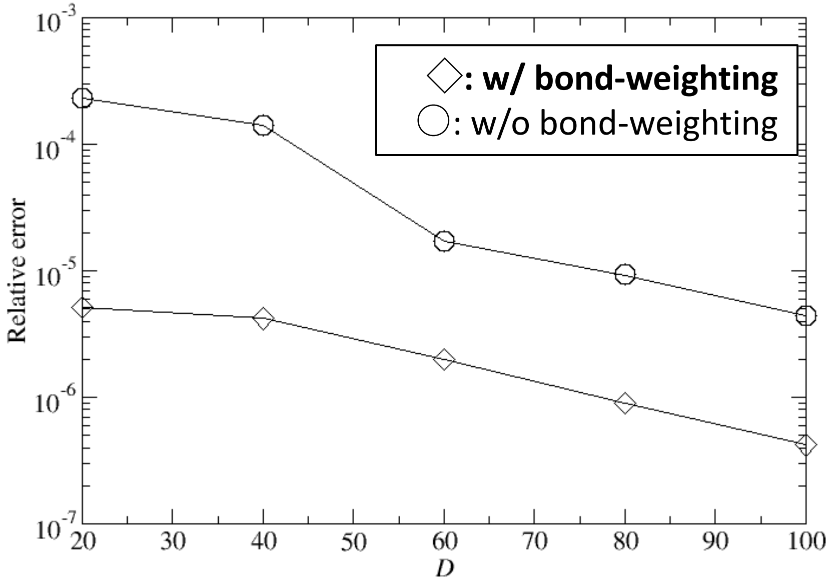
SA, JHEP11(2022)030

✔ **A sample code is available on GitHub.**

https://github.com/akiyama-es/Grassmann-BTRG

2d massless free Wilson fermion





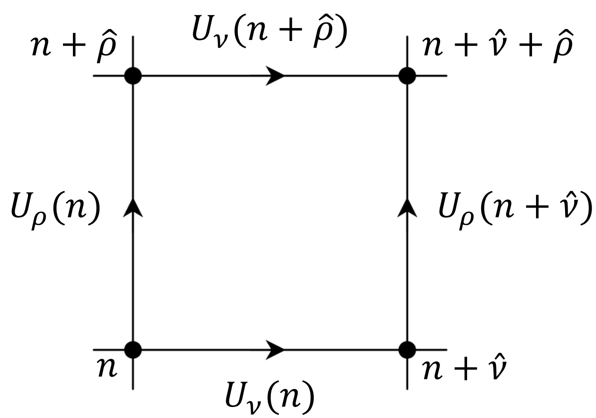
First application of TRG to 4d LGT

S. A. and Y. Kuramashi, JHEP05(2022)102

$\mathbb{Z}_2$ gauge-Higgs model in the unitary gauge

✔ Action of the $(d+1)$-dimensional $\mathbb{Z}_2$ gauge-Higgs model

$$S = -\beta \sum_n \sum_{\nu>\rho} U_\nu(n) U_\rho(n+\hat{\nu}) U_\nu(n+\hat{\rho}) U_\rho(n)$$
$$-\eta \sum_n \sum_\nu \left[\mathrm{e}^{\mu \delta_{\nu,d+1}} \sigma(n) U_\nu(n) \sigma(n+\hat{\nu}) + \mathrm{e}^{-\mu\delta_{\nu,d+1}} \sigma(n) U_\nu(n-\hat{\nu}) \sigma(n-\hat{\nu}) \right]$$



$U_\nu(n) (\in \mathbb{Z}_2)$: link variable (gauge field)
$\sigma(n) (\in \mathbb{Z}_2)$: matter field

✔ **Choosing the unitary gauge, all the matter fields are eliminated**

$$\sigma(n) U_\nu(n) \sigma(n+\hat{\nu}) \mapsto U_\nu(n)$$

$$\boxed{S = -\beta \sum_n \sum_{\nu>\rho} U_\nu(n) U_\rho(n+\hat{\nu}) U_\nu(n+\hat{\rho}) U_\rho(n) - 2\eta \sum_n \sum_\nu \cosh\left(\mu \delta_{\nu,d+1}\right) U_\nu(n)}$$

Motivation of studying $\mathbb{Z}_2$ gauge-Higgs model

✔ **The simplest lattice gauge theory coupling to a matter field**

A good target to see whether the TRG is efficient for the 4D lattice gauge theory or not.
In this study, we employ the ATRG algorithm.

