

Inclusive semi-leptonic decays of charmed mesons with Möbius domain wall fermions

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in collaboration with

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Introduction and Motivation

Inclusive semileptonic decay rate

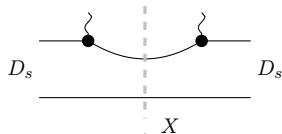
$$\sum_X \left| \begin{array}{c} \text{Diagram: } D_s \text{ decaying to } X_s \text{ via } W^- \text{ to } \ell \text{ and } \bar{\nu}_\ell \\ \hline D_s \quad X_s \end{array} \right|^2$$

The weight over X has to be adjusted by

$$\int d\omega_X K(\omega_X) \left[\right]_{\text{Lattice}}$$

kinematics; requires some approximation \rightarrow
source of systematic error

Lattice: **4pt functions**



contributions from
all possible states

This work

- Understand systematic errors
- Validate using experimental data (BESIII) [BESIII, arXiv:2104.07311 [hep-ex]]

Related works

- Comparison Chebyshev vs. Backus-Gilbert [A. Barone, PoS(LATTICE2022)403]
- Inclusive semi-leptonic B -decays [A. Smecca, PoS(LATTICE2022)423]

Hadronic Tensor

contains all non-perturbative information

$$W^{\mu\nu}(p, q) = \sum_{X_s} (2\pi)^3 \delta^{(4)}(p - q - r) \\ \times \frac{1}{2E_{D_s}} \langle D_s(\mathbf{p}) | \tilde{J}^{\mu\dagger}(-\mathbf{q}) | X_s(\mathbf{r}) \rangle \langle X_s(\mathbf{r}) | \tilde{J}^\nu(\mathbf{q}) | D_s(\mathbf{p}) \rangle$$

Total Decay Rate [Gambino, Hashimoto, arXiv:2005.13730]

$$\Gamma = \frac{G_F^2 |V_{cs}|^2}{24\pi^3} \int_0^{\mathbf{q}_{\max}^2} d\mathbf{q}^2 \sqrt{\mathbf{q}^2} \bar{X}$$

$$\begin{aligned} \bar{X} &= \int_{\omega_{\min}}^{\omega_{\max}} d\omega K_{\mu\nu}(\mathbf{q}, \omega) \times W^{\mu\nu} \\ &= \int_0^\infty d\omega K_{\mu\nu}(\omega, \mathbf{q}) \langle D_s(0) | \tilde{J}_\mu^\dagger(-\mathbf{q}) \delta(\hat{H} - \omega) \tilde{J}_\nu(\mathbf{q}) | D_s(0) \rangle \\ &= \langle D_s(\mathbf{0}) | \tilde{J}_\mu^\dagger(-\mathbf{q}) K_{\mu\nu}(\mathbf{q}, \hat{H}) \tilde{J}_\nu(\mathbf{q}) | D_s(\mathbf{0}) \rangle \end{aligned}$$

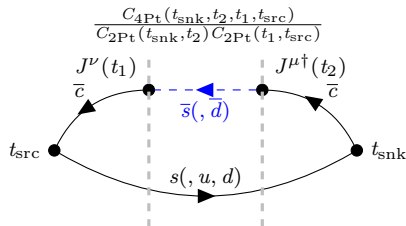
Reconstruction from lattice data

lattice data

$$C_{\mu\nu}^{JJ}(t) \sim \langle D_s | \tilde{J}_\mu^\dagger(-\mathbf{q}) e^{-\hat{H}t} \tilde{J}_\nu(\mathbf{q}) | D_s \rangle$$

energy integral in incl. rate

$$\bar{X} = \langle D_s(\mathbf{0}) | \tilde{J}_\mu^\dagger K_{\mu\nu}(\mathbf{q}, \hat{H}) \tilde{J}_\nu | D_s(\mathbf{0}) \rangle$$



