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Introduction to '100 Years of Spin Physics'

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Why 100 years?

In 1920's, there were lots of efforts and discussions on
a possible new intrinsic degree of freedom of the particles.

The concept of spin emerged and was established.

In 1928, Dirac equation was formulated. $(i\hbar\gamma^{\mu}\partial_{\mu} - m)\,\psi = 0.$
a free particle

If we use α_i and β for the coefficients, it can be written as

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi, \qquad \hat{H} = \sum_{i=1}^{3} \alpha_i\, \hat{p}_i + m\beta$$

This equation has to be consistent with Klein-Gordon equation which is based on $E^2 = p^2 + m^2$:

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 \psi = \left[\sum_{i=1}^{3} \hat{p}_i^2 + m^2\right] \psi.$$

The conditions on the coefficients $\alpha_i,\ \beta$ are then

$$\alpha_i\,\alpha_j + \alpha_j\,\alpha_i = 0 \quad (i \neq j), \qquad \alpha_i\,\beta + \beta\,\alpha_i = 0, \tag{1}$$

$$\alpha_i^2 = I, \quad \beta^2 = I. \tag{2}$$

1) Eq.1: α_i and β are not just numbers, but matrices.
2) Eq.2: The eigenvalues of the matrices α_i, β are $+1$ and -1.
3) Eq.1: α_i and β are traceless. $\mathrm{tr}\,(\alpha_i)$=$\mathrm{tr}\,(\beta) = 0$: the sum of the diagonal elements is 0. ($\rightarrow$ note 1)
4) Four independent matrices $\alpha_i,\ \beta$ are needed.

$\longrightarrow$ The size of matrices is 4x4, 6x6, or ....

The 2x2 matrix such as Pauli matrix has only three independent matrices, and does not satisfy the condition.

The matrices α_i and β have, when diagonalized, $+1$ and -1 as the diagonal elements

$$\begin{bmatrix} +1 & & & & \\ & +1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & \cdots \end{bmatrix}$$

for example. The numbers of $+1$'s and -1's have to be equal in order to be traceless.
The minimum is 4x4 matrix.

If we adopt 4x4 matrices for α_i and β, the Hamiltonian becomes a 4x4 matrix:

$$\hat{H} = \alpha \cdot \hat{p} + m\beta = \begin{bmatrix} mI & \sigma \cdot \hat{p} \\ \sigma \cdot \hat{p} & -mI \end{bmatrix}$$

This is the Pauli-Dirac representation. σ is the Pauli matrix. The Hamiltonian can be equivalently expressed by other representations as well.

As a result, the wave function has four components.

The plane wave is expressed as

$$\psi = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} e^{i\,(k\cdot x - \omega t)}, \qquad k = p/\hbar, \quad \omega = E/\hbar.$$

spinor and space-time wave function

In this case, the momentum operator $\hat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i}$ and the energy operator $\hat{E} = i\hbar\frac{\partial}{\partial t}$ act on $e^{i\,(k\cdot x - \omega t)}$. After these operations,

$$E \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} mI & \sigma \cdot p \\ \sigma \cdot p & -mI \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}.$$

The Hamiltonian can have four eigenvalues.

The condition

$$\det(EI - H) = 0, \quad \text{where } H = \begin{bmatrix} mI & \sigma \cdot p \\ \sigma \cdot p & -mI \end{bmatrix}$$

leads to the eigen-function which is a quartic equation of E:

$$\{E^2 - (p^2 + m^2)\}^2 = 0.$$

The eigenvalues are $E = \sqrt{p^2 + m^2}, \quad E = -\sqrt{p^2 + m^2}$.

Two states are degenerate at $E = \sqrt{p^2 + m^2}$, and
two other states are degenerate at $E = -\sqrt{p^2 + m^2}$.

The Dirac equation contains a new intrinsic degree of freedom of the particle.

The electromagnetic potentials (ϕ, A_x, A_y, A_z) are included by replaceing the operators $\hat{p} \to \hat{p} - qA, \;\; \hat{E} \to \hat{E} - q\phi$.

In the low energy limit, the Dirac equation is reduced to

$$T\, u_A = \left\{ \frac{1}{2m}(p - qA)^2 + q\phi - \frac{q\hbar}{2m}\sigma \cdot B \right\} u_A$$

u_A : upper two components,
T : kinetic energy, $E = m + T$,
$q = -e > 0$ for electron, $e > 0$.

$$\mu = \frac{q\,\hbar}{2m}\sigma = g\frac{q}{2m} \cdot \frac{\hbar}{2}\sigma = g\frac{q}{2m}\,s, \qquad s = \frac{\hbar}{2}\sigma, \;\; g = 2.$$

The $-\mu \cdot B$ term causes the energy split.

The Dirac equation describes the spin-$\frac{1}{2}$ particle.
Spin physics has made a big progress since then.

note 1

When $CD + DC = 0$ holds,

$$
\begin{aligned}
&D^{-1}CD + D^{-1}DC = D^{-1}CD + C = 0, \\
&Tr(D^{-1}CD + C) = 0, \\
&Tr(D^{-1}CD) + Tr(C) = 0, \\
&Tr(CDD^{-1}) + Tr(C) = 0, \\
&\longrightarrow \; 2Tr(C) = 0.
\end{aligned}
$$