✔ **The model possesses the critical endpoint (CEP)**

QCD at finite temperature and density also has the CEP.
Can we use the TRG to specify the precise location of CEP?

✔ **We can consider the model at finite density**

We can investigate how the CEP moves by introducing the chemical potential.
Note that the model is free from the sign problem even at finite density.

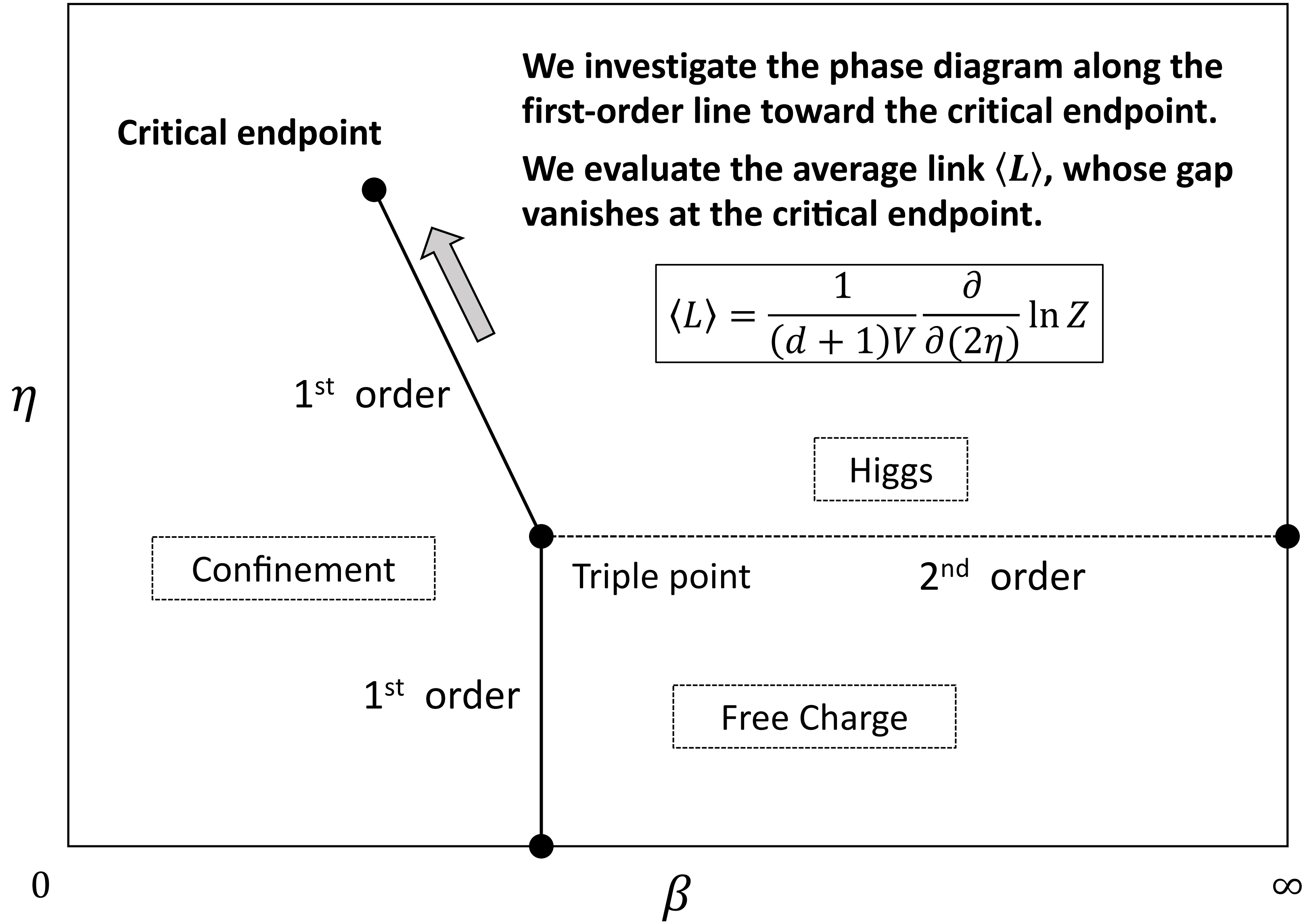
(TRG calculation for the 4D lattice gauge theory with the sign problem is an important future work, which is in progress)