One can construct an approximation of $K(\mathbf{q}, \hat{H})$ in terms of $e^{-\hat{H}}$

$$K(\mathbf{q}, \hat{H}) = k_0 + k_1 e^{-\hat{H}} + \dots + k_N e^{-N\hat{H}},$$

which then allows us to write

$$\begin{aligned} \bar{X} \sim & k_0 \underbrace{\langle D_s | \tilde{J}_\mu^\dagger(-\mathbf{q}) \tilde{J}_\nu(\mathbf{q}) | D_s \rangle}_{C_{\mu\nu}^{JJ}(0)} + k_1 \underbrace{\langle D_s | \tilde{J}_\mu^\dagger(-\mathbf{q}) e^{-\hat{H}} \tilde{J}_\nu(\mathbf{q}) | D_s \rangle}_{C_{\mu\nu}^{JJ}(1)} + \dots \\ & + k_N \underbrace{\langle D_s | \tilde{J}_\mu^\dagger(-\mathbf{q}) e^{-\hat{H}N} \tilde{J}_\nu(\mathbf{q}) | D_s \rangle}_{C_{\mu\nu}^{JJ}(N)} \end{aligned}$$

Chebyshev Approximation

Modified Chebyshev polynomials $T_j^*(e^{-\omega})$ to approximate $K(\omega)$ in the range $[\omega_0, \infty]$, with $0 \leq \omega_0 < \omega_{\min}$

$$K(\omega) \simeq \sum_j c_j^* T_j^*(e^{-\omega})$$

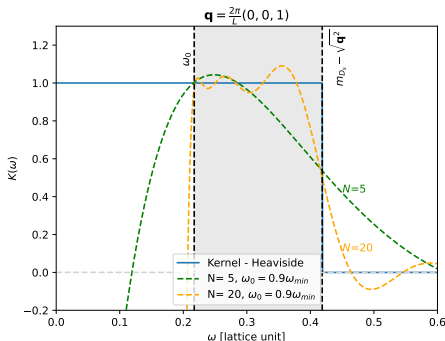
$$T_1^*(x) = 2x - 1, \quad T_2^*(x) = 8x^2 - 8x + 1, \quad T_3^*(x) = 32x^3 - 48x^2 + 18x - 1, \quad \dots$$

Kernel function

$$K(\omega) = \theta(m_{D_s} - \sqrt{q^2} - \omega)$$



step function to implement the upper limit of the ω integral; $\omega_{\max} = m_{D_s} - \sqrt{q^2}$

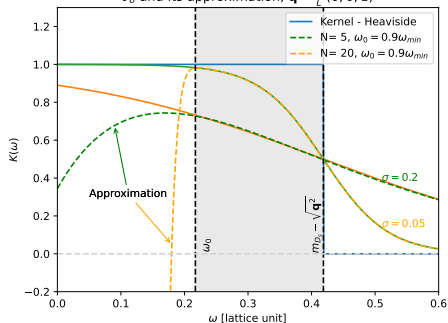


Chebyshev Approximation – Kernel Function

Stabilize approximation by applying a **smearing with a width σ**

$$K_\sigma(\omega) = \theta_\sigma (m_{D_s} - \sqrt{q^2} - \omega)$$

θ_σ and its approximation, $\mathbf{q} = \frac{2\pi}{L}(0, 0, 1)$



Limits to be taken

- smearing of the Kernel, $\sigma \rightarrow 0$
- Order of Chebyshev polynomials, $N \rightarrow \infty$

Constructing the Approximation

The ω -integral can be approximated as

$$\frac{\langle \psi_\mu | K(\hat{H}) | \psi_\nu \rangle}{\langle \psi_\mu | \psi_\nu \rangle} = \frac{c_0^*}{2} + \sum_{j=1}^N c_j^* \underbrace{\frac{\langle \psi_\mu | T_j^*(e^{-\hat{H}}) | \psi_\nu \rangle}{\langle \psi_\mu | \psi_\nu \rangle}}_{C(t+2t_0)/C(2t_0)},$$

with $|\psi_\nu\rangle \equiv e^{-\hat{H}t_0} \tilde{J}_\nu(\mathbf{q}) |D_s(\mathbf{0})\rangle$. Coefficients are given by

$$c_j^* = \frac{2}{\pi} \int_0^\pi d\theta K\left(-\ln \frac{1 + \cos\theta}{2}\right) \cos(j\theta) \quad , \quad \text{when } \omega_0 = 0$$

