On the equivalence between Yang-Mills Gradient flow and stout smearing

Members : Masato Nagatsuka, Keita Sakai, Shoichi Sasaki

Affiliation : Tohoku University

Summary

In this study, we showed that the Yang-Mills gradient flow and the stout smearing remain equivalent within some numerical precision if we take the proper combination of the parameters. Especially, we verified that $\rho \leq 0.1$ for $\beta = 6.42$, $\rho \leq 0.025$ for $\beta = 5.96, 6{,}17$, and $\rho \leq 0.01$ for $\beta = 5.76$ are enough small to identify the stout smearing with the Yang-Mills gradient flow.

We also note that the computational speed of the stout smearing is about 10 times faster than that of the Yang-Mills gradient flow. Therefore, we suggest that the stout smearing should be used as an alternative to the Yang-Mills gradient flow within the parameters we showed above.

Introduction

The similarity between the Yang-Mills gradient flow and the stout smearing was implied by M. Lüscher[1] and the rigorous proof of the equivalence was recently given by K. Sakai and S. Sasaki at the zero limit of the lattice spacing a and the smearing parameter ρ [2].

However, it is not obvious that they remain equivalent even with finite parameters within some numerical precision, therefore we verified the equivalence by comparing the energy density $\langle E \rangle$ measured in numerical simulations.

Outline of the proof given in Ref.[2]

One can demonstrate the proof of the equivalence between the two methods in the following three steps. Detailed definitions and proof are given in the reference [2].

1. Derive a continuous version of the stout smearing procedure given below (note that $\rho \to 0$ is taken here):

$$U_\mu^{(n+1)}(x) = e^{i\rho Q_\mu^{(n)}(x)} U_\mu^{(n)}(x) \quad \to \quad \frac{\partial}{\partial s} \ln U_\mu(x,s) = iQ_\mu(x,s)\,.$$

2. Derive the explicit form of the link derivative of the Wilson gauge action S_W given below:

$$g_0^2 \partial_{x,\mu} S_W(U) = -iQ_\mu(x,s)\,.$$

3. Prove the relation given below:

$$\frac{\partial U_\mu(x,s)}{\partial s} U_\mu^{-1}(x,s) = \frac{\partial}{\partial s} \ln U_\mu(x,s) + \frac{1}{2!}\left[\ln U_\mu(x,s), \frac{\partial}{\partial s} \ln U_\mu(x,s)\right] + \cdots\,.$$

Combining these equations, one can get the following equation and then find that it is reduced to the Yang-Mills Gradient flow equation in the limit of $a \to 0$.

$$\frac{\partial U_\mu(x,s)}{\partial s} U_\mu^{-1}(x,s) = -g_0^2 \partial_{x,\mu} S_W(U) + O(a)\,.$$

Thus, the two methods are equivalent in the zero limit of two parameters. Remark that the flow time t and the number of smearing iterations n satisfy the relation $t = \rho n$.

However, there must be a finite difference depending on a and ρ in numerical simulations.

Numerical results

In this study, we prepare four sets of gauge configurations as summarized in table 1. We perform the stout smearing with $\rho =$ 0.1, 0.025, and 0.01 for each configuration to evaluate $X(t) \equiv t^2 \langle E(t) \rangle$ where t indicates the flow time. We also evaluate $X(t)$ with the Yang-Mills gradient flow. Fig. 1 shows the typical behavior of $X(t)$. To discuss the detail of the difference, we calculate

$$R(t) \equiv \frac{X_{\rm stout}(t) - X_{\rm flow}(t)}{X_{\rm flow}(t)}\,.$$

Table 1.

β	$L^3 \times T$	$\sim a/r_0$	$N_{\rm conf}$
5.76	$16^3 \times 32$	0.297	100
5.96	$24^3 \times 48$	0.200	100
6.17	$32^3 \times 64$	0.142	100
6.42	$48^3 \times 96$	0.100	100