Phase diagram of the (3+1)D model at $\mu = 0$

We investigate the phase diagram along the first-order line toward the critical endpoint.

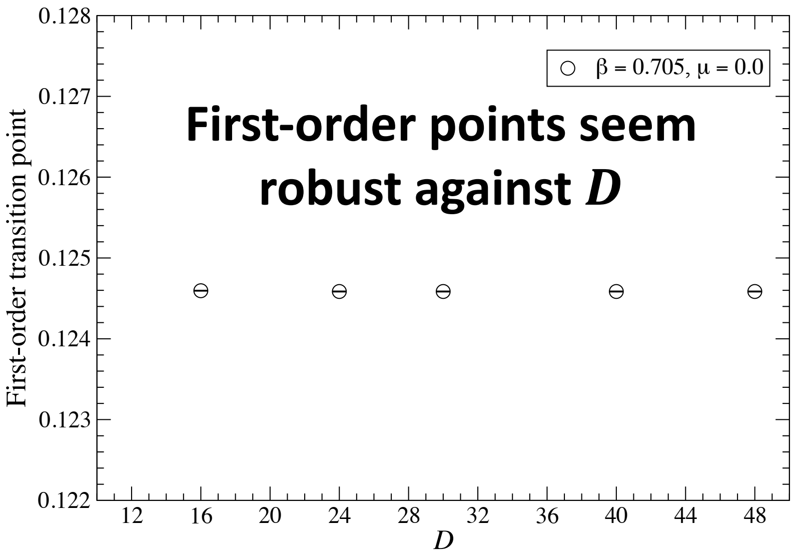
We evaluate the average link $\langle L \rangle$, whose gap vanishes at the critical endpoint.

$$\boxed{\langle L \rangle = \frac{1}{(d+1)V} \frac{\partial}{\partial(2\eta)} \ln Z}$$



