The Radiative M1 Transition of Charmonia and Bottomonia in the Light-Front Quark Model

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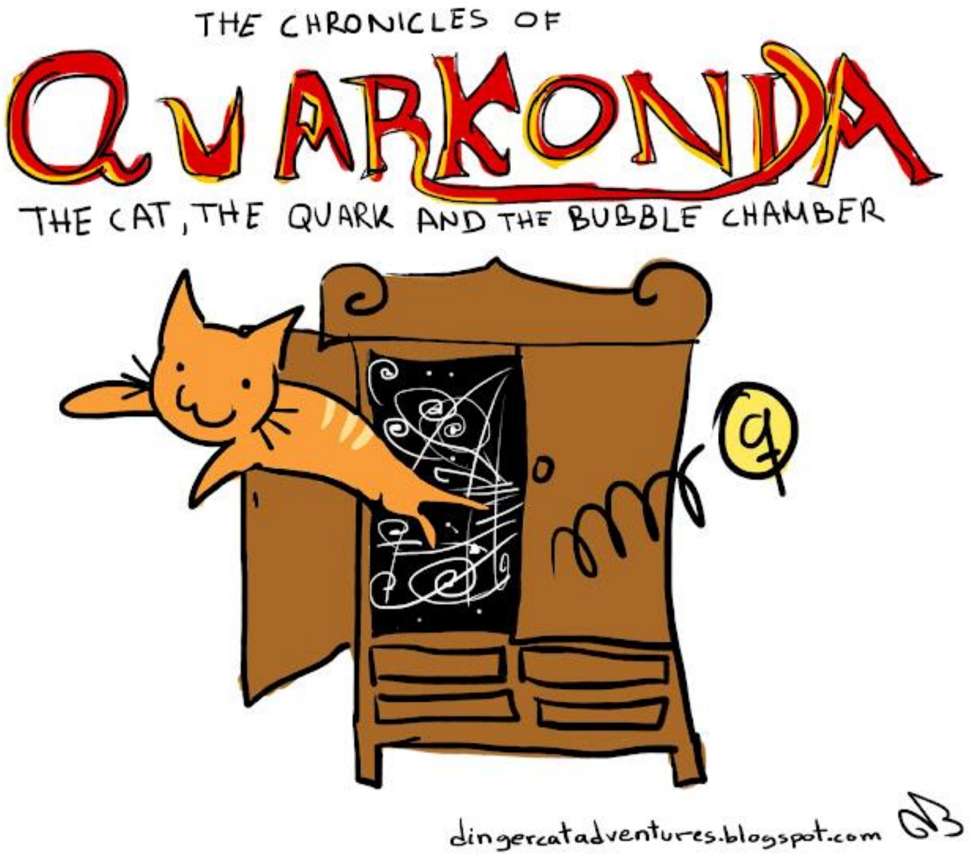
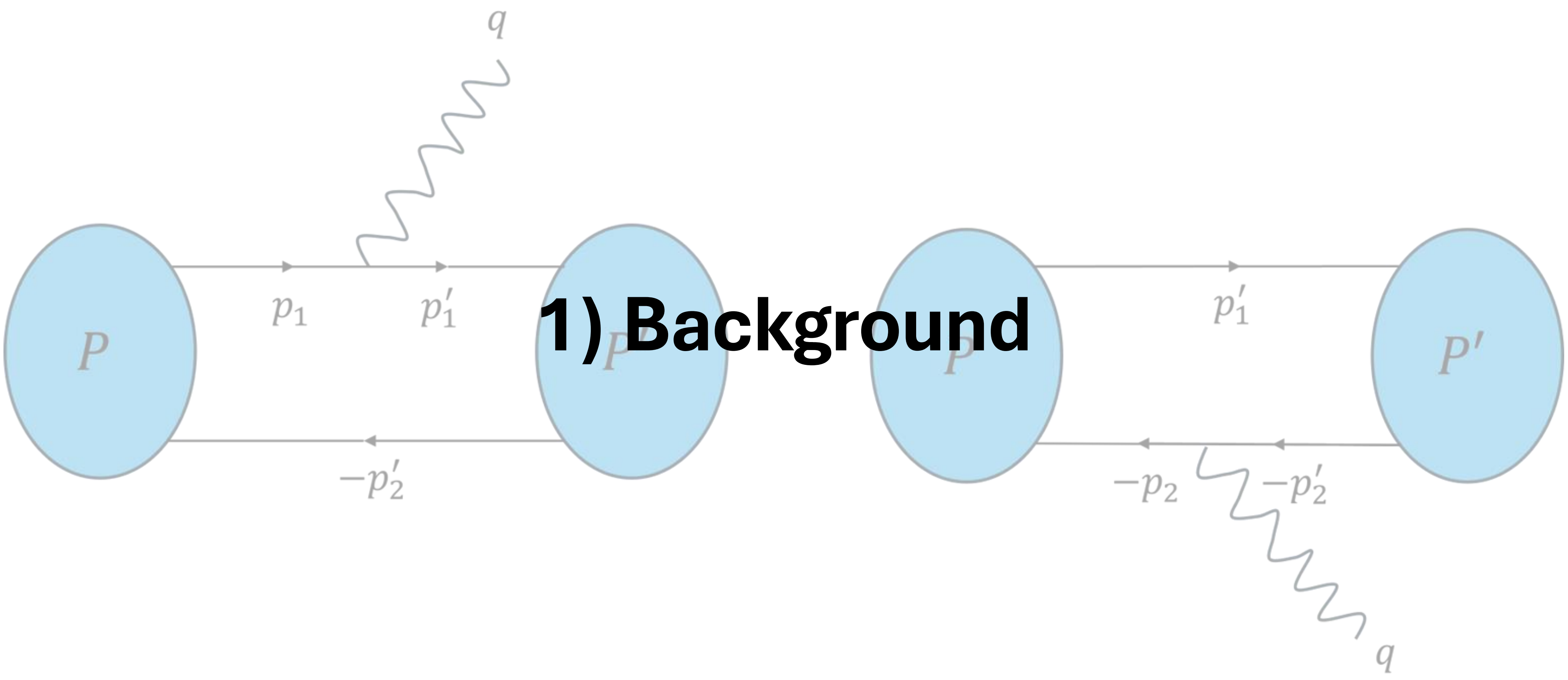
Outline

1) Background
2) Formalism
3) Result
4) Conclusion & Outlook



1) Background



Backgrounds

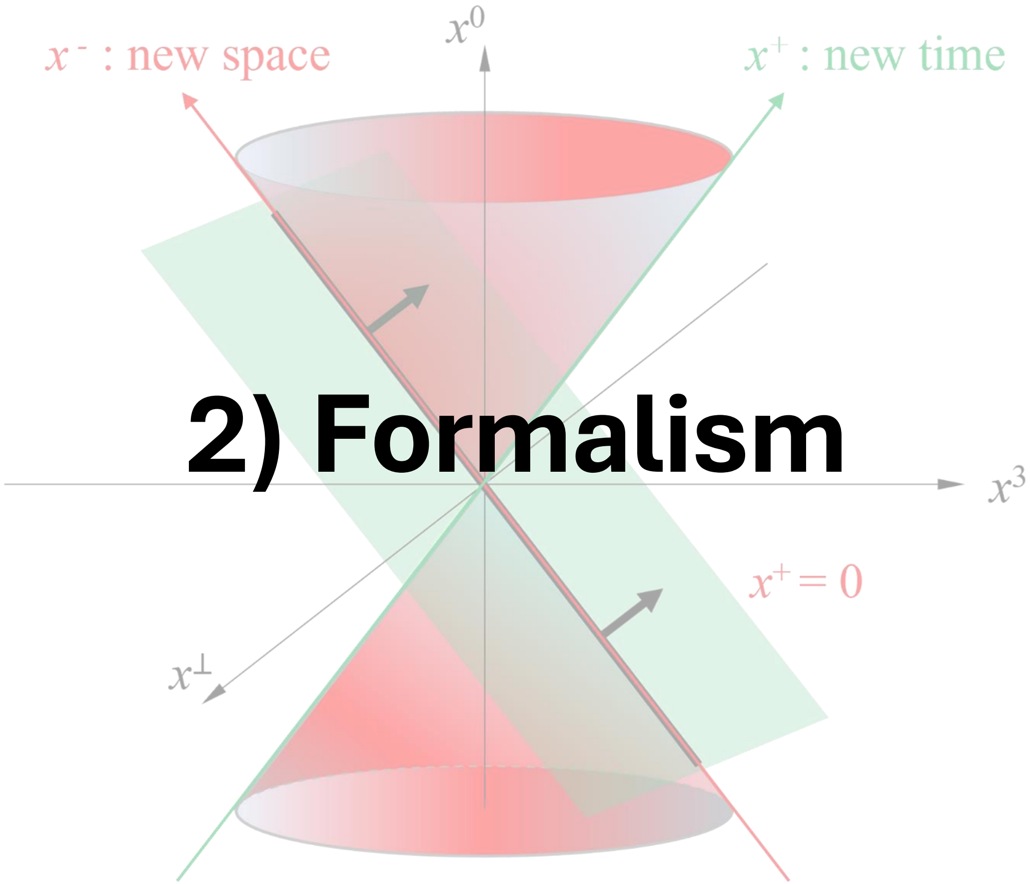
- Heavy quarkonia is the simplest hadron to observe and study their properties → e.g., charmonia ($c\bar{c}$) and bottomonia ($b\bar{b}$)

- A study about the radiative M1 transition[1] with good (J^+) and transverse ($\vec{\boldsymbol{j}}_\perp$) currents in BLFQ have been conducted
  → yielding different results in coupling constant on $c\bar{c}$ and $b\bar{b}$,
  → different $\vec{k}_\perp$ distribution on these two currents

- In this work, Light-front quark model (LFQM) will be used to derive the radiative transition → good (J^+) and transverse ($\vec{\boldsymbol{j}}_\perp$) current

[1] Li et al., PRD 98 (2018), 034024

2) Formalism



Light-Front Quark Model

- Light-Front Dynamics (LFD) and Constituent Quark Model (CQM)
  → treat hadron as a bound state $(q\bar{q}, qqq)$

- The light-front wave function (LFWF)

$$\Psi_{nS}^{JJ_z}(x, \boldsymbol{k}_\perp, \lambda_i) = \Phi_{nS}(x, \boldsymbol{k}_\perp)\ \mathcal{R}_{\lambda_q\lambda_{\bar{q}}}^{JJ_z}(x, \boldsymbol{k}_\perp),$$

→ Lorentz invariant variable $x_i = p_i^+/P^+; \boldsymbol{k}_\perp$

→ Radial WF: $\Phi_{nS}(x, \boldsymbol{k}_\perp)$ (the trial wave function)

→ Spin-orbit WF: $\mathcal{R}_{\lambda_q\lambda_{\bar{q}}}^{JJ_z} = \frac{1}{\sqrt{2}\tilde{M}_0}\bar{u}_{\lambda_q}(p_q)\Gamma_{\mathcal{P}(\mathcal{V})}v_{\lambda_{\bar{q}}}(p_{\bar{q}}),$

Operator

$$\Gamma_{\mathcal{P}} = \gamma_5, \quad \text{(pseudoscalar)}$$

$$\Gamma_{\mathcal{V}} = -\not{\epsilon}(J_z) + \frac{\epsilon\cdot(p_q - p_{\bar{q}})}{M_0 + m_q + m_{\bar{q}}}, \quad \text{(vector)}$$

Radiative M1 Transition

$\Delta L = 0, \Delta S = 1$, parity change

- The transition form factor (TFF) $F_{\mathcal{VP}}(q^2)$ is defined by

$$\langle \mathcal{P}(P')| J^{\mu}_{\rm em}(0) |\mathcal{V}(P,h)\rangle = ie\varepsilon^{\mu\nu\rho\sigma}\epsilon_\nu q_\rho P_\sigma F_{\mathcal{VP}}(Q^2),$$

the matrix element $\mathcal{J}^{\mu}_{h} = \langle \mathcal{P}(P')|J^{\mu}_{em}|\mathcal{V}(P,h)\rangle$

$$\mathcal{J}^{\mu}_{h} = \sum_{\lambda,\bar\lambda,j} \left\langle \Psi^{00\dagger}_{\lambda'\bar\lambda} \left| \frac{\bar u_{\lambda'}(p'_1)}{\sqrt{x'}} e^j_Q \gamma^\mu \frac{u_\lambda(p_1)}{\sqrt{x}} \right| \Psi^{1h}_{\lambda\bar\lambda} \right\rangle$$

$$= \sum_j e^j_Q \int \frac{{\rm d}x\ {\rm d}^2\boldsymbol{k}_\perp}{2(2\pi)^3} \Phi(x,\boldsymbol{k}'_\perp)\Phi(x,\boldsymbol{k}_\perp)$$

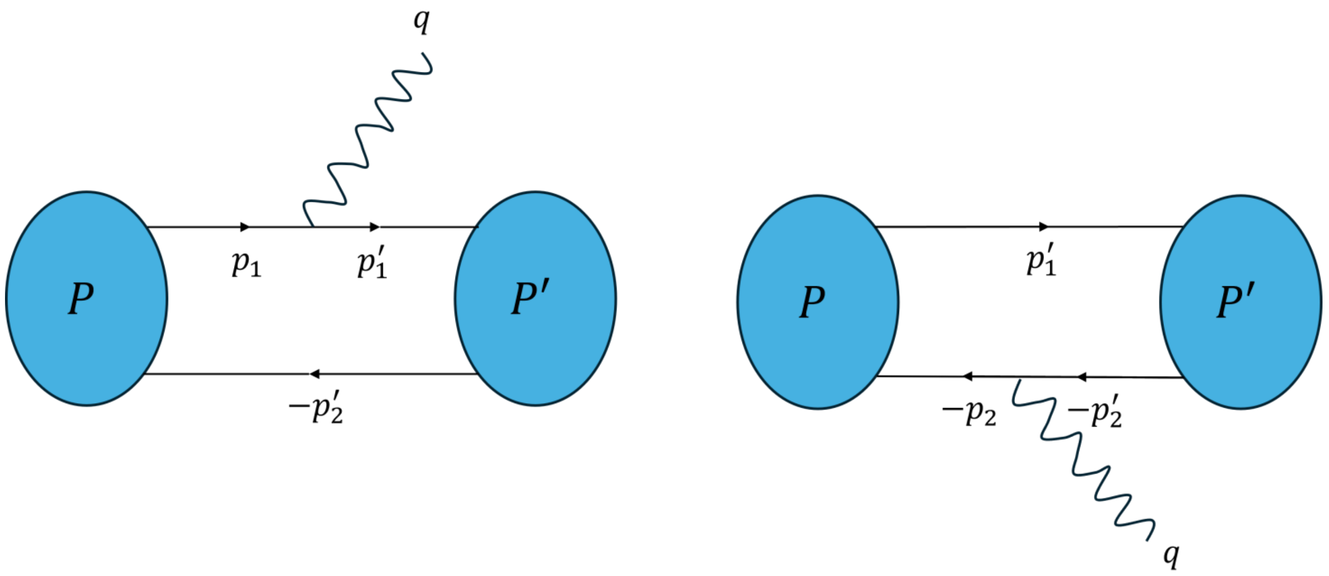
$$\times \sum_{\lambda,\bar\lambda} \mathcal{R}^{00\dagger}_{\lambda'\bar\lambda}(x,\boldsymbol{k}'_\perp) \frac{\bar u_{\lambda'}(p'_1)}{\sqrt{x'}} \gamma^\mu \frac{u_\lambda(p_1)}{\sqrt{x}} \mathcal{R}^{1h}_{\lambda\bar\lambda}(x,\boldsymbol{k}_\perp)$$

the tensor term

$$ie\varepsilon^{\mu\nu\rho\sigma}\epsilon_\nu q_\rho P_\sigma F_{\mathcal{VP}}(Q^2),$$

→ Good ($\mu = +$) and transverse ($\mu = \perp$) current are used to explore the TFF