Property of the Chebyshev polynomials

$$\left| \frac{\langle \psi_\mu | T_j^*(e^{-\hat{H}}) | \psi_\nu \rangle}{\langle \psi_\mu | \psi_\nu \rangle} \right| \leq 1.$$

- Use as a constraint: suppress statistical noise for large j , for which huge cancellations are expected
- Use as an upper limit of the error: bound by 0 ± 1 , for all j

Lattice setup

[Colquhoun et al., arXiv:2203.04938]

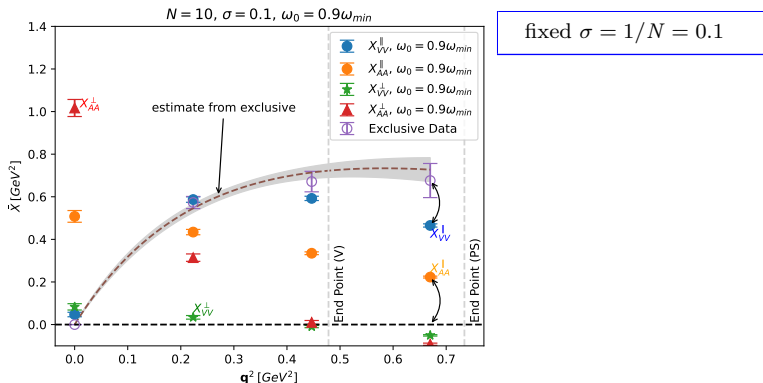
A first study with JLQCD ensembles

- Lattice size: $48^3 \times 96$
- Lattice spacing: $a = 0.055$ fm
- $2 + 1$ Möbius domain-wall fermions
- u, d quarks at $m_\pi \simeq 300$ MeV
- s, c quark simulated at near-physical values
- 4 choices of momentum insertion corresponding to $\mathbf{q} = (0, 0, 0) \rightarrow (1, 1, 1)$
- Numerical computation on Fugaku
- Used Grid/Hadrons

ID	a (fm)	β	$L^3 \times N_T \times L_s$	N_{cfg}	am_l	am_s	am_Q
C-ud5-sa	0.080	4.17	$32^3 \times 64 \times 12$	100	0.019	0.04	0.44037 0.68808
C-ud5-sb	0.080	4.17	$32^3 \times 64 \times 12$	100	0.019	0.03	0.44037 0.68808
C-ud4-sa	0.080	4.17	$32^3 \times 64 \times 12$	100	0.012	0.04	0.44037 0.68808
C-ud4-sb	0.080	4.17	$32^3 \times 64 \times 12$	100	0.012	0.03	0.44037 0.68808
C-ud3-sa	0.080	4.17	$32^3 \times 64 \times 12$	100	0.007	0.04	0.44037 0.68808
C-ud3-sb	0.080	4.17	$32^3 \times 64 \times 12$	100	0.007	0.03	0.44037 0.68808
C-ud2-sa-L	0.080	4.17	$48^3 \times 96 \times 12$	100	0.0035	0.04	0.44037 0.68808
M-ud5-sa	0.055	4.35	$48^3 \times 96 \times 8$	50	0.012	0.025	0.27287 0.42636 0.68808
M-ud4-sa	0.055	4.35	$48^3 \times 96 \times 8$	50	0.008	0.025	0.27287 0.42636 0.68808
M-ud3-sa	0.055	4.35	$48^3 \times 96 \times 8$	42	0.0042	0.025	0.27287 0.42636 0.68808
F-ud3-sa	0.044	4.47	$64^3 \times 128 \times 8$	50	0.003	0.015	0.210476 0.328869 0.5138574

First Numerical Results

$$\bar{X} = \langle D_s(\mathbf{0}) | \tilde{J}_\mu^\dagger(-\mathbf{q}) K_{\mu\nu}(\mathbf{q}, \hat{H}) \tilde{J}_\nu(\mathbf{q}) | D_s(\mathbf{0}) \rangle$$