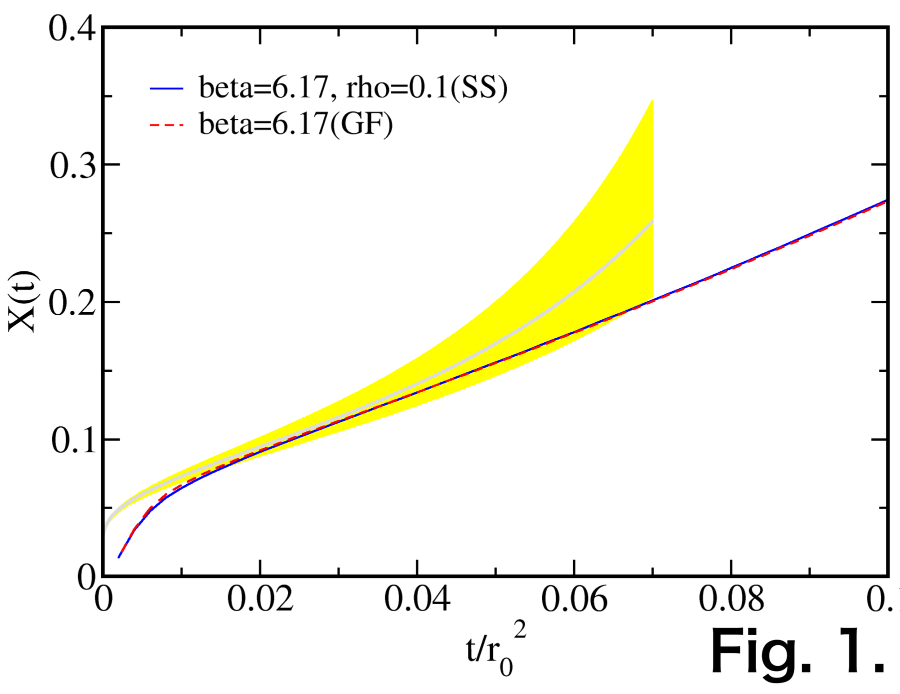

Fig. 1.

Fig. 2, 3 show the typical behaviors of $R(t)$ where t is scaled by t_0 defined to be $X(t_0) = 0.3$. Noise/Signal ratio of $X_{\rm flow}$ is also expressed as the grayed region. These figures implies that $R(t)$ saturates at some flow time.

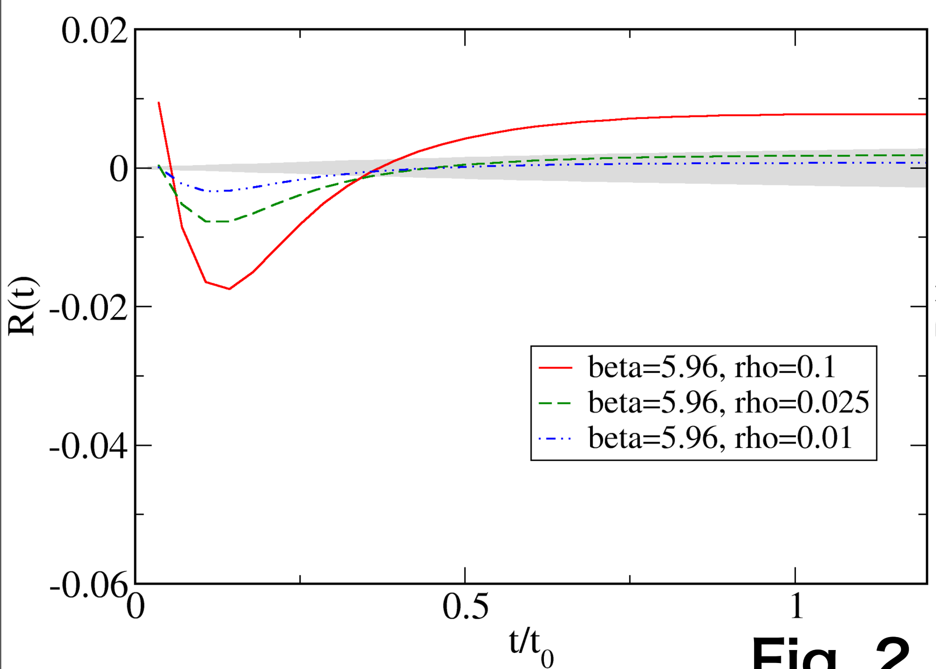

Fig. 2.

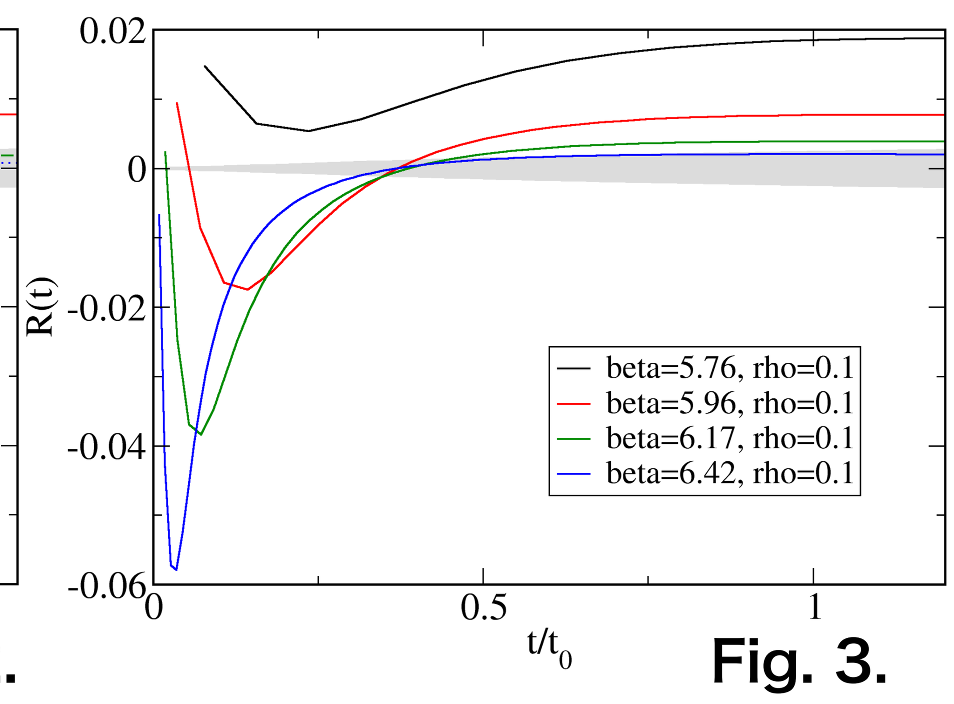

Fig. 3.