- The lowest Feynman diagram $\mathcal{V}(P) \to \mathcal{P}(P')\gamma$



$$F_{\mathcal{VP}}(Q^2) = e_q I^\mu(m_q, m_{\bar{q}}, Q^2) + e_{\bar{q}} I^\mu(m_{\bar{q}}, m_q, Q^2),$$

quark — anti-quark

where $I_h^\mu(m_1, m_2, Q^2)$ is the one-loop integral

$$I_h^\mu = \int \frac{\mathrm{d}x\mathrm{d}^2\boldsymbol{k}_\perp}{2(2\pi)^3} \frac{\Phi(x,\boldsymbol{k}'_\perp)\Phi(x,\boldsymbol{k}_\perp)}{\sqrt{\mathcal{A}^2+\boldsymbol{k}'^2_\perp}\sqrt{\mathcal{A}^2+\boldsymbol{k}^2_\perp}} \mathcal{O}_h^\mu(x,\boldsymbol{k}_\perp)$$

operator

- Coupling constant,
  $g_{\mathcal{VP}} = F_{\mathcal{VP}}(0)$

- Decay width

$$\Gamma(\mathcal{V} \to \mathcal{P}\gamma) = \frac{\alpha_{\mathrm{em}}}{(2J_{\mathcal{V}}+1)} g^2_{\mathcal{VP}\gamma} k^3_\gamma,$$

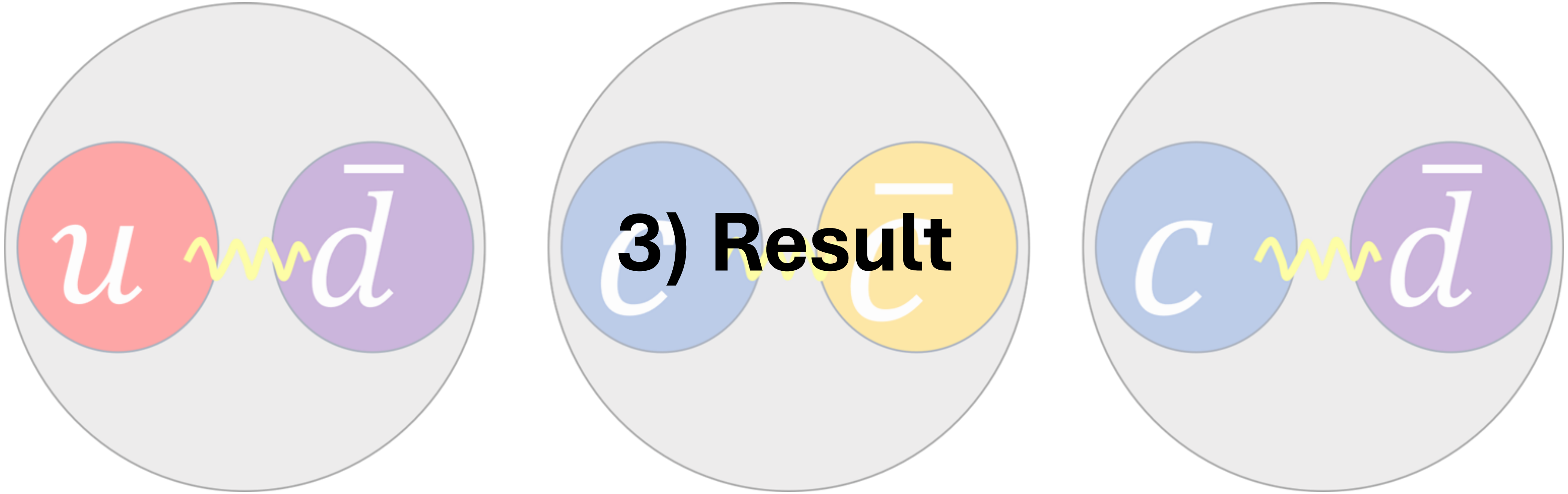
where

$$k_\gamma = \frac{(M^2_{\mathcal{V}} - M^2_{\mathcal{P}})}{2M_{\mathcal{V}}}$$

- Branching ratio

$$\mathrm{Br}(\mathcal{V} \to \mathcal{P}\gamma) = \frac{\Gamma(\mathcal{V} \to \mathcal{P}\gamma)}{\Gamma_{\mathrm{Total}}}$$

3) Result



Determining the parameter → Variational principle

- Trial wave function → Gaussian (H.O Basis)
  - Defining the mixing state (1S – 3S)

$$\begin{pmatrix}\Phi_{1S}\\ \Phi_{2S}\\ \Phi_{3S}\end{pmatrix} = \begin{pmatrix}c_1^{1S} & c_2^{1S} & c_3^{1S}\\ c_1^{2S} & c_2^{2S} & c_3^{2S}\\ c_1^{3S} & c_2^{3S} & c_3^{3S}\end{pmatrix}\begin{pmatrix}\phi_{1S}^{\mathrm{HO}}\\ \phi_{2S}^{\mathrm{HO}}\\ \phi_{3S}^{\mathrm{HO}}\end{pmatrix}$$

- Assuming some interactions
  - Screened pot., Coulomb, Hyperfine

$$V_{\text{Conf.}}^{\text{Scr.}} = a + \frac{b(1-e^{-\mu r})}{\mu} \qquad V_{\text{Coul.}} = -\frac{4\alpha_s}{3r}, \qquad V_{\text{Hyp.}} = \frac{2(\mathbf{S}_q\cdot\mathbf{S}_{\bar{q}})}{3m_q m_{\bar{q}}}\nabla^2 V_{\text{Coul.}}$$

- Variational analysis → mass spectra on 1S [2]
  - Treat pertubatively, i.e., neglecting Hyp. term

$$\frac{\partial M_{q\bar{q}}}{\partial\beta} = \frac{\partial\langle\Psi_{q\bar{q}}|(H_0+V_{\text{conf}}+V_{\text{coul}})|\Psi_{q\bar{q}}\rangle}{\partial\beta_{q\bar{q}}} = 0,$$

⟶ determine a, β, α, etc.

[2] Ridwan et al., ITM Web Conf. (2024), 01016

Parameter result → Variational Principle

$$\frac{\partial M_{q\bar{q}}}{\partial \beta} = \frac{\partial \langle \Phi | [H_0 + V_{q\bar{q}}] | \Phi \rangle}{\partial \beta} = \frac{\partial H_0}{\partial \beta} + \frac{\partial V_{q\bar{q}}}{\partial \beta} = 0$$

η_c and Υ in 1S as inputs

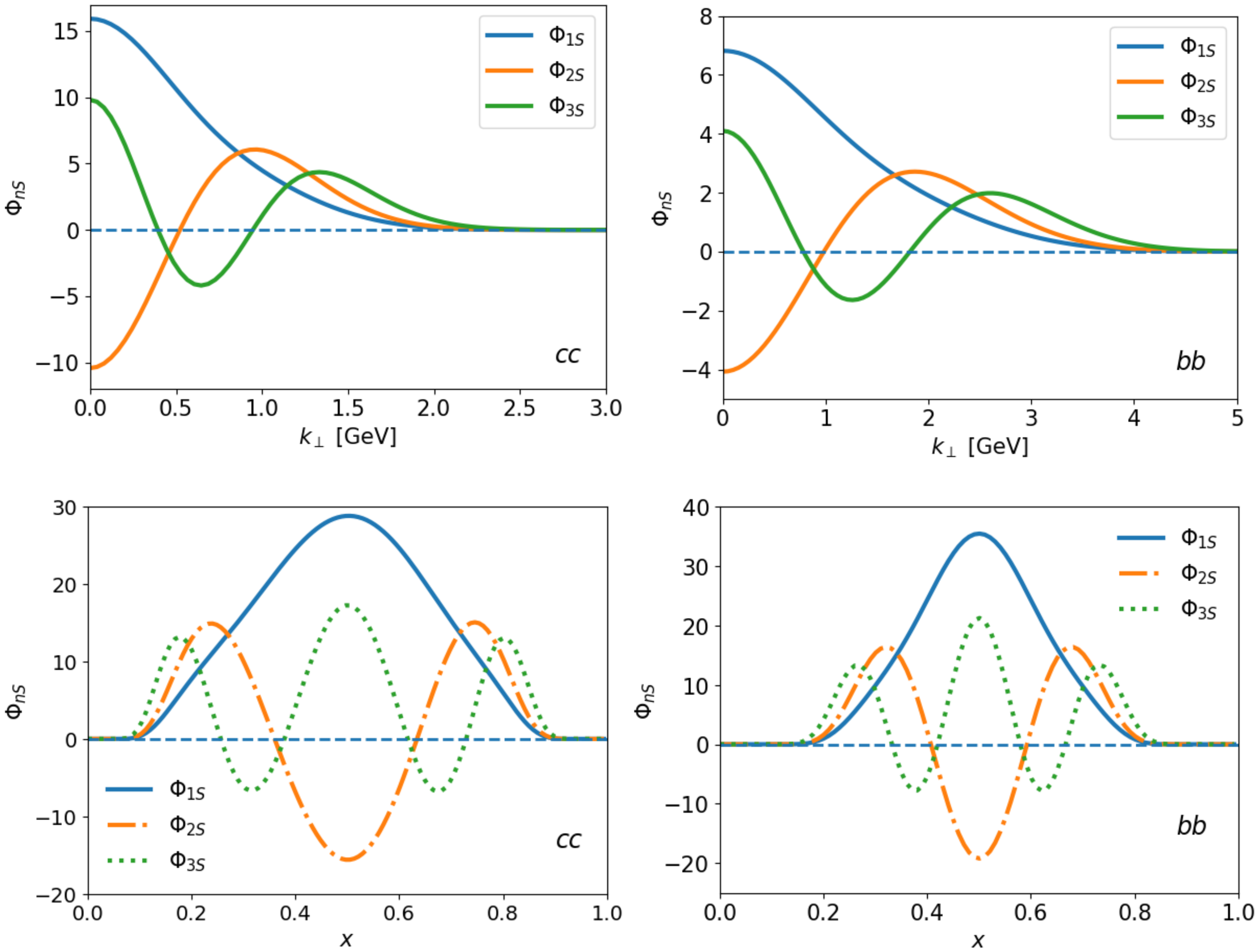
θ_{12}	$\theta_{13} = \theta_{23}$	m_c	m_b	a	b
12.12	8.44	1.61	4.97	−0.41	0.18

Fixed parameters

μ	α_s	$\beta_{c\bar{c}}$	$\beta_{c\bar{b}}$	$\beta_{b\bar{b}}$
0.027	0.402	0.5417	0.7019	1.0595

- 6 parameters are obtained

Wave function in the mixing state

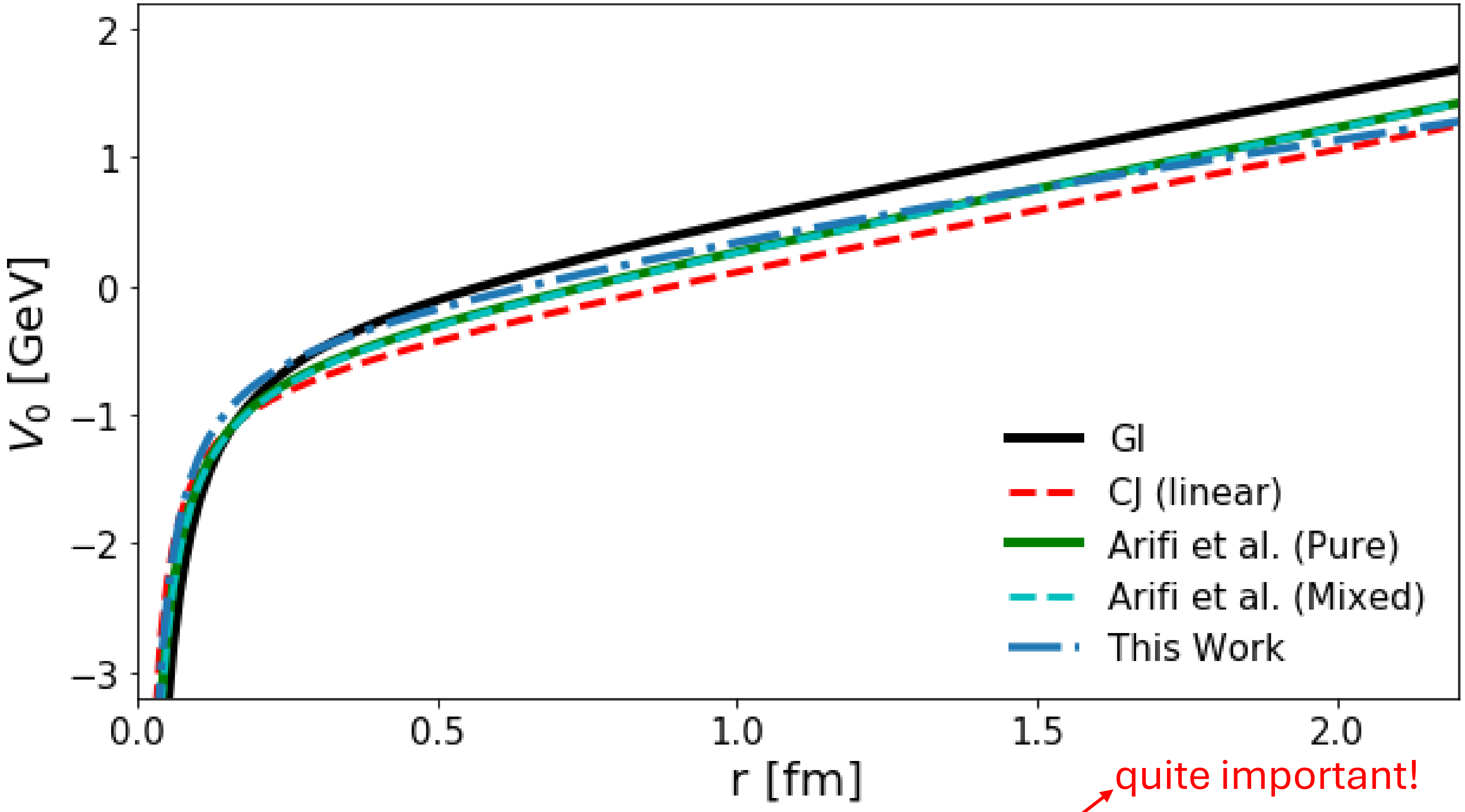


$$\begin{pmatrix}\Phi_{1S}\\ \Phi_{2S}\\ \Phi_{3S}\end{pmatrix} = \begin{pmatrix} c_1^{1S} & c_2^{1S} & c_3^{1S} \\ c_1^{2S} & c_2^{2S} & c_3^{2S} \\ c_1^{3S} & c_2^{3S} & c_3^{3S}\end{pmatrix} \begin{pmatrix}\phi_{1S}^{\rm HO}\\ \phi_{2S}^{\rm HO}\\ \phi_{3S}^{\rm HO}\end{pmatrix}$$

$$\Phi_{nS}(x, \boldsymbol{k}_\perp) = \sqrt{2(2\pi)^3}\sqrt{\frac{\partial k_z}{\partial x}}\phi_{nS}(\boldsymbol{k}),$$

$$\frac{\partial k_z}{\partial x} = \frac{M_0}{4x(1-x)}\left[1 - \frac{(m_q^2 - m_{\bar{q}}^2)^2}{M_0^4}\right]$$

➢ V_{Conf} in every model



Our V_{Conf} → Screened effect.