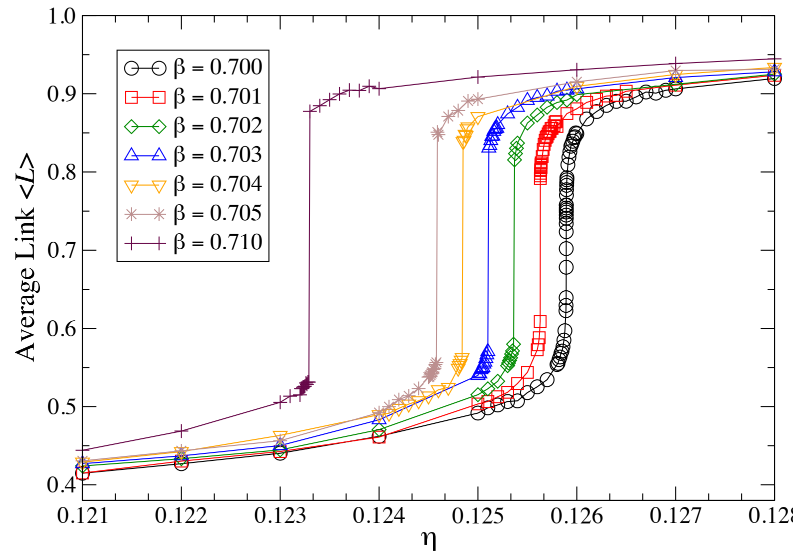
Study of the (2+1)D model at $\mu = 0$

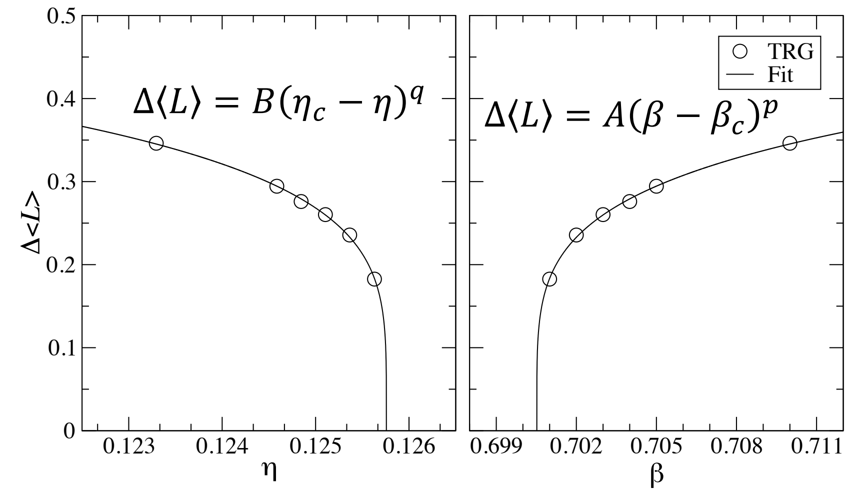
with $D \leq 48,\ \eta_+ - \eta_- = O(10^{-5})$



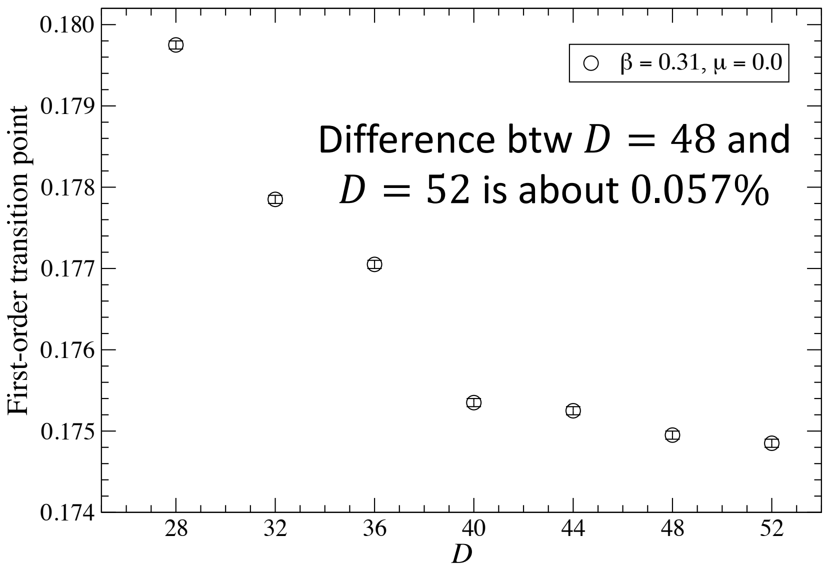
Bond dimension

MC Somoza+, PRX11(2021)041008	$\beta_c \approx 0.701$
TRG this work	$\beta_c = 0.70051(7)$