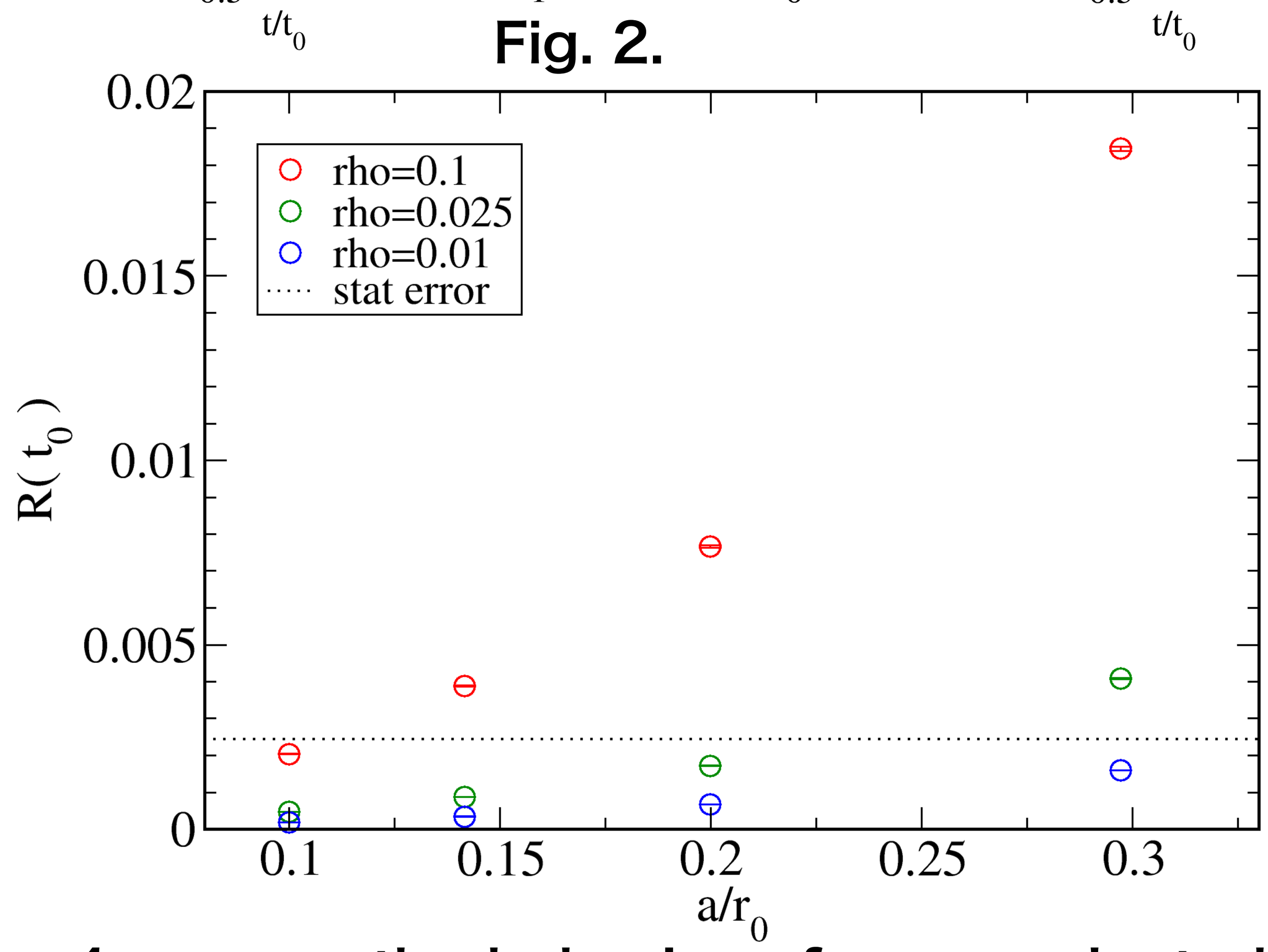

Fig. 4.

In Fig. 4, we see the behavior of $R(t)$ evaluated at t_0 for every combinations of β and ρ. The dotted line corresponds to the grayed domain appearing in Fig. 2, 3. Then we conclude that the gradient flow can be replace by the stout smearing if the parameter ρ is satisfied with a certain condition given in summary.

References

[1] M. Lüscher, JHEP 08 (2010) 071.

[2] K. Sakai and S. Sasaki, arXiv:2211.15176 [hep-lat], (2022).