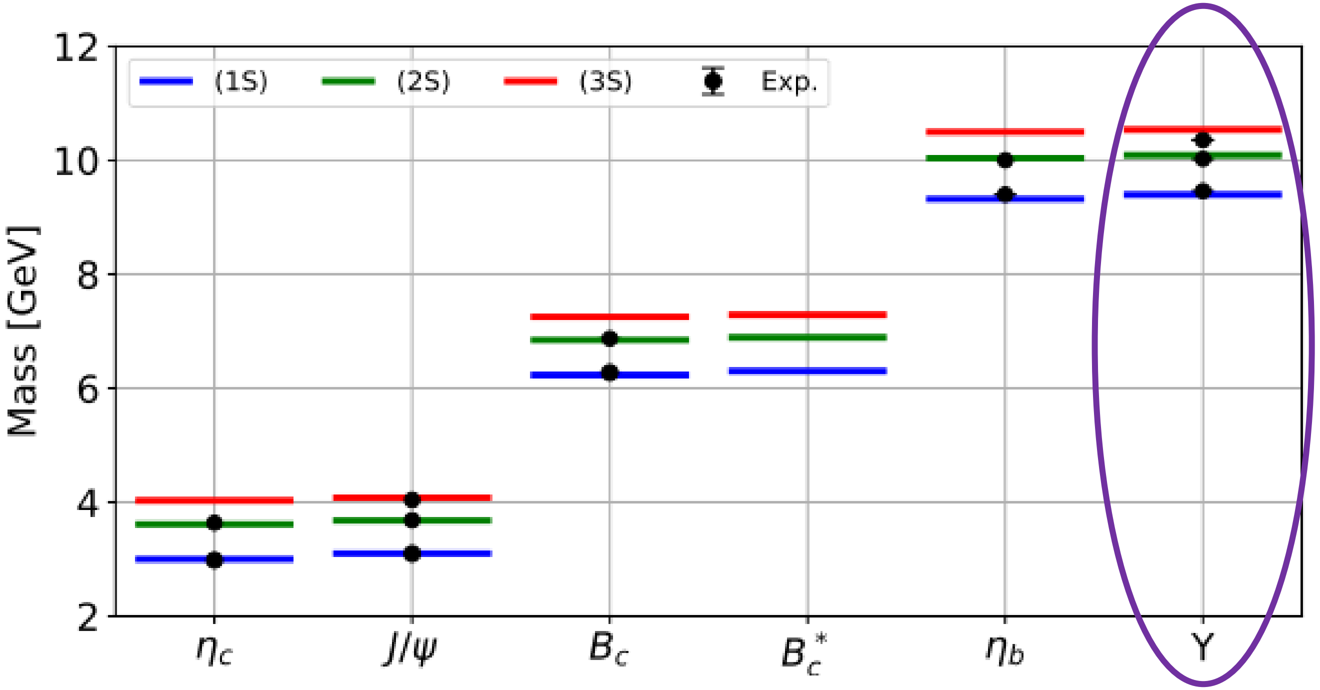
$$V_{\mathrm{Conf}} = a + \frac{b(1 - e^{-\mu r})}{\mu},$$

$$V_{\mathrm{Lin.}} = a + br,$$

quite important!

$$M_{q\bar{q}} = \langle H_0 \rangle + \langle V_{\mathrm{Conf}} \rangle + \langle V_{\mathrm{Coul}} \rangle + \langle V_{\mathrm{Hyp}} \rangle$$

➢ Mass Spectra

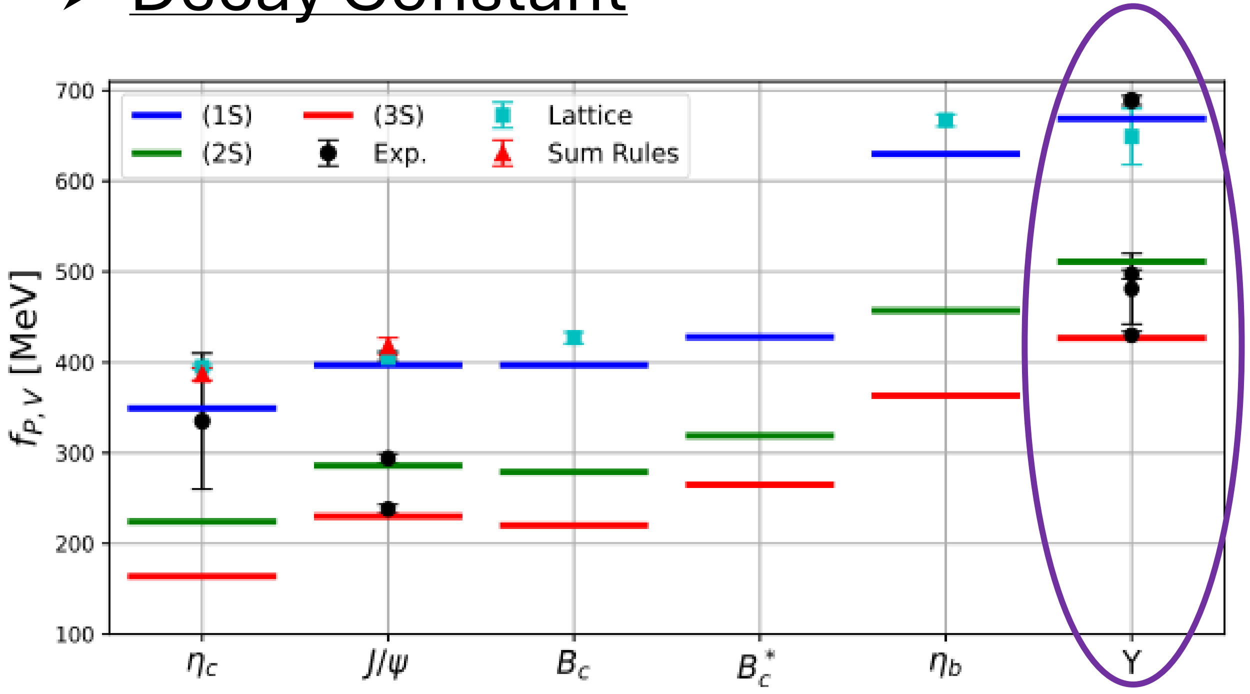


$$M_{q\bar{q}} = \langle \Psi^M_{q\bar{q}} | \hat{H} | \Psi^M_{q\bar{q}} \rangle = \langle \Phi | [H_0 + V_{q\bar{q}}] | \Phi \rangle$$

$$M_{q\bar{q}} = \langle H_0 \rangle + \langle V_{\text{Conf}} \rangle + \langle V_{\text{Coul}} \rangle + \langle V_{\text{Hyp}} \rangle$$

→ All heavy mesons are in good agreement with experimental data, except $\Upsilon(3S)$.

➢ Decay Constant



$$\langle 0|\bar{q}\gamma^\mu\gamma_5 q|P\rangle = i f_P P^\mu$$ Pseudoscalar

$$\langle 0|\bar{q}\gamma^\mu q|V(P,\lambda)\rangle = f_V M \epsilon^\mu(P,\lambda)$$ Vector

Explicit form

$$f_{P,V} = \sqrt{6}\int_0^1 dx \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^3} \frac{\Phi(x,\mathbf{k}_\perp)}{\sqrt{\mathcal{A}^2+\mathbf{k}_\perp^2}} \mathcal{O}_{P,V}$$

→ Only Υ's which have moderately difference value between ours and experimental data.

→ The hierarchy: $f_{1S} > f_{2S} > f_{3S}$

One-loop integral

$$I_h^\mu = \int \frac{\mathrm{d}x\mathrm{d}^2\boldsymbol{k}_\perp}{2(2\pi)^3} \frac{\Phi(x,\boldsymbol{k}'_\perp)\Phi(x,\boldsymbol{k}_\perp)}{\sqrt{\mathcal{A}^2+\boldsymbol{k}'^2_\perp}\sqrt{\mathcal{A}^2+\boldsymbol{k}^2_\perp}} \mathcal{O}_h^\mu(x,\boldsymbol{k}_\perp).$$

operator

μ	$\epsilon(h)$	$\mathcal{O}$
$+$	$\epsilon(0)$	$\dots$
	$\epsilon(\pm1)$	$2(1-x)\left[\mathcal{A}+\frac{2}{\mathcal{D}_0}\left(\boldsymbol{k}_\perp^2-\frac{(\boldsymbol{k}_\perp\cdot\boldsymbol{q}_\perp)^2}{\boldsymbol{q}_\perp^2}\right)\right]$
$\perp$	$\epsilon(0)$	$\frac{1}{xM_0}\left\{\mathcal{A}\left(\mathcal{A}+\frac{2\boldsymbol{k}_\perp^2}{\mathcal{D}_0}\right)+\frac{\mathcal{M}}{\mathcal{D}_0}\left[(1-2x)\boldsymbol{k}_\perp^2+(1-x)\left((\boldsymbol{k}_\perp\cdot\boldsymbol{q}_\perp)-\frac{2(\boldsymbol{k}_\perp\cdot\boldsymbol{q}_\perp)^2}{\boldsymbol{q}_\perp^2}\right)\right]\right\}$

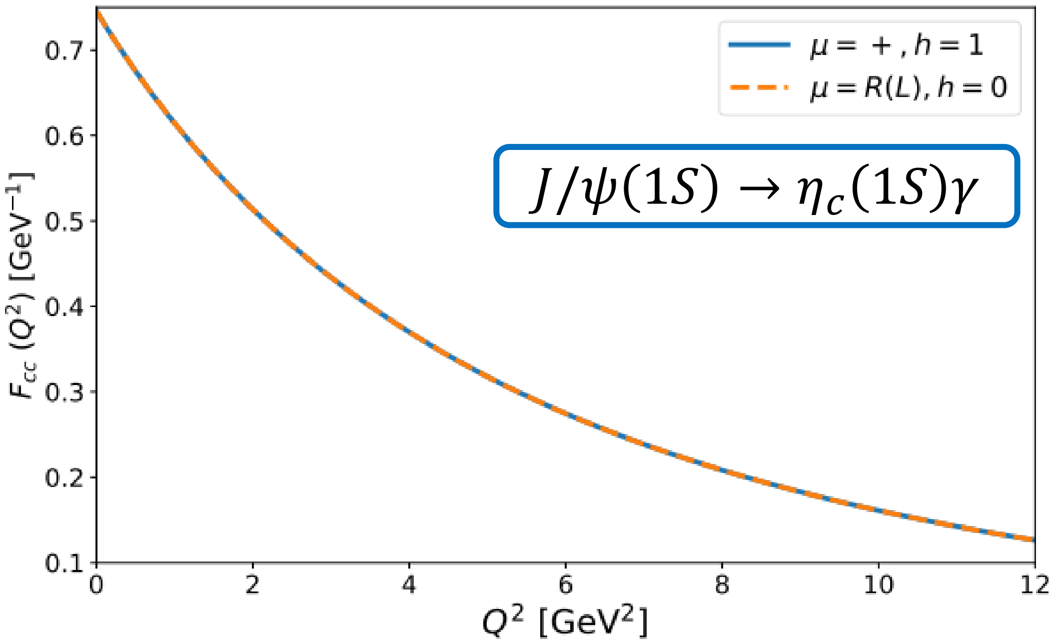
➡ Only good current ($h=0$) does not have operator since $\mathcal{G}_0^+ = 0$

- Simplifying tensor and matrix element term will establish the operator.
- We use these to find the properties and compare each other between current

➢Coupling constant ($g_{\mathcal{VP}}$): good vs transverse

Charmonia	$g_{\mathcal{VP}}$ (GeV^-1)
$J/\psi(1S) \to \eta_c(1S)\gamma$	0.74523 $[\mu = +, h = 1]$
	0.74523 $[\mu = R(L), h = 0]$
$\psi(2S) \to \eta_c(2S)\gamma$	0.71341 $[\mu = +, h = 1]$
	0.71341 $[\mu = R(L), h = 0]$
$\psi(3S) \to \eta_c(3S)\gamma$	0.68771 $[\mu = +, h = 1]$
	0.68771 $[\mu = R(L), h = 0]$

Bottomonia	$g_{\mathcal{VP}}$ (GeV^-1)
$\Upsilon(1S) \to \eta_b(1S)\gamma$	-0.12792 $[\mu = +, h = 1]$
	-0.12792 $[\mu = R(L), h = 0]$
$\Upsilon(2S) \to \eta_b(2S)\gamma$	-0.12508 $[\mu = +, h = 1]$
	-0.12508 $[\mu = R(L), h = 0]$
$\Upsilon(3S) \to \eta_b(3S)\gamma$	-0.12265 $[\mu = +, h = 1]$
	-0.12265 $[\mu = R(L), h = 0]$



$\mu = +$ and $\mu = R(L)$ give the same result!

➢Coupling constant for: (in GeV^-1)

Charmonia

Transition	Our	BLFQ [3]	GI [4]
$J/\psi(1S) \to \eta_c(1S)\gamma$	0.745	0.873	0.69
$\psi(2S) \to \eta_c(2S)\gamma$	0.713	0.739	0.68
$\psi(2S) \to \eta_c(1S)\gamma$	-0.060	-0.144	-0.056
$\psi(3S) \to \eta_c(3S)\gamma$	0.688	...	...
$\psi(3S) \to \eta_c(2S)\gamma$	-0.059	...	...
$\psi(3S) \to \eta_c(1S)\gamma$	-0.011	...	...
$\eta_c(2S) \to J/\psi(1S)\gamma$	-0.060	...	...
$\eta_c(3S) \to \psi(2S)\gamma$	-0.059	...	...
$\eta_c(3S) \to J/\psi(1S)\gamma$	-0.011	...	...

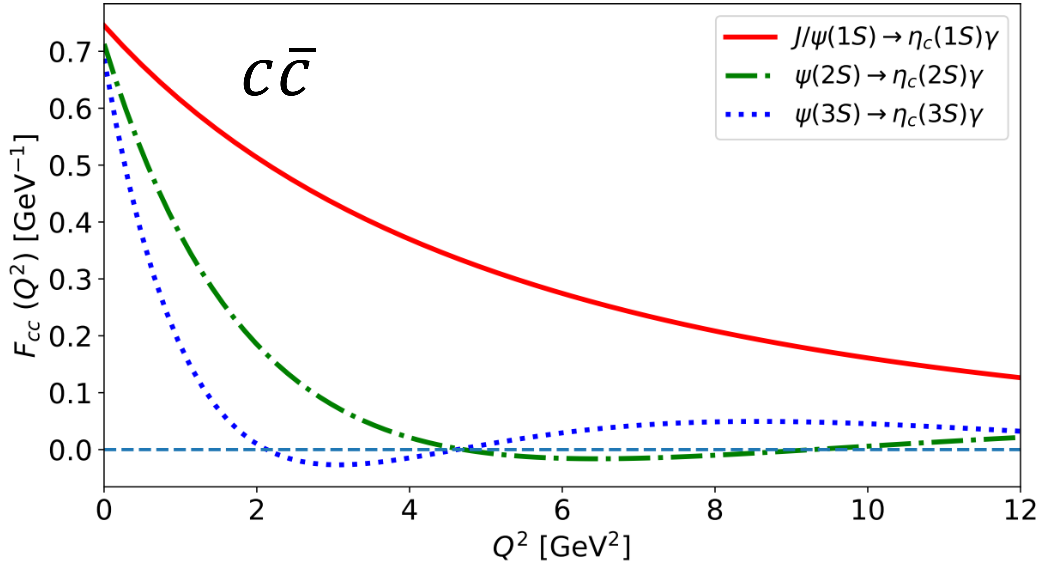
Bottomia

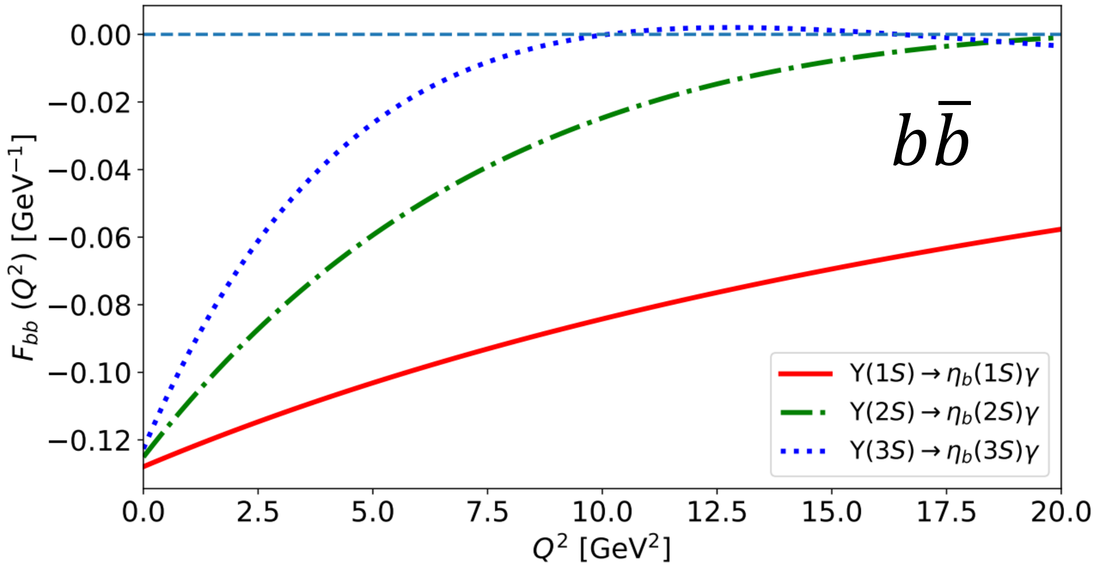
Transition	Our	BLFQ [3]	GI [4]
$\Upsilon(1S) \to \eta_b(1S)\gamma$	-0.128	-0.141	-0.130
$\Upsilon(2S) \to \eta_b(2S)\gamma$	-0.125	-0.134	-0.120
$\Upsilon(2S) \to \eta_b(1S)\gamma$	0.005	0.011	0.007
$\Upsilon(3S) \to \eta_b(3S)\gamma$	-0.123	-0.134	-0.120
$\Upsilon(3S) \to \eta_b(2S)\gamma$	0.005	0.009	0.007
$\Upsilon(3S) \to \eta_b(1S)\gamma$	0.001	0.005	0.004
$\eta_b(2S) \to \Upsilon(1S)\gamma$	0.005	0.006	...
$\eta_b(3S) \to \Upsilon(2S)\gamma$	0.005	0.004	...
$\eta_b(3S) \to \Upsilon(1S)\gamma$	0.001	0.002	...