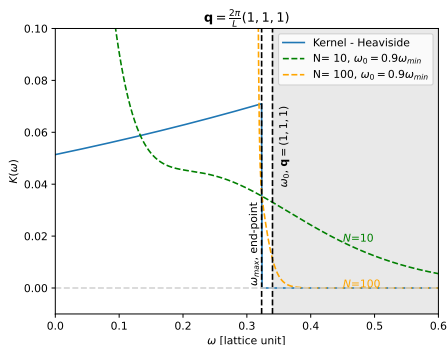
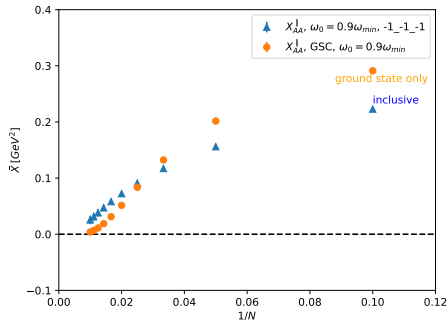


- Decompose \bar{X} into different channels of V and A ; \parallel and \perp
- Comparison of $VV \parallel$ with exclusive $D \rightarrow K$ decay data indicates that values are in the right ballpark
- further analysis on *dangerous region*, i.e close to the end of the phase space

Above the end-point, $AA \parallel$

At $\mathbf{q} = (1, 1, 1)$, where X_{AA}^{\parallel} receives contributions only from the vector state. $E_{V,\min}$ is above the threshold $\rightarrow \bar{X}$ expected to be zero.

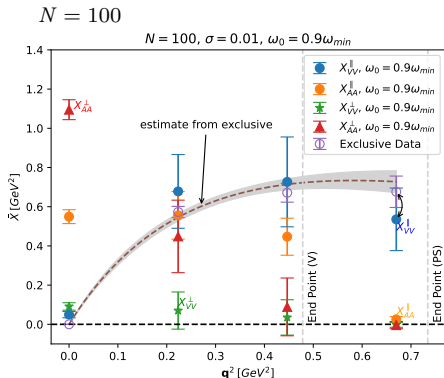
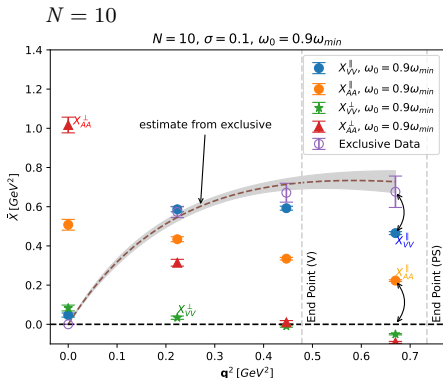
- set $\sigma = \frac{1}{N}$; enables us to take both limits simultaneously
- due to statistical uncertainties higher orders in the Chebyshev approximation are basically 0 ± 1 ; only add errors to \bar{X} by $\sum_j |c_j^*|$



Approaches zero for sufficiently large N

Taking the $N \rightarrow \infty$ limit

Conservative error estimate from the N (or σ) dependence



- central values of the points remain stable
- error bars show the mathematical upper limit, when the $N \rightarrow \infty$ limit is taken
→ likely an overestimation; real error expected to be smaller
- proper estimate requires knowledge on the spectrum
 - ▶ With a flat spectrum, errors cancel around the threshold
 - ▶ Real problem might occur when the spectrum rapidly changes; expected only near the ground state

The Influence of finite volume effects

Infinite volume provides continuous spectrum, **but** finite volume on the lattice results in a sum of delta peaks.

Proof of Concept

Use freedom to choose ω_{th}

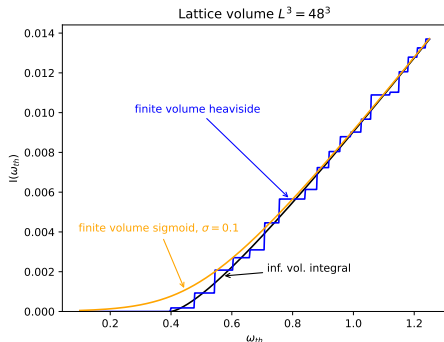
- consider the spectrum (inf. vol.)

$$\rho(\epsilon) = \frac{1}{16\pi} \sqrt{1 - \frac{4m^2}{\epsilon^2}}$$

- finite vol. formulation

$$\rho(\epsilon) = \frac{\pi}{V} \sum_q \frac{1}{(2\epsilon_q)^2} \delta(\epsilon - 2\epsilon_q)$$