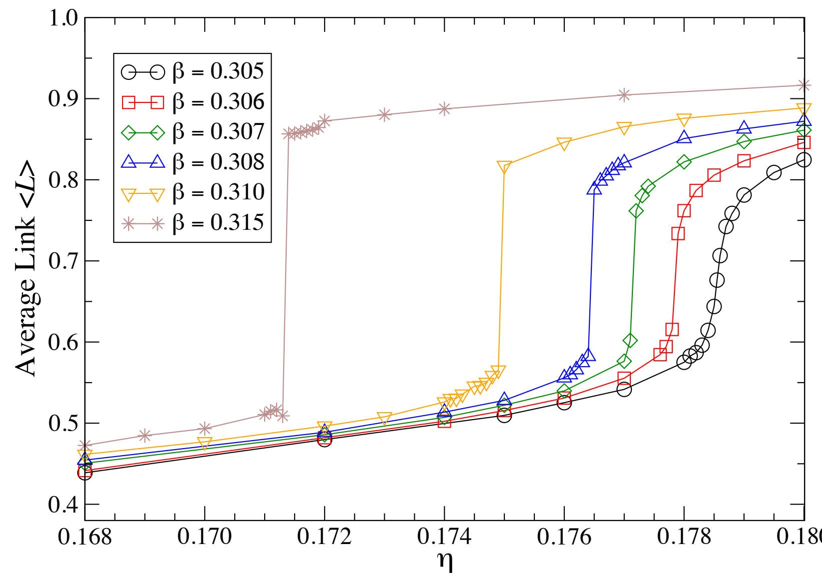
(3+1)D model at vanishing density



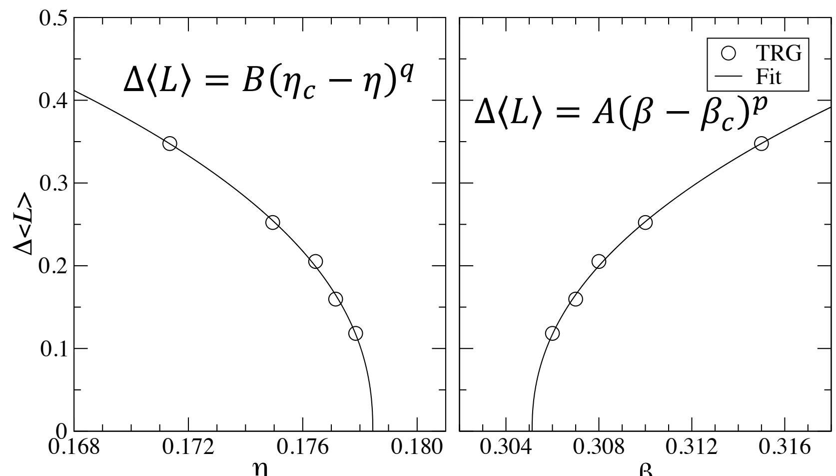
Difference btw $D = 48$ and $D = 52$ is about 0.057%