→ Minus sign indicates a destructive in the transition

[3] Li et al., PRD **98** (2024), 034024

[4] Godfrey & Isgur, PRD **32** (1985), 189

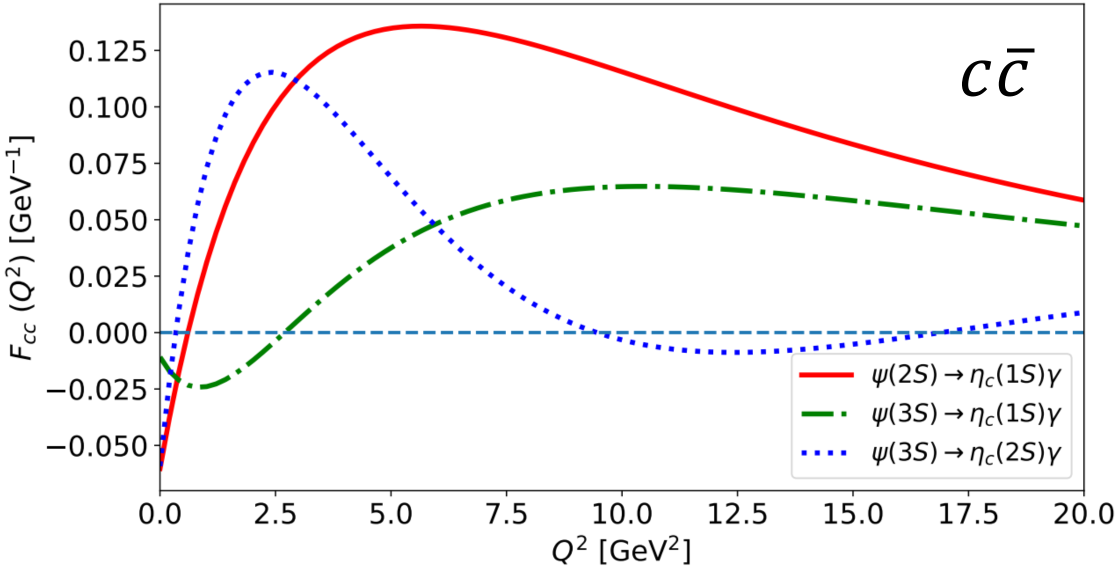


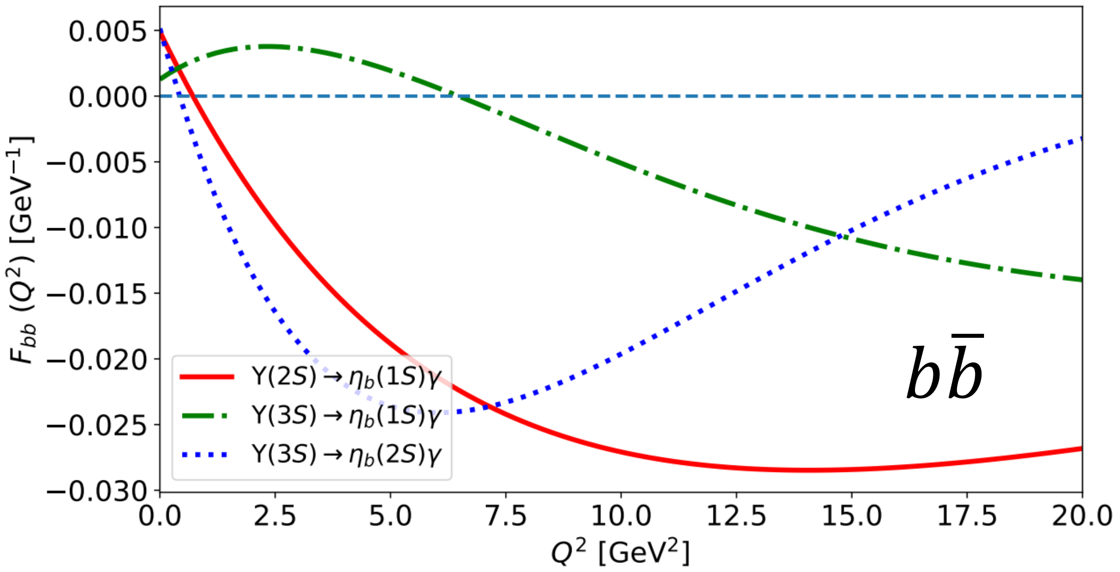


$$\mathcal{V}(nS) \to \mathcal{P}(n'S)\gamma \quad (n = n')$$

- the *allowed* transition due to the overlap of the initial and final wave function.

$$I_h^\mu = \int \frac{\mathrm{d}x\mathrm{d}^2\boldsymbol{k}_\perp}{2(2\pi)^3} \frac{\Phi(x,\boldsymbol{k}'_\perp)\Phi(x,\boldsymbol{k}_\perp)}{\sqrt{\mathcal{A}^2+\boldsymbol{k}'^2_\perp}\sqrt{\mathcal{A}^2+\boldsymbol{k}^2_\perp}} \mathcal{O}_h^\mu(x,\boldsymbol{k}_\perp).$$





$$\mathcal{V}(nS) \to \mathcal{P}(n'S)\gamma \quad (n \neq n')$$

- the *hindered* transition due to the orthogonality of the wave functions.

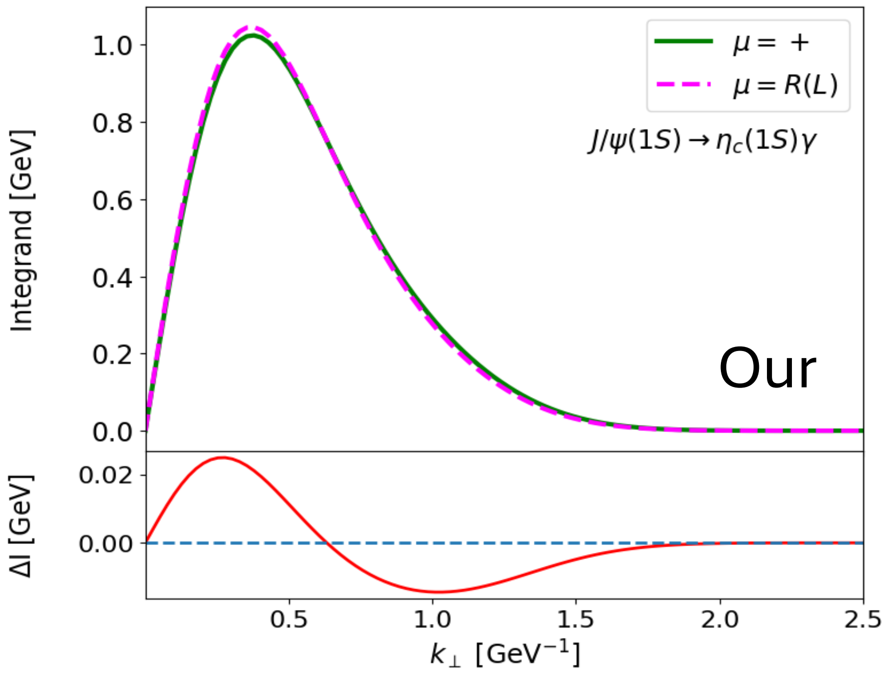
$$I_h^\mu = \int \frac{\mathrm{d}x\mathrm{d}^2\boldsymbol{k}_\perp}{2(2\pi)^3} \frac{\boxed{\Phi(x, \boldsymbol{k}'_\perp)\Phi(x, \boldsymbol{k}_\perp)}}{\sqrt{\mathcal{A}^2 + \boldsymbol{k}'^2_\perp}\sqrt{\mathcal{A}^2 + \boldsymbol{k}^2_\perp}} \mathcal{O}_h^\mu(x, \boldsymbol{k}_\perp).$$

- To see the contribution of $k_\perp$ in two frame, we plot the integrands (those inside {...}) vs $k_\perp$

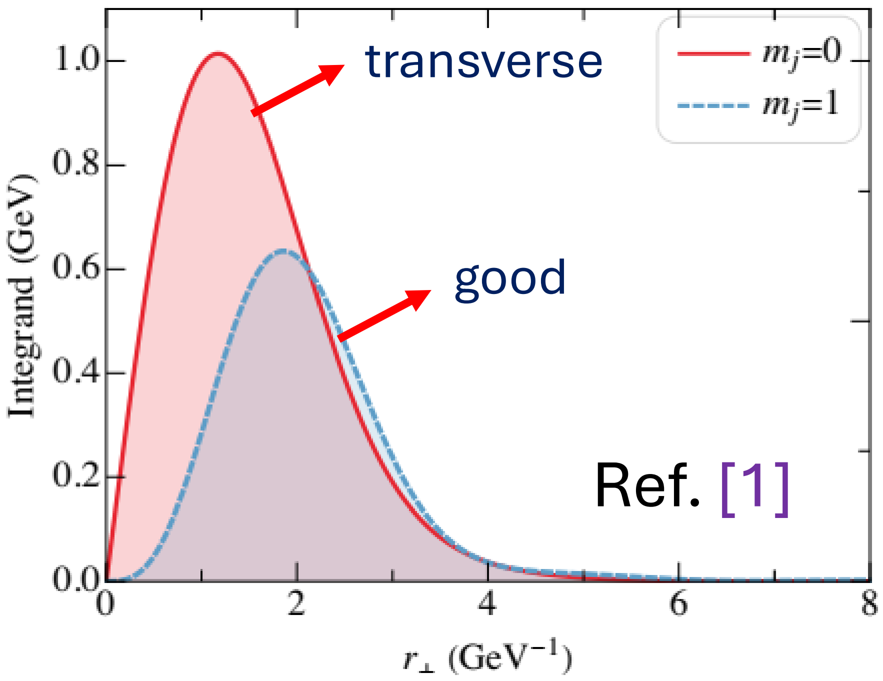
$$I_h^\mu = \int_0^\infty k_\perp \Big\{ \int_0^{2\pi} \theta \int_0^1 x \frac{k_\perp}{2(2\pi)^3} \frac{\Phi(x, k'_\perp)\Phi(x, k_\perp)}{\sqrt{\mathcal{A}^2 + k'^2_\perp}\sqrt{\mathcal{A}^2 + k^2_\perp}} \mathcal{O}_h^\mu(x, k_\perp) \Big\}$$

depends on μ and h!
→ expect different curve

However....