- smearing results in a "wash-out" of the finite volume effects
- for large ω_{th} good agreement between smeared and heaviside kernel
→ extrapolate back to reduce error



Summary and Outlook

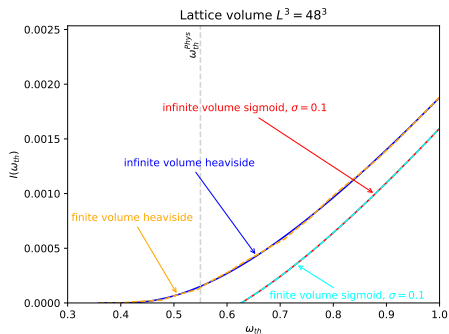
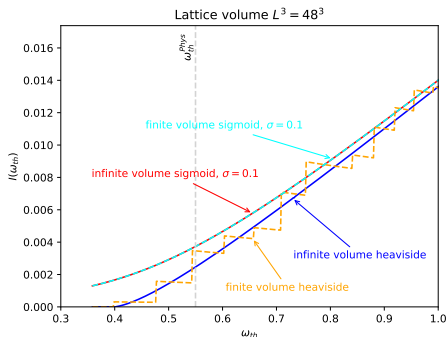
- Inclusive $D \rightarrow X\ell\nu$: towards a lattice computation with fully controlled systematics
- Important systematic error due to the approximation of the kernel function
 - Conservative error estimate employing the mathematical properties of the Chebyshev polynomials; mathematically robust but not too helpful
 - Concept of estimating errors due to finite volume effects; further study necessary
- Extend to different ensembles (two more lattice spacings)
- Extend to different inclusive channels $D \rightarrow X_s$, $D_s \rightarrow X_d$ and $D \rightarrow X_d$
- Estimates for the total decay rate and compare the results with the experiment

Back-Up Slides

Applying the Concept - An example

We combine our kernel function and our spectrum to have a more realistic example

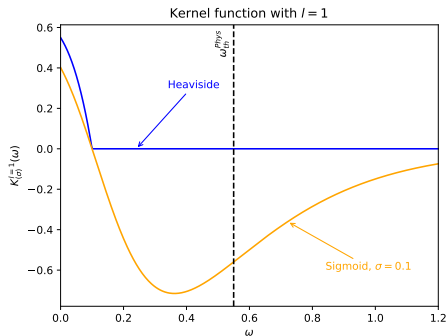
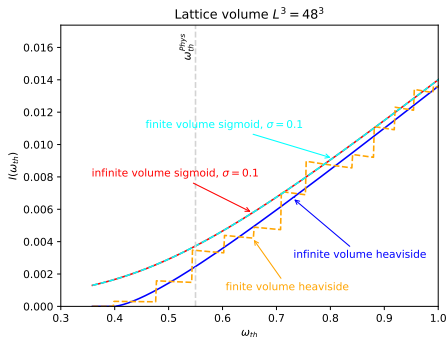
- possible reduction of errors by extrapolating backwards from higher ω_{th}
- (large) discrepancy between heaviside and smeared functions
→ especially for $l = 1$; smeared function gets negative



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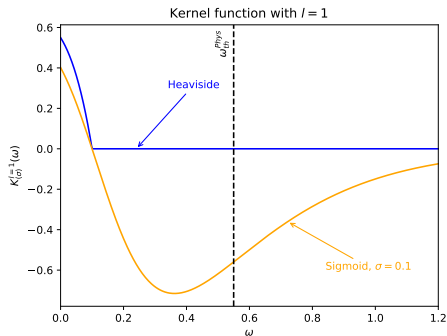
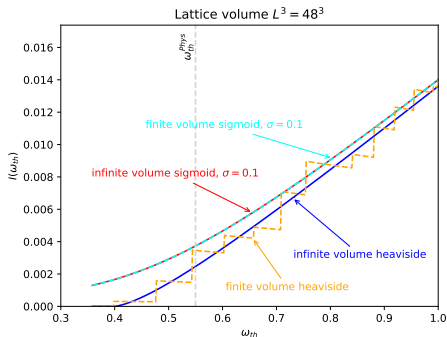
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Idea: Reduce error by changing/altering the way we approach the kernel approximation