Bond dimension

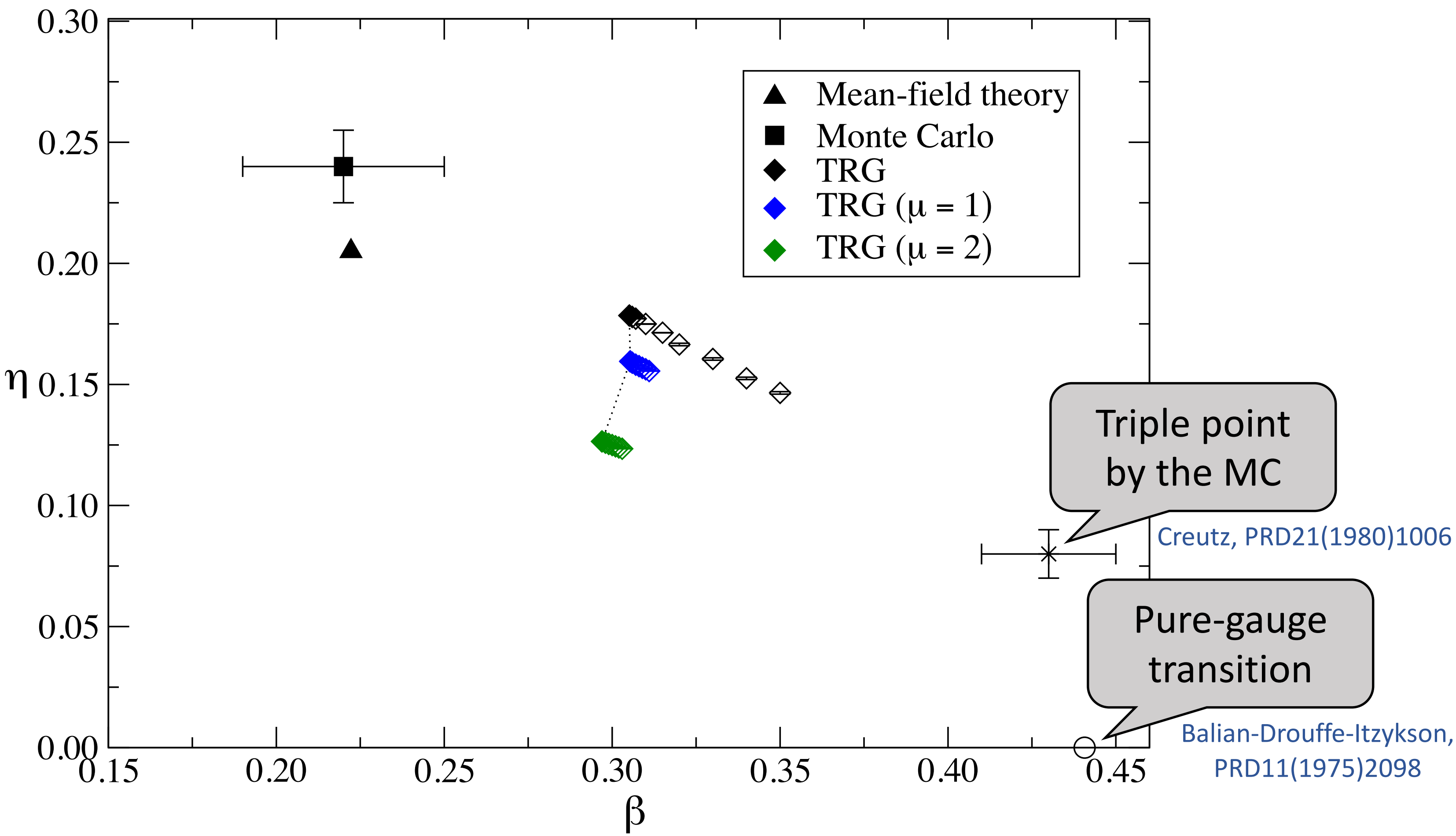
with $D \leq 52$, $\eta_+ - \eta_- = O(10^{-4})$



Mean-field Brezin-Drouffe, NPB200(1982)93	$(\beta_c, \eta_c) = (0.22, 0.205)$
MC on $V = 8^4$ Creutz, PRD21(1980)1006	$(\beta_c, \eta_c) = (0.22(3), 0.24(2))$
TRG w/ $D = 52$ this work	$(\beta_c, \eta_c) = (0.3051(2), 0.1784(2))$



Current status of the phase diagram near the CEP



It seems that TRG and MC share a similar first-order line at $\mu = 0$

A deviation about the location of the CEP

Summary & Outlook

- ✔ The TRG approach does not suffer from the sign problem and allows us to investigate the thermodynamic limit.

- ✔ TRG algorithms are useful to investigate fermionic systems.

- ✔ The ATRG algorithm w/ parallel computation has been a good way to investigate higher-dimensional QFTs on a lattice.

- ✔ A next target is the (3+1)d LGT coupled to fermions.

  cf) Variational approach based on the tree TN for the(3+1)d lattice QED ($L \leq 8$).

  Magnifico+, Nature Commun. 12(2021)1

- ✔ Aiming to establish a TRG algorithm to deal w/ two- and three-flavor (or more?) lattice fermions. SA, in progress