VS

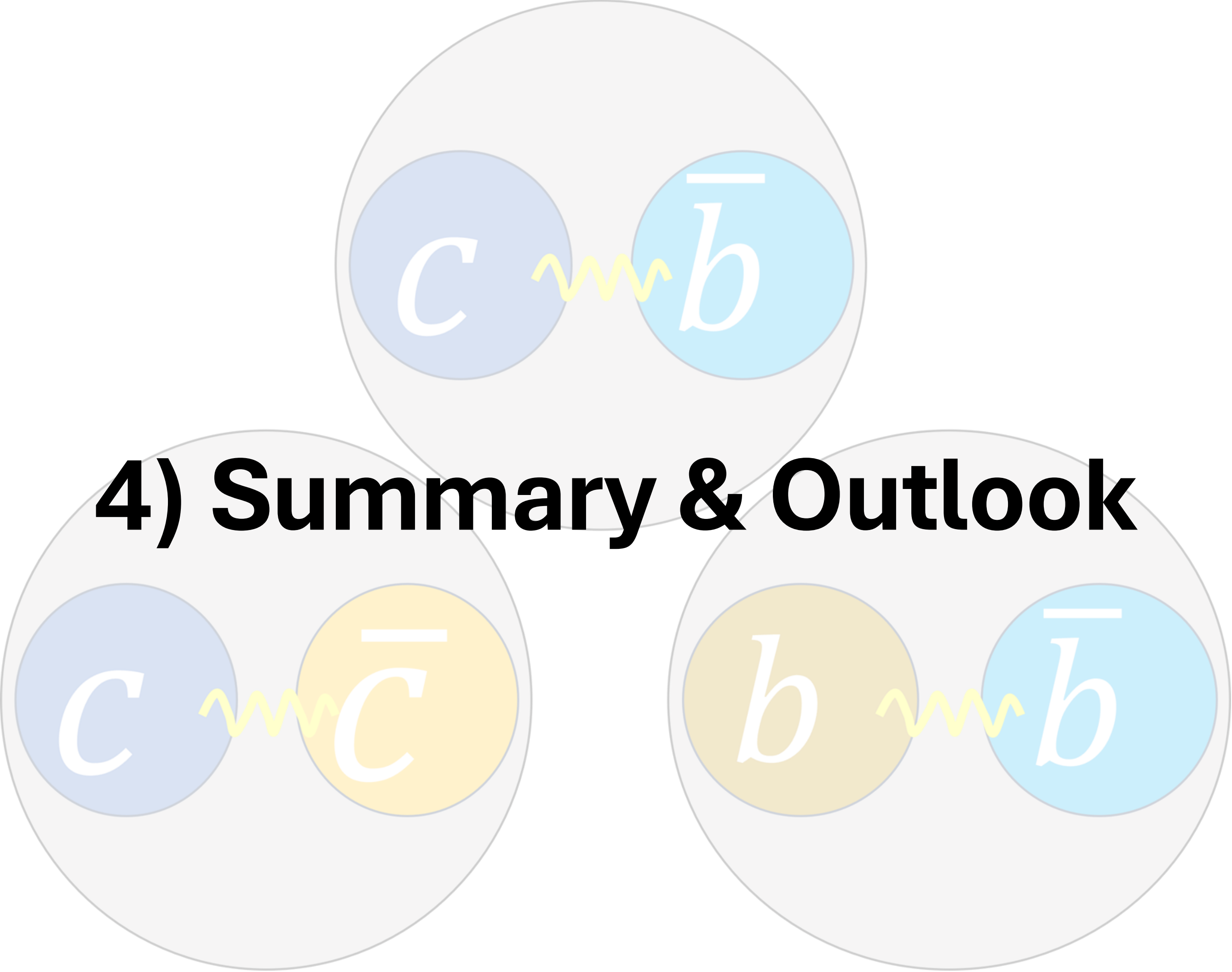


[1] Li et al., PRD 98 (2018), 034024

➢Γ and Br in *cc* and *bb*

Transition	Γ(Our)	Expt. [16]	NRQM [20]	RQM [21]	Br(Our)	Expt. [16]	NRQM [20]	RQM [21]
$J/\psi \to \eta_c(1S)\gamma$	1.94	1.57	2.72	1.05	2.10×10^{-2}	$1.7(0.4)\times10^{-2}$	2.94×10^{-2}	1.13×10^{-2}
$\psi(2S) \to \eta_c(2S)\gamma$	0.13	0.206	1.17	0.99	4.56×10^{-4}	$7(5)\times10^{-4}$	3.98×10^{-3}	3.37×10^{-3}
$\psi(3S) \to \eta_c(3S)^{\dagger}\gamma$	1.53×10^{-3}	...	9.93	...	1.91×10^{-8}	...	1.24×10^{-4}	...
$\psi(2S) \to \eta_c(1S)\gamma$	2.27	0.99	7.51	0.95	7.72×10^{-3}	$3.4(0.5)\times10^{-3}$	2.55×10^{-2}	3.23×10^{-3}
$\psi(3S) \to \eta_c(2S)\gamma$	0.47	...	...	...	5.94×10^{-6}	...	...	...
$\psi(3S) \to \eta_c(1S)\gamma$	0.22	...	...	...	2.75×10^{-6}	...	...	...
$\eta_c(2S) \to J/\psi\gamma$	3.30	...	...	...	2.34×10^{-4}	$< 1.39\times10^{-2}$	...	...
$\eta_c(3S)^{\dagger} \to \psi(2S)\gamma$	0.90	...	...	...	...	...	...	...
$\eta_c(3S)^{\dagger} \to J/\psi\gamma$	0.47	...	...	...	...	...	...	...
$\Upsilon(1S) \to \eta_b(1S)\gamma$	8.96×10^{-3}	...	3.77×10^{-4}	5.8×10^{-3}	1.66×10^{-4}	...	6.98×10^{-6}	1.07×10^{-4}
$\Upsilon(2S) \to \eta_b(2S)\gamma$	5.24×10^{-4}	...	5.62×10^{-3}	1.4×10^{-3}	1.64×10^{-5}	...	1.76×10^{-4}	4.38×10^{-5}
$\Upsilon(3S)^{\dagger} \to \eta_b(3S)^{\dagger}\gamma$	2.34×10^{-3}	...	2.85×10^{-3}	0.8×10^{-3}	1.15×10^{-4}	...	1.40×10^{-4}	3.94×10^{-5}
$\Upsilon(2S) \to \eta_b(1S)\gamma$	1.29×10^{-2}	1.76×10^{-2}	7.72×10^{-4}	6.4×10^{-3}	4.03×10^{-4}	$5.5^{+1.1}_{-0.9}\times10^{-4}$	2.41×10^{-5}	2.00×10^{-4}
$\Upsilon(3S) \to \eta_b(2S)\gamma$	2.77×10^{-3}	$< 1.26\times10^{-2}$	3.62×10^{-4}	1.5×10^{-3}	1.36×10^{-4}	$<6.7\times10^{-4}$	1.78×10^{-4}	7.38×10^{-5}
$\Upsilon(3S) \to \eta_b(1S)\gamma$	3.02×10^{-3}	1.03×10^{-2}	7.70×10^{-4}	1.05×10^{-4}	1.48×10^{-4}	$5.1(0.7)\times10^{-4}$	3.79×10^{-4}	5.17×10^{-5}
$\eta_b(2S) \to \Upsilon(1S)\gamma$	2.54×10^{-2}	...	...	1.18×10^{-4}	1.06×10^{-6}	...	...	1.18×10^{-8}
$\eta_b(3S)^{\dagger} \to \Upsilon(2S)\gamma$	1.91×10^{-2}	...	...	2.8×10^{-3}	...	...	...	...
$\eta_b(3S)^{\dagger} \to \Upsilon(1S)\gamma$	1.14×10^{-2}	...	...	2.4×10^{-4}	...	...	...	...

4) Summary & Outlook



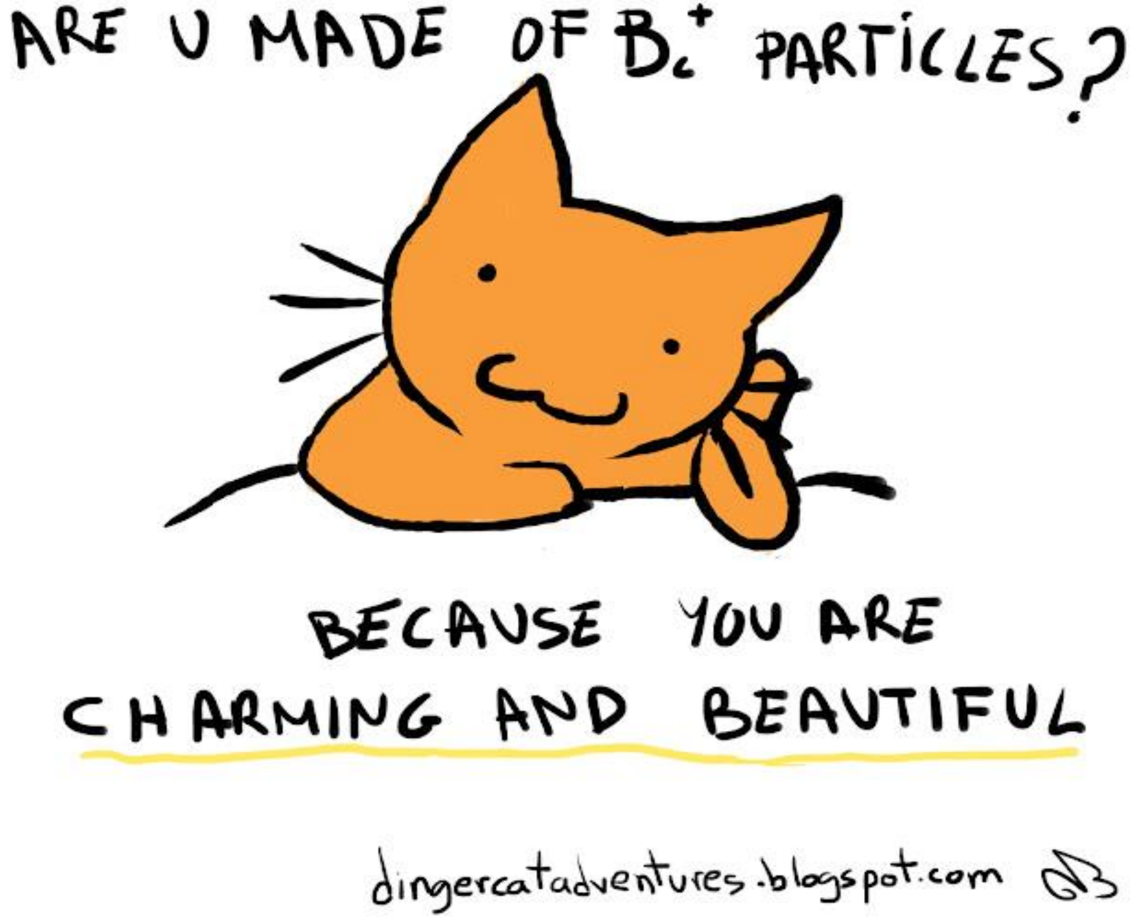
Summary and Outlook

- We have studied the radiative M1 transition for charmonia and bottomonia,
  - We obtained the same coupling constant $g_{\mathcal{VP}}$ for the good and transverse current,
  - We found that our results yield the same $\vec{k}_\perp$ distributions for the good and transverse current

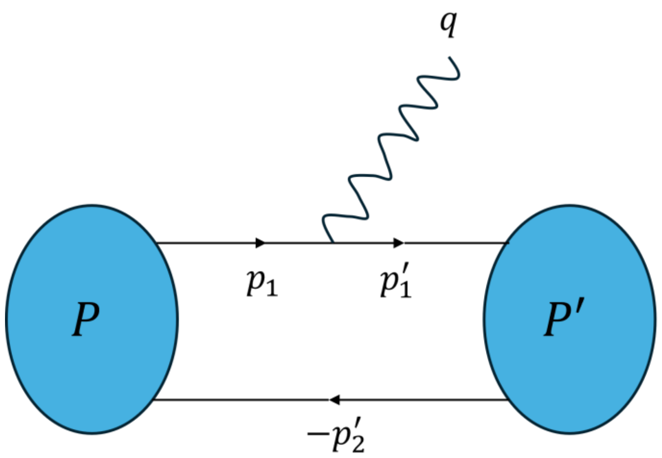
- We would consider the bad current ($\mu = -$) for the sake of completeness.

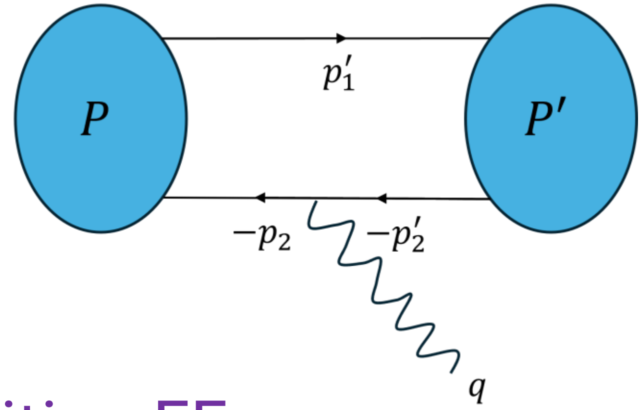
Thank You For Your Attention!

Thank You For Your Attention!



Backup Slide





→The lowest Feynman diagram
$\mathcal{V}(P) \to \mathcal{P}(P')\gamma$

Transition FF:

$$F_{\mathcal{VP}}(Q^2) = e_q I^\mu(m_q, m_{\bar{q}}, Q^2) + e_{\bar{q}} I^\mu(m_{\bar{q}}, m_q, Q^2),$$

quark anti-quark

where $I_h^\mu(m_1, m_2, Q^2)$ is the one-loop integral:

$$I_h^\mu = \int \frac{\mathrm{d}x\mathrm{d}^2\boldsymbol{k}_\perp}{2(2\pi)^3} \frac{\Phi(x, \boldsymbol{k}'_\perp)\Phi(x, \boldsymbol{k}_\perp)}{\sqrt{\mathcal{A}^2 + \boldsymbol{k}'^2_\perp}\sqrt{\mathcal{A}^2 + \boldsymbol{k}^2_\perp}} \mathcal{O}_h^\mu(x, \boldsymbol{k}_\perp)$$

operator

Coupling constant: $g_{\mathcal{VP}} = F_{\mathcal{VP}}(0)$

➢ Γ in cb mesons

Transition	Γ(Our)	NRQM1 [20]	RQM [19]
$B_c^*(1S)^\dagger \to B_c(1S)\gamma$	4.74×10^{-3}	4.04×10^{-2}	3.30×10^{-2}
$B_c^*(2S)^\dagger \to B_c(2S)\gamma$	1.77×10^{-3}	3.30×10^{-3}	1.70×10^{-2}
$B_c^*(3S)^\dagger \to B_c(3S)^\dagger\gamma$	8.09×10^{-3}	...	...
$B_c^*(2S)^\dagger \to B_c(1S)\gamma$	0.58	0.56	0.43
$B_c^*(3S)^\dagger \to B_c(2S)\gamma$	0.17	...	...
$B_c^*(3S)^\dagger \to B_c(1S)\gamma$	4.97×10^{-2}	...	...
$B_c(2S) \to B_c^*(1S)^\dagger\gamma$	1.36	0.14	0.49
$B_c(3S)^\dagger \to B_c^*(2S)^\dagger\gamma$	0.33	...	...
$B_c(3S)^\dagger \to B_c^*(1S)^\dagger\gamma$	0.12	...	...

→ We cannot compute Br in cb mesons due to the unavailable of total width in PDG

μ	$\epsilon(h)$	$\mathcal{G}_h^\mu(\boldsymbol{P}_\perp \neq 0)$	$\mathcal{G}_h^\mu(\boldsymbol{P}_\perp = 0)$
$+$	$\epsilon(0)$	0	0
	$\epsilon(+1)$	$\frac{eP^+q^R}{\sqrt{2}}$	$\frac{eP^+q^R}{\sqrt{2}}$
	$\epsilon(-1)$	$\frac{eP^+q^L}{\sqrt{2}}$	$\frac{eP^+q^L}{\sqrt{2}}$
R	$\epsilon(0)$	$-eMq^R$	$-eMq^R$
	$\epsilon(+1)$	$\frac{-eP^Rq^R}{\sqrt{2}}$	0
	$\epsilon(-1)$	$\frac{e\left(q^-P^+ + P^Lq^R\right)}{\sqrt{2}}$	$\frac{eq^-P^+}{\sqrt{2}}$
L	$\epsilon(0)$	eMq^L	eMq^L
	$\epsilon(+1)$	$\frac{e\left(q^-P^+ + P^Rq^L\right)}{\sqrt{2}}$	$\frac{eq^-P^+}{\sqrt{2}}$
	$\epsilon(-1)$	$\frac{-eP^Lq^L}{\sqrt{2}}$	0

the tensor term → $\mathcal{G}_h^\mu$ $\quad ie\varepsilon^{\mu\nu\rho\sigma}\epsilon_\nu q_\rho P_\sigma F_{\mathcal{VP}}(Q^2),$

Transverse → $q^{R(L)} = q_x \pm iq_y$

- We consider $\boldsymbol{P}_\perp = 0$ and $\boldsymbol{P}_\perp \neq 0$ contribution.

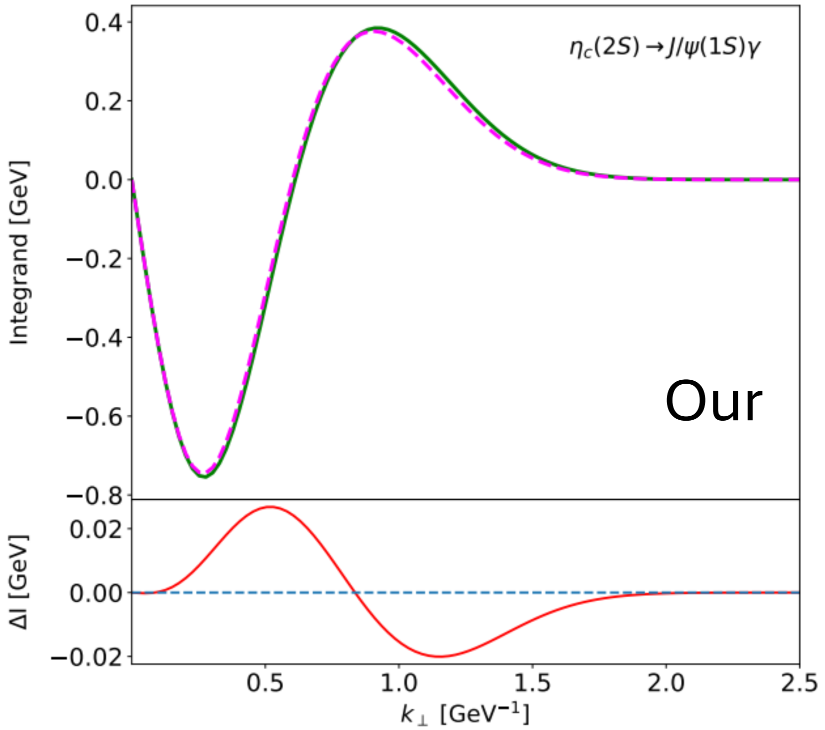
Matrix elements in $\mathcal{J}_h^\mu$

$$\sum_{\lambda,\bar{\lambda}} \mathcal{R}^{00\dagger}_{\lambda'\bar{\lambda}}(x,\boldsymbol{k}'_\perp)\frac{\bar{u}_{\lambda'}(p'_1)}{\sqrt{x'}}\gamma^\mu\frac{u_\lambda(p_1)}{\sqrt{x}}\mathcal{R}^{1h}_{\lambda\bar{\lambda}}(x,\boldsymbol{k}_\perp)$$

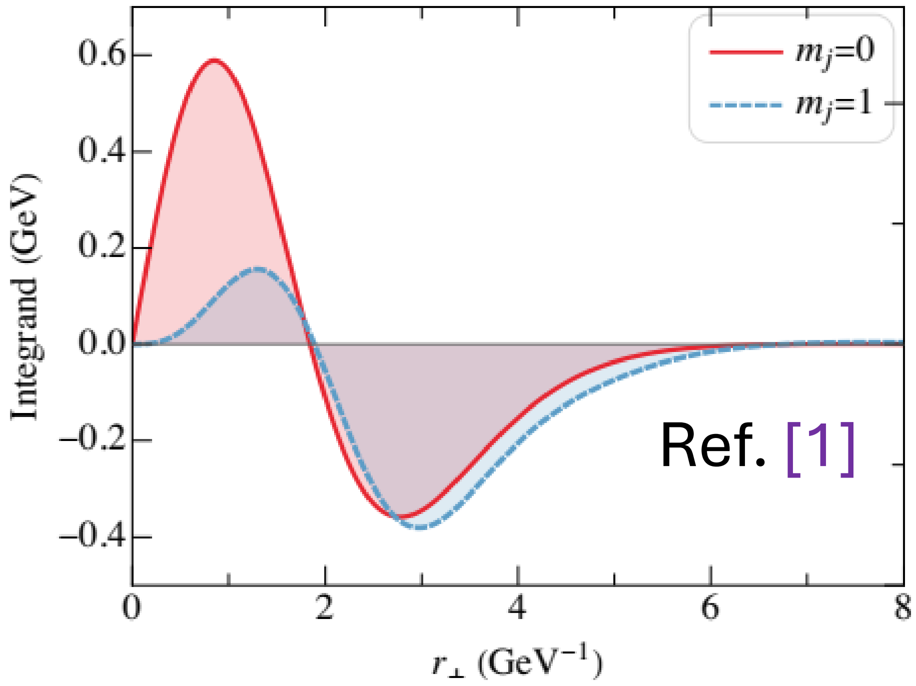
Matrix element	↑→↑	↑→↓	↓→↑	↓→↓
$\bar{u}_{\lambda'}(p'_1)\gamma^+u_\lambda(p_1)$	$2\sqrt{p_1^+p_1'^+}$	0	0	$2\sqrt{p_1^+p_1'^+}$
$\bar{u}_{\lambda'}(p'_1)\gamma^Ru_\lambda(p_1)$	$2p_1^R$	0	0	$2p_1'^R$
$\bar{u}_{\lambda'}(p'_1)\gamma^Lu_\lambda(p_1)$	$2p_1'^L$	0	0	$2p_1^L$

- Only different helicity which make matrix element zero.

- To see the contribution of $k_\perp$ in two frame, we plot the integrands (those inside {...}) vs $k_\perp$ (continued)



Vs



Numerical Calculation

- Parameter result → Variational Principle

$$\frac{\partial M_{q\bar{q}}}{\partial \beta} = \frac{\partial \langle \Phi | [H_0 + V_{q\bar{q}}] | \Phi \rangle}{\partial \beta} = \frac{\partial H_0}{\partial \beta} + \frac{\partial V_{q\bar{q}}}{\partial \beta} = 0 \quad (3.18)$$

η_c and Υ in 1S as inputs

θ_{12}	$\theta_{13} = \theta_{23}$	m_c	m_b	a	b
12.12	8.44	1.61	4.97	−0.41	0.18

μ	α_s	$\beta_{c\bar{c}}$	$\beta_{c\bar{b}}$	$\beta_{b\bar{b}}$
0.027	0.402	0.5417	0.7019	1.0595

Analytical Calculation

- Light-Front Wave Function

$$
\begin{aligned}
\phi_{1S}(x,\mathbf{k}_\perp) &= \sqrt{2(2\pi)^3}\sqrt{\frac{\partial k_z}{\partial x}}\frac{1}{\pi^{3/4}\beta^{3/2}}e^{-\mathbf{k}^2/2\beta^2} \\
\phi_{2S}(x,\mathbf{k}_\perp) &= \sqrt{2(2\pi)^3}\sqrt{\frac{\partial k_z}{\partial x}}\frac{1}{\sqrt{6}\pi^{3/4}\beta^{7/2}}(2\mathbf{k}^2-3\beta^2)e^{-\mathbf{k}^2/2\beta^2} \\
\phi_{3S}(x,\mathbf{k}_\perp) &= \sqrt{2(2\pi)^3}\sqrt{\frac{\partial k_z}{\partial x}}\frac{1}{2\sqrt{30}\pi^{3/4}\beta^{11/2}}(15\beta^4-20\beta^2\mathbf{k}^2+4\mathbf{k}^2) \\
&\times\; e^{-\mathbf{k}^2/2\beta^2}
\end{aligned}
\qquad (3.1)
$$

where we define the mixing state

$$
\begin{aligned}
\begin{pmatrix}\Phi_{1S}\\ \Phi_{2S}\\ \Phi_{3S}\end{pmatrix} &= \begin{pmatrix} c_{12}c_{13} & s_{12}s_{13} & s_{13} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13} & c_{12}c_{23}-s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}c_{23}-c_{12}c_{23}s_{13} & -c_{12}s_{23}-s_{12}c_{23}s_{13} & c_{23}c_{13}\end{pmatrix}\begin{pmatrix}\phi_{1S}\\ \phi_{2S}\\ \phi_{3S}\end{pmatrix} \\
&= \begin{pmatrix} c_1^{1S} & c_2^{1S} & c_3^{1S} \\ c_1^{2S} & c_2^{2S} & c_3^{2S} \\ c_1^{3S} & c_2^{3S} & c_3^{3S}\end{pmatrix}\begin{pmatrix}\phi_{1S}\\ \phi_{2S}\\ \phi_{3S}\end{pmatrix} \qquad c_{ij}(s_{ij}) = \cos\theta_{ij}(\sin\theta_{ij})
\end{aligned}
\qquad (3.2)
$$

- The mixing mass formula

$$M_{q\bar{q}} = \langle \Psi^M_{q\bar{q}} | \hat{H} | \Psi^M_{q\bar{q}} \rangle = \langle \Phi | [H_0 + V_{q\bar{q}}] | \Phi \rangle$$

$$M_{q\bar{q}} = \langle H_0 \rangle + \langle V_{\text{Conf}} \rangle + \langle V_{\text{Coul}} \rangle + \langle V_{\text{Hyp}} \rangle \quad (3.3)$$

- Kinetic term

$$\langle H_0 \rangle = \frac{\beta}{120\sqrt{\pi}} \sum_{j=1,2} \Big[(g_1 + g_2 z_i^2) z_i e^{z_i/2} K_1 \left[\frac{z_i}{2}\right] + g_3 z_i^2 (z_i - 3) e^{z_i/2} K_2 \left[\frac{z_i}{2}\right] + 15\sqrt{\pi} \Big(g_4 U(-1/2, -2, z_i) + g_5 U(-1/2, -4, z_i) + g_6 U(-1/2, -5, z_i) \Big) \Big], \quad (3.4)$$

$$g_1 = 120c_1^2 - 120\sqrt{6}c_1c_2 + 180c_2^2 + 60\sqrt{30}c_1c_3 - 180\sqrt{5}c_2c_3 + 225c_3^2,$$

$$g_2 = 40c_2^2 + 8\sqrt{30}c_1c_3 - 104\sqrt{5}c_2c_3 + 260c_3^2,$$

$$g_3 = -4(10c_2^2 - 26\sqrt{5}c_2c_3 + 2\sqrt{30}c_1c_3 + 65c_3^2),$$

$$g_4 = 4(-6c_2^2 + 9\sqrt{5}c_2c_3 - 15c_3^2 + 2\sqrt{6}(c_1c_2 - \sqrt{5}c_1c_3)),$$

$$g_5 = 28(\sqrt{5}c_2c_3 - 5c_3^2),$$

$$g_6 = 63c_3^2,$$

- Confinement pot. → exponential (screened) pot. : $V^{\text{Scr.}}_{\text{Conf.}} = a + \frac{b(1-e^{-\mu r})}{\mu}$

$$\langle V^{\exp}_{\text{Conf}} \rangle = a + \frac{b}{\mu} + \frac{b}{3840\sqrt{\pi}\beta^{10}\mu}\Big[2\beta\mu(d_1\beta^8 + d_2\beta^6\mu^2 + d_3\beta^4\mu^4 + d_4\beta^2\mu^6 + d_5\mu^8) - \sqrt{\pi}e^{\mu^2/4\beta^2}(f_1\beta^{10} + f_2\beta^8\mu^2 + f_3\beta^6\mu^4 + f_4\beta^4\mu^6 + f_5\beta^2\mu^8 + f_6\mu^{10})\text{erfc}\left[\frac{\mu}{2\beta}\right]\Big], \quad (3.5)$$

$$d_1 = 32(60c_1^2 - 4\sqrt{6}(10c_1c_2 + \sqrt{5}c_1c_3) + 15(8c_2^2 - 4\sqrt{5}c_2c_3 + 11c_3^2)),$$
$$d_2 = 32(40c_2^2 - 48\sqrt{5}c_2c_3 + 115c_3^2 + 2\sqrt{6}(-5c_1c_2 + 2\sqrt{5}c_1c_3),$$
$$d_3 = 8(10c_2^2 - 28\sqrt{5}c_2c_3 + 2\sqrt{30}c_1c_3 + 89c_3^2),$$
$$d_4 = -8(\sqrt{5}c_2c_3 - 6c_3^2),$$
$$d_5 = c_3^2.$$

$$f_1 = 3840,$$
$$f_2 = 1920(c_1^2 + 3c_2^2 + 5c_3^2 - \sqrt{6}c_1c_2 - 2\sqrt{5}c_2c_3).$$
$$f_3 = 160(9c_2^2 + 30c_3^2 + \sqrt{30}c_1c_3 - 2\sqrt{6}c_1c_2 - 12\sqrt{5}c_2c_3),$$
$$f_4 = 16(5c_2^2 + 50c_3^2 - 15\sqrt{5}c_2c_3 + \sqrt{30}c_1c_3).$$
$$f_5 = 50c_3^2 - 8\sqrt{5}c_2c_3,$$
$$f_6 = c_3^2.$$

- Coulomb and Hyperfine pot.

$$\langle V_{\text{Coul}} \rangle = -\frac{\beta\alpha_s}{45\sqrt{\pi}}\left(120c_1^2 + 100c_2^2 + 89c_3^2 + 40\sqrt{6}c_1c_2 + 12\sqrt{30}c_1c_3 + 44\sqrt{5}c_2c_3\right),$$
$$\langle V_{\text{Hyp}} \rangle = \frac{\langle \mathbf{S}_q.\mathbf{S}_{\bar{q}} \rangle \beta^3\alpha_s}{3m_1m_2\sqrt{\pi}}\left(\frac{32}{3}c_1^2 + 16c_2^2 + 20c_3^2 + 32\sqrt{\frac{2}{3}}c_1c_2 + 16\sqrt{\frac{10}{3}}c_1c_3 + 16\sqrt{5}c_2c_3\right). \quad (3.6)$$

- Decay constant (DC) of pseudoscalar and vector meson are defined by

$$\begin{aligned} \langle 0|\bar{q}\gamma^\mu\gamma_5 q|P\rangle &= i f_P P^\mu \\ \langle 0|\bar{q}\gamma^\mu q|V(P,\lambda)\rangle &= f_V M \epsilon^\mu(P,\lambda) \end{aligned} \quad (3.7)$$

Pseudoscalar

Vector

- The explicit form:

$$f_{P,V} = \sqrt{6}\int_0^1 dx \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^3} \frac{\Phi(x,\mathbf{k}_\perp)}{\sqrt{\mathcal{A}^2+\mathbf{k}_\perp^2}} \mathcal{O}_{P,V} \quad (3.8)$$

$$\begin{aligned} \mathcal{O}_P &= \mathcal{A} \qquad & \mathcal{A} = (1-x)m_q + x m_{\bar{q}} \\ \mathcal{O}_V &= \mathcal{A} + \frac{2\mathbf{k}_\perp^2}{D} \qquad & D = M_0 + m_q + m_{\bar{q}}. \end{aligned}$$

- To see the contribution of $k_\perp$ in two frame, we plot the integrands (those inside {...}) vs $k_\perp$

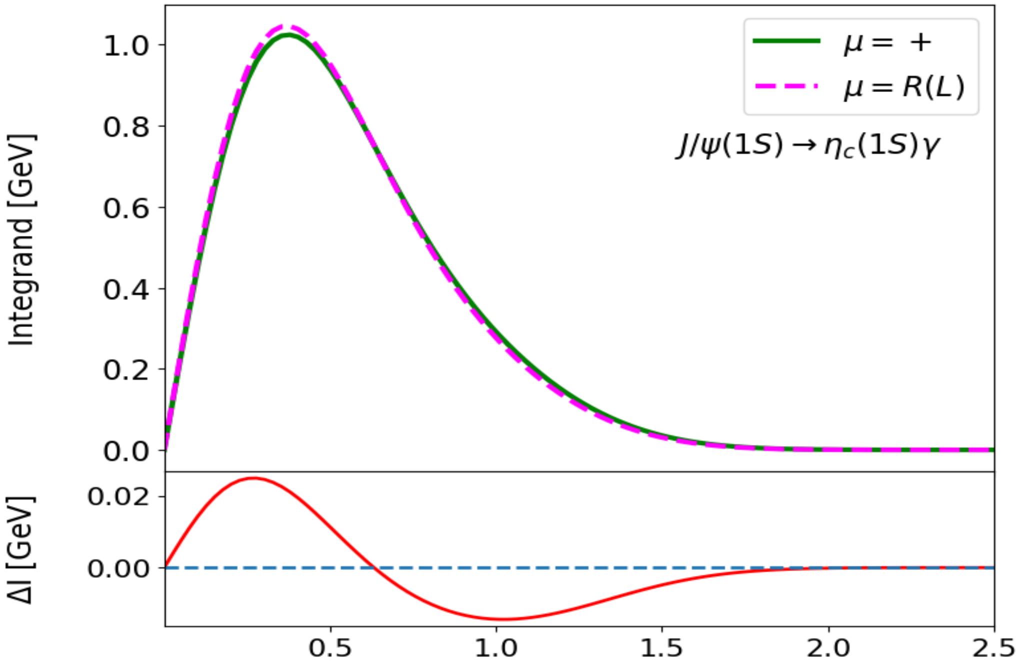
Good current

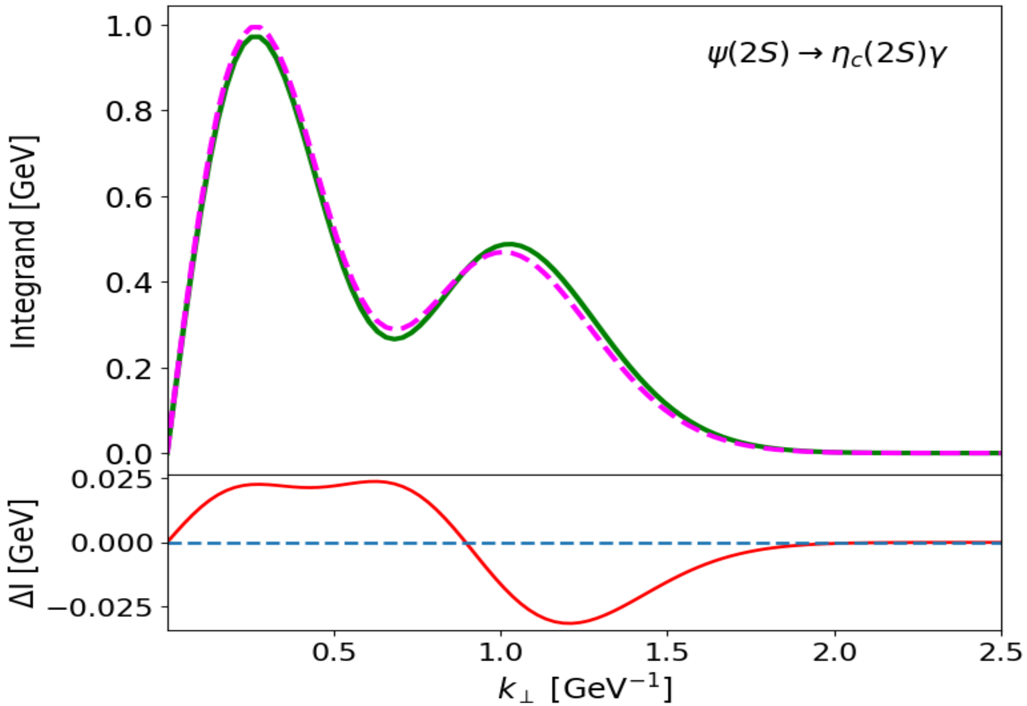
$$I_1^+ = \int dk_\perp \left\{ \int \frac{dx d\theta}{16\pi^3} \frac{\phi(x,\mathbf{k}'_\perp)\phi(x,\mathbf{k}_\perp)}{x\tilde{M}_0\tilde{M}'_0} \left(\mathcal{A} + \frac{2}{\mathcal{D}} \left[\mathbf{k}_\perp^2 - \frac{(\mathbf{k}_\perp \cdot \mathbf{q}_\perp)^2}{\mathbf{q}_\perp^2} \right] \right) \right\} \quad (4.1)$$

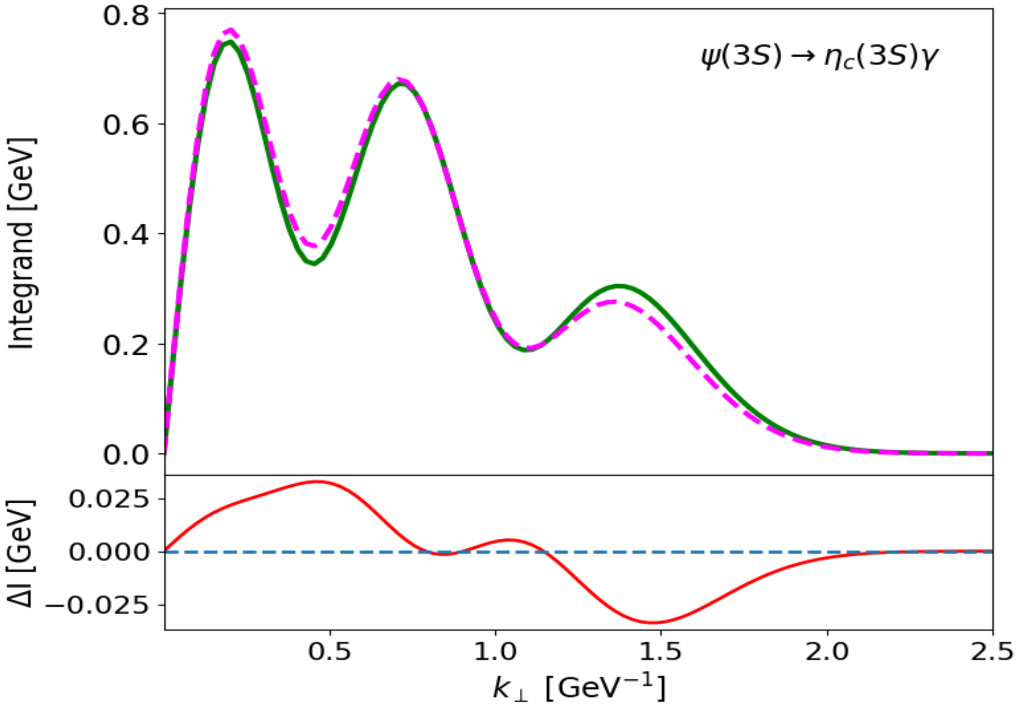
Transverse current

$$I_0^{RL} = \int dk_\perp \left\{ \int \frac{dx d\theta}{16\pi^3} \frac{\phi(x,\mathbf{k}'_\perp)\phi(x,\mathbf{k}_\perp)}{x_1 x(1-x)\tilde{M}_0\tilde{M}'_0 M_0} \left[\mathcal{A} \left(\mathcal{A} + \frac{2\mathbf{k}_\perp^2}{\mathcal{D}} \right) + \frac{\mathcal{M}}{\mathcal{D}} \left((1-2x)\mathbf{k}_\perp^2 + (1-x) \left\{ (\mathbf{k}_\perp \cdot \mathbf{q}_\perp) - \frac{2(\mathbf{k}_\perp \cdot \mathbf{q}_\perp)^2}{\mathbf{q}_\perp^2} \right\} \right) \right] \right\} \quad (4.2)$$

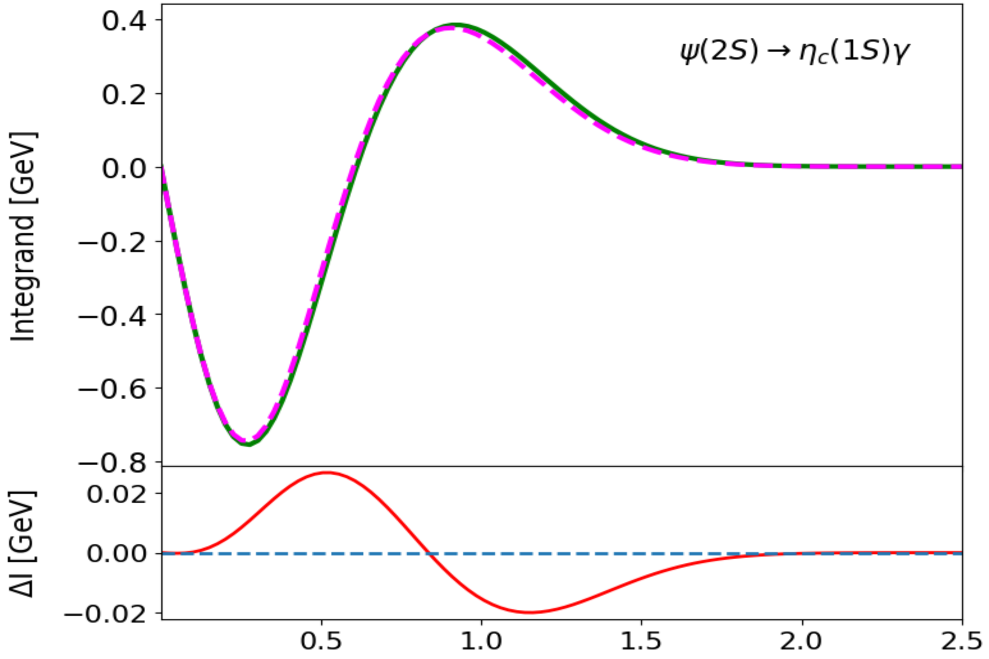
$$V(nS) \to P(n'S)\gamma \quad (n = n')$$

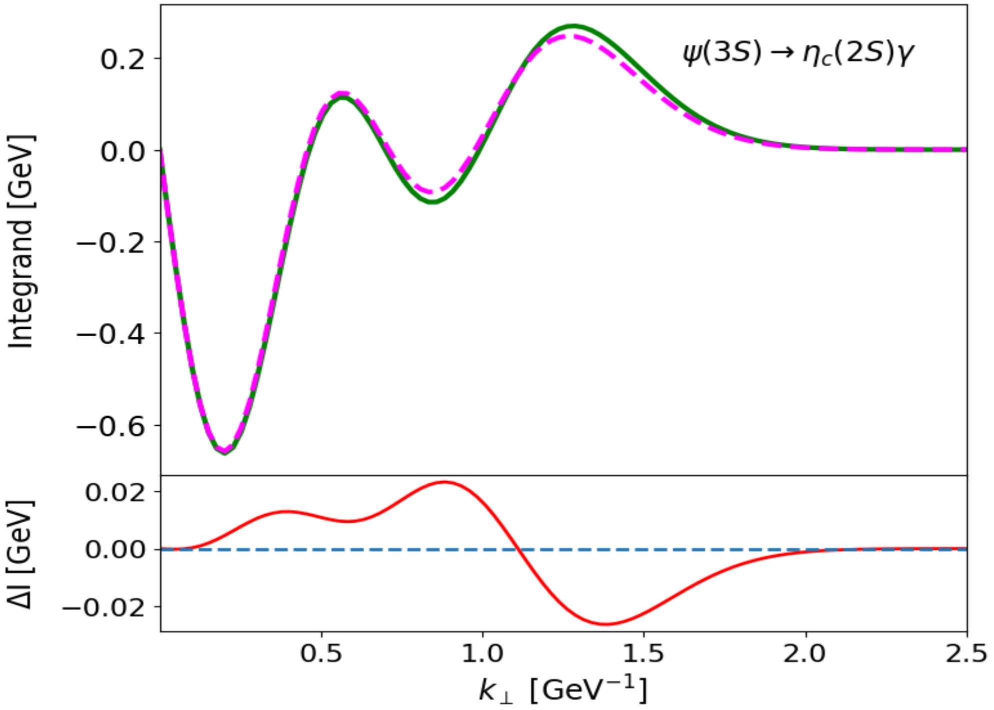


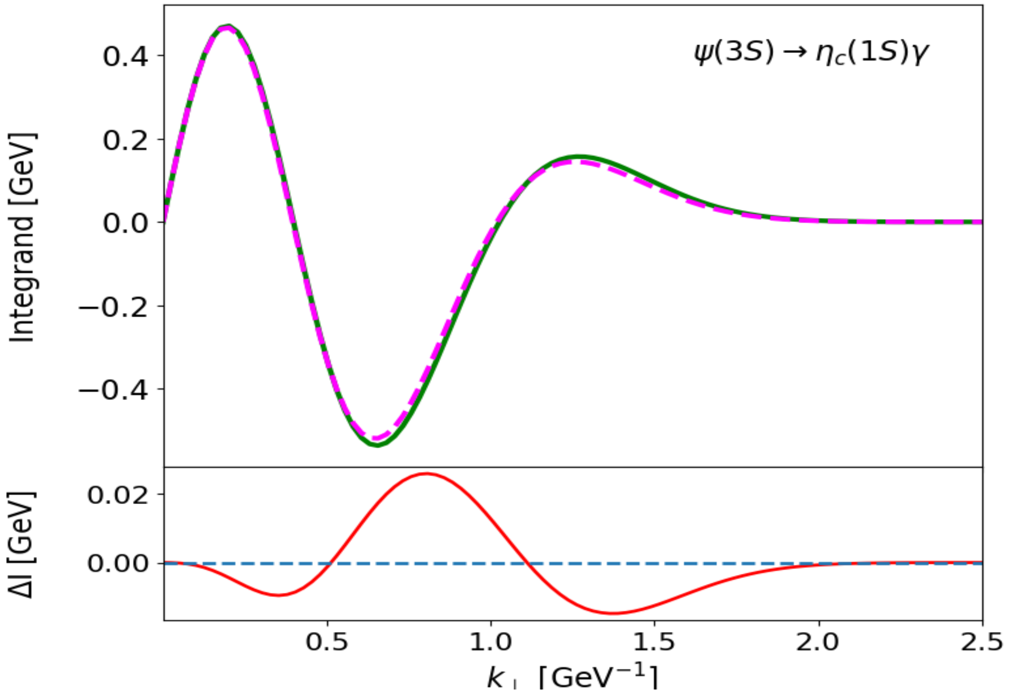




$$V(nS) \to P(n'S)\gamma$$
$$(n \neq n')$$







- The transition form factor $F_{VP}(q^2)$ can be defined as:

$$\langle P(P')|\, J^{\mu}_{em}\, |V(P,h)\rangle = ie\epsilon^{\mu\nu\rho\sigma}\epsilon_{\nu}(P,h)q_{\rho}P_{\sigma}F_{VP}(q^2), \tag{3.9}$$

LHS **RHS**

1. The left-hand side (LHS):

$$\begin{aligned}\langle J^{\mu}_{h}\rangle &= e\sum_{j}Q_j\int_0^1\frac{dx}{16\pi^3}\int d^2\mathbf{k}_{\perp}\phi(x,\mathbf{k}'_{\perp})\phi(x,\mathbf{k}_{\perp})\sum_{\lambda'\lambda\bar{\lambda}}\mathcal{R}^{00\dagger}_{\lambda'\bar{\lambda}}(x,\mathbf{k}'_{\perp})\frac{\bar{u}_{\lambda'}(p'_1)}{\sqrt{x'_1}}\gamma^{\mu}\frac{u_{\lambda}(p_1)}{\sqrt{x_1}}\mathcal{R}^{1h}_{\lambda\bar{\lambda}}(x,\mathbf{k}_{\perp})\\ &= e\sum_{j}Q_j\int_0^1\frac{dx}{16\pi^3}\int d^2\mathbf{k}_{\perp}\phi(x,\mathbf{k}'_{\perp})\phi(x,\mathbf{k}_{\perp})\frac{S}{x}\end{aligned} \tag{3.10}$$

2. The right-hand side (RHS), we define $\mathcal{G}_h^\mu$:

- Good current ($\mu = +$)

$$\mathcal{G}_{+1}^{+} = F_{VP}(Q^2)\frac{eP^+q^R}{\sqrt{2}}.$$

$h = +1$

$$\mathcal{G}_{-1}^{+} = F_{VP}(Q^2)\frac{eP^+q^L}{\sqrt{2}}.$$

$h = -1$

- Transverse current ($\mu = R(L)$)

$$\mathcal{G}_0^L = eFM_0q^L$$
$$\mathcal{G}_0^R = -eFM_0q^R$$

$h = 0$

$$\mathcal{G}_{+1}^R = \frac{-eF}{\sqrt{2}}P^Rq^R$$
$$\mathcal{G}_{+1}^L = \frac{eF}{\sqrt{2}}(q^-P^+ + P^Rq^L)$$

$h = +1$

$$\mathcal{G}_{-1}^R = \frac{eF}{\sqrt{2}}(q^-P^+ + P^Lq^R)$$
$$\mathcal{G}_{-1}^L = \frac{-eF}{\sqrt{2}}P^Lq^L$$

$h = -1$

- The one-loop integral :

1. Good current ($\langle J_1^+ \rangle = F P^+ q_\perp^2 / 2$)

$$I^+(m_1, m_2, q^2) = \int_0^1 \frac{dx}{8\pi^3} \int d^2\mathbf{k}_\perp \frac{\phi(x, \mathbf{k}'_\perp)\phi(x, \mathbf{k}_\perp)}{x_1 \tilde{M}_0 \tilde{M}'_0} \times \left\{ \mathcal{A} + \frac{2}{D_0}\left[\mathbf{k}_\perp^2 - \frac{(\mathbf{k}_\perp \cdot \mathbf{q}_\perp)^2}{\mathbf{q}_\perp^2} \right] \right\} \quad (3.11)$$

2. Transverse current ($\langle J_0^{RL} \rangle = F M_0 q_\perp^2$)

$$I_0^{RL}(m_1, m_2, q^2) = \int \frac{dx d^2\mathbf{k}_\perp}{16\pi^3} \frac{\phi(x, \mathbf{k}'_\perp)\phi(x, \mathbf{k}_\perp)}{x_1 x(1-x)\tilde{M}_0 \tilde{M}'_0 M_0} \left[\mathcal{A}\left(\mathcal{A} + \frac{2\boldsymbol{k}_\perp^2}{D} \right) + \frac{\mathcal{M}}{D}\left((1-2x)\boldsymbol{k}_\perp^2 + (1-x)\left\{ (\boldsymbol{k}_\perp \cdot \boldsymbol{q}_\perp) - \frac{2(\boldsymbol{k}_\perp \cdot \boldsymbol{q}_\perp)^2}{\boldsymbol{q}_\perp^2} \right\} \right) \right] \quad (3.12)$$

3. Transverse current ($\langle J_{1(-1)}^{RL} \rangle = FP_\perp^2 q_\perp^2/\sqrt{2}$)

$$\begin{aligned}
I_{1(-1)}^{RL}(m_1,m_2,q^2) &= \int \frac{dx d^2\mathbf{k}_\perp}{8\pi^3}\frac{\phi(x,\mathbf{k}'_\perp)\phi(x,\mathbf{k}_\perp)}{x^2(1-x)\tilde{M}_0\tilde{M}'_0}\left[\frac{(\mathbf{P}_\perp\cdot\mathbf{k}_\perp)}{\mathbf{q}_\perp^2}\left\{(1-x)\mathcal{A}-\frac{\mathcal{A}\mathcal{M}_2}{D}-x\frac{\mathbf{k}_\perp^2}{D}\right\}\right.\\
&+\frac{(1-x)}{2\mathbf{P}_\perp^2 D}\left\{\frac{4\mathbf{k}_\perp^2}{\mathbf{q}_\perp^2}(\mathbf{q}_\perp\cdot\mathbf{k}_\perp)(\mathbf{q}_\perp\cdot\mathbf{P}_\perp)+4(\mathbf{k}_\perp\cdot\mathbf{P}_\perp)(\mathbf{k}_\perp\cdot\mathbf{q}_\perp)-\frac{8}{\mathbf{q}_\perp^2}(\mathbf{k}_\perp\cdot\mathbf{q}_\perp)^2(\mathbf{k}_\perp\cdot\mathbf{P}_\perp)\right.\\
&\left.\left.+\mathbf{k}_\perp^2(3(\mathbf{k}_\perp\cdot\mathbf{P}_\perp)-(\mathbf{P}_\perp\cdot\mathbf{q}_\perp))\right\}-\frac{(1-x)x}{D}\left\{\frac{2(\mathbf{k}_\perp\cdot\mathbf{q}_\perp)^2}{\mathbf{q}_\perp^2}-\mathbf{k}_\perp^2\right\}+(1-x)x\left(\mathcal{A}+\frac{\mathbf{k}_\perp^2}{D}\right)\right]
\end{aligned}$$

(3.13)

Surprisingly for the transverse current ($I_{1(-1)}^{RL}$), if we assume $P_\perp = 0$, the one loop integral will be the same as good current (I_1^+),

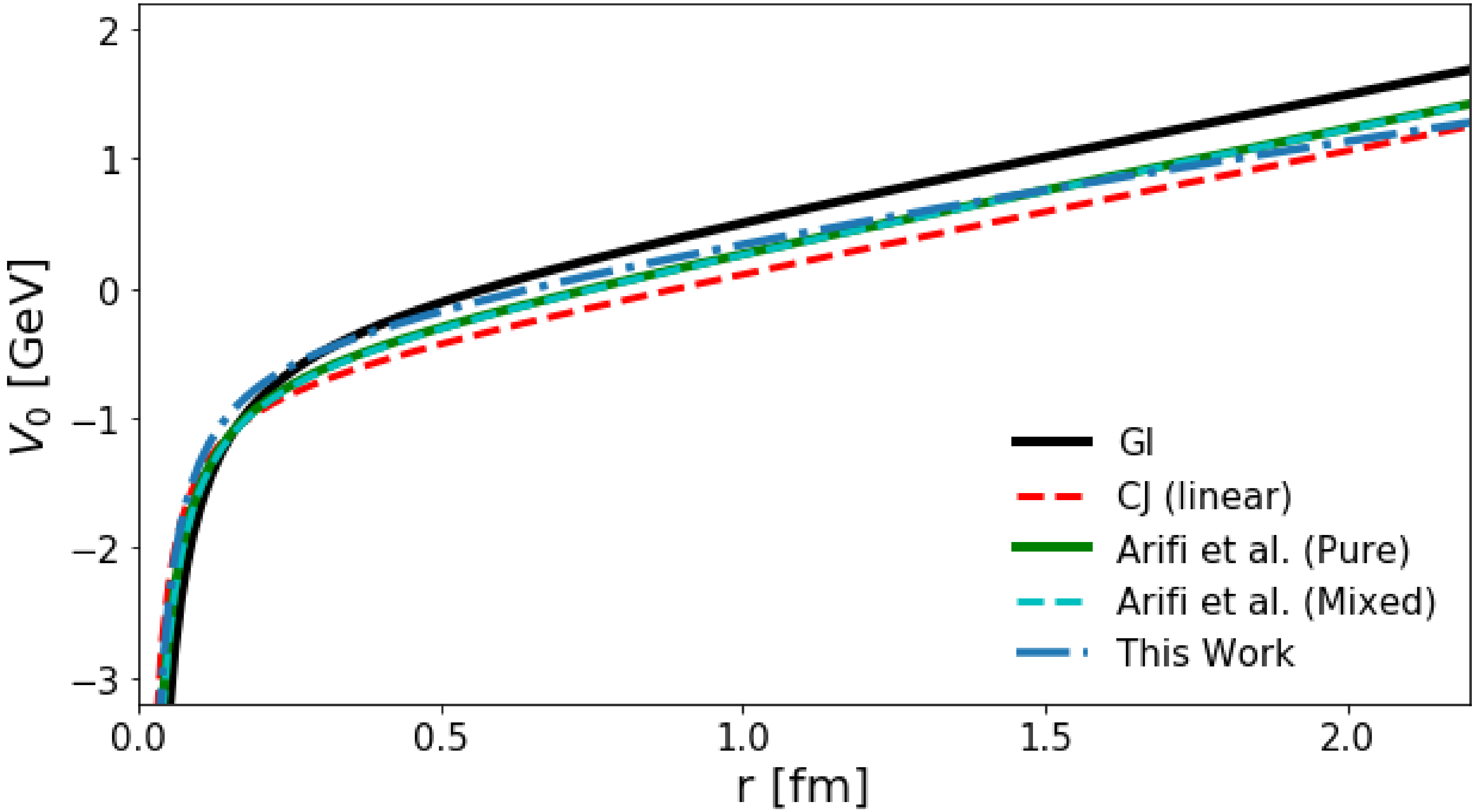
$$\begin{aligned}
I_{1(-1)}^{RL}(m_1,m_2,q^2) &= \int \frac{dx d^2\mathbf{k}_\perp}{8\pi^3}\frac{\phi(x,\mathbf{k}'_\perp)\phi(x,\mathbf{k}_\perp)}{x^2(1-x)\tilde{M}_0\tilde{M}'_0}\left[\frac{(1-x)x}{D}\left\{\mathbf{k}_\perp^2-\frac{2(\mathbf{k}_\perp\cdot\mathbf{q}_\perp)^2}{\mathbf{q}_\perp^2}\right\}+(1-x)x\left(\mathcal{A}+\frac{\mathbf{k}_\perp^2}{D}\right)\right]\\
&= \int \frac{dx d^2\mathbf{k}_\perp}{8\pi^3}\frac{\phi(x,\mathbf{k}'_\perp)\phi(x,\mathbf{k}_\perp)}{x^2(1-x)\tilde{M}_0\tilde{M}'_0}(1-x)x\left[\frac{\mathbf{k}_\perp^2}{D}-\frac{2(\mathbf{k}_\perp\cdot\mathbf{q}_\perp)^2}{\mathbf{q}_\perp^2 D}+\mathcal{A}+\frac{\mathbf{k}_\perp^2}{D}\right]\\
&= \int \frac{dx d^2\mathbf{k}_\perp}{8\pi^3}\frac{\phi(x,\mathbf{k}'_\perp)\phi(x,\mathbf{k}_\perp)}{x\tilde{M}_0\tilde{M}'_0}\left[\mathcal{A}+\frac{2}{D}\left(\mathbf{k}_\perp^2-\frac{(\mathbf{k}_\perp\cdot\mathbf{q}_\perp)^2}{\mathbf{q}_\perp^2}\right)\right]
\end{aligned}$$

(3.14)

4. Result & Discussion

1) Mass Spectra

- V_{Conf} in every model



Our V_{Conf} → Screened effect.

$$V_{\text{Conf}} = a + \frac{b(1 - e^{-\mu r})}{\mu},$$

$$V_{\text{Lin.}} = a + br,$$

$$M_{q\bar{q}} = \langle H_0 \rangle + \langle V_{\text{Conf}} \rangle + \langle V_{\text{Coul}} \rangle + \langle V_{\text{Hyp}} \rangle$$

- Mass Spectra Result for $c\bar{c}$ and $b\bar{b}$ (in MeV)

Discrepancy between $M_{theo.}$ and $M_{exp.}$

States	$M_{\text{Theo.}}$	Exp.	GI	RQM	NRQM	Discrepancy between $M_{theo.}$ and $M_{exp.}$
$\eta_c(1S)$	3002.3	2983.9 ± 0.4	2970	2979	2989	0,62%
$\eta_c(2S)$	3614.5	3637.5 ± 1.1	3620	3588	3602	0,63%
$\eta_c(3S)$	4028.1	...	4060	3991	4058	
$J/\psi(1S)$	3102.4	3098.9 ± 0.01	3100	3096	3094	0,11%
$\psi(2S)$	3679.8	3686.1 ± 0.06	3680	3686	3681	0,17%
$\psi(3S)$	4079.1	4039 ± 1	4100	4088	4129	0,99%
$\eta_b(1S)$	9319.2	9398.7 ± 2	9400	9400	9428	0,85%
$\eta_b(2S)$	10038	9999 ± 4	9980	9993	9955	0,39%
$\eta_b(3S)$	10495	...	10340	10328	10338	
$\Upsilon(1S)$	9397.8	9460.3 ± 0.3	9460	9460	9463	0,66%
$\Upsilon(2S)$	10089	10023.3 ± 0.3	10000	10023	9979	0,66%
$\Upsilon(3S)$	10535	10355.2 ± 0.5	10350	10355	10359	1.7%

GI: (Godfrey & Isgur, 1985), RQM: (Ebert et al., 2003), NRQM: (Soni et al., 2018)

- Mass Spectra Result for $c\bar{b}$ (in MeV)

States	$M_{\text{Theo.}}$	Exp.[21]	GI[22]	RQM[23]	NRQCD[24]
$B_c(1S)$	6231.6	6274.5 ± 0.32	6270	6270	6272
$B_c(2S)$	6846.4	6871.2 ± 1	6850	6835	6864
$B_c(3S)$	7251.2	...	...	7193	7306
$B_c^*(1S)$	6302.1	...	6340	6332	6321
$B_c^*(2S)$	6892.4	...	6890	6881	6900
$B_c^*(3S)$	7287.2	...	...	7235	7338

Discrepancy between $M_{theo.}$ and $M_{exp.}$

→ 0,68%

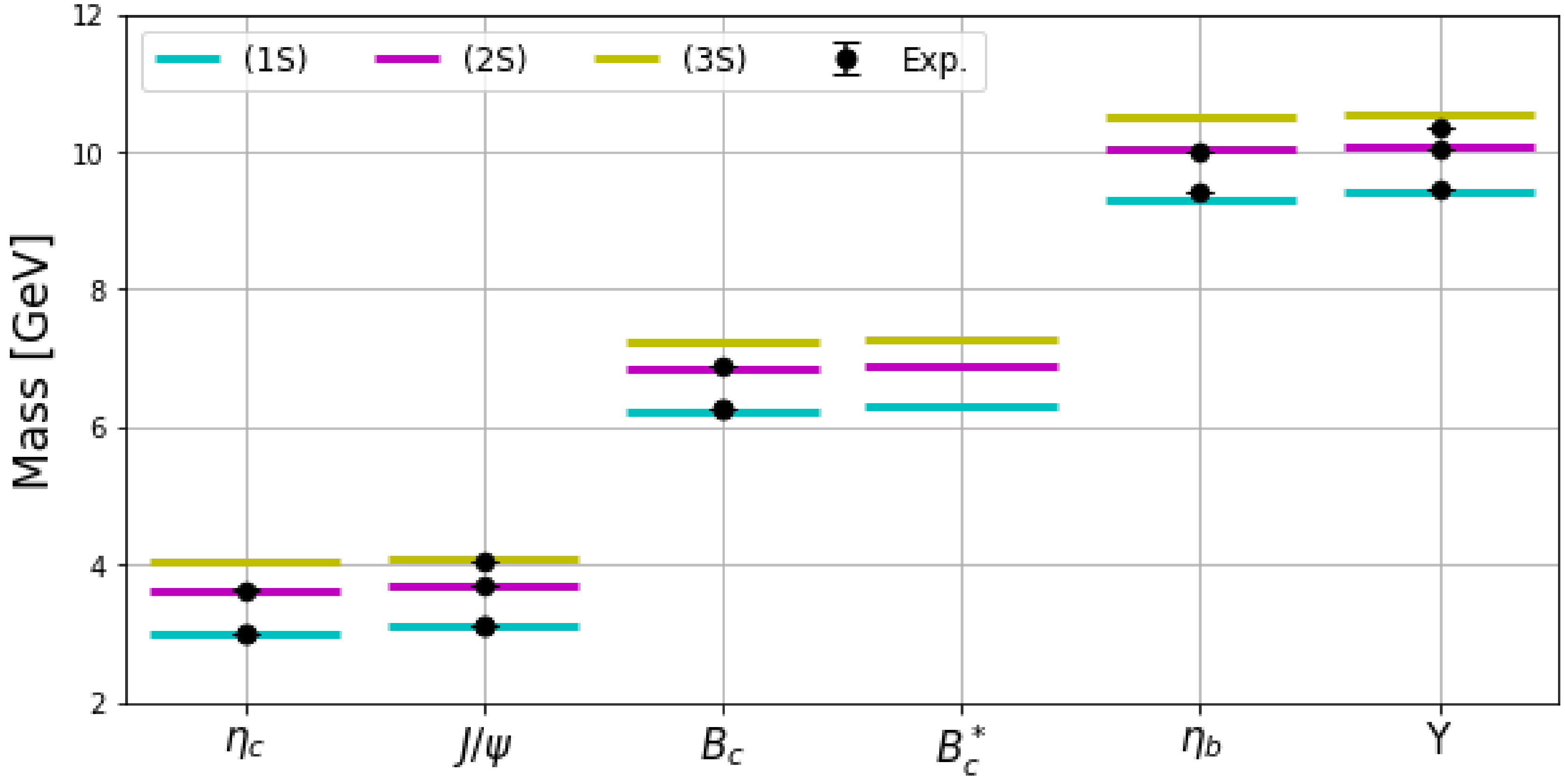
→ 0,36%

Discrepancy:

$$\left|\frac{M_{theo.} - M_{exp.}}{M_{exp.}}\right| \times 100\%$$

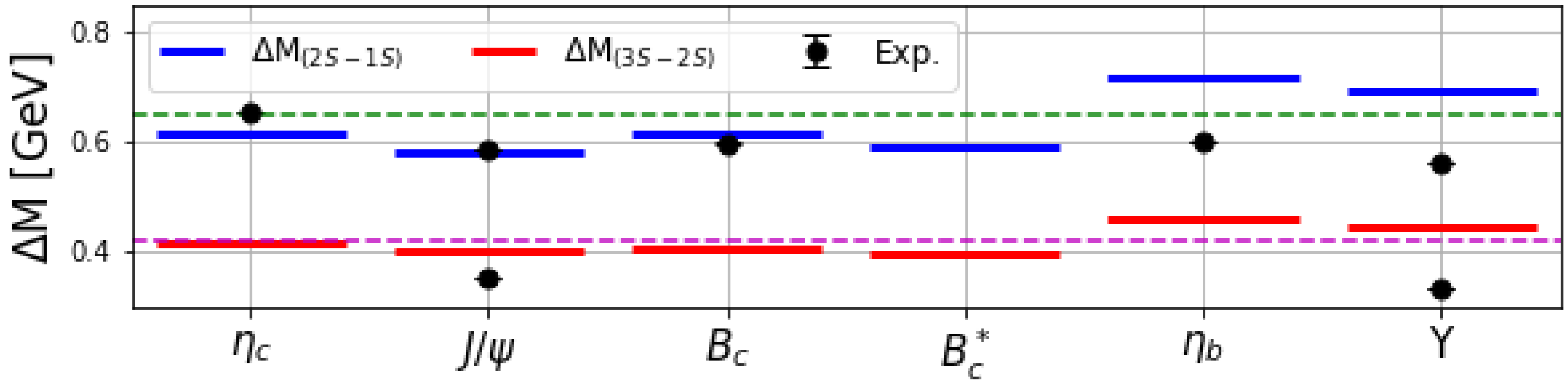
GI: (Godfrey & Isgur, 1985), RQM: (Ebert et al., 2003), NRQM: (Soni et al., 2018)

- Mass Spectra in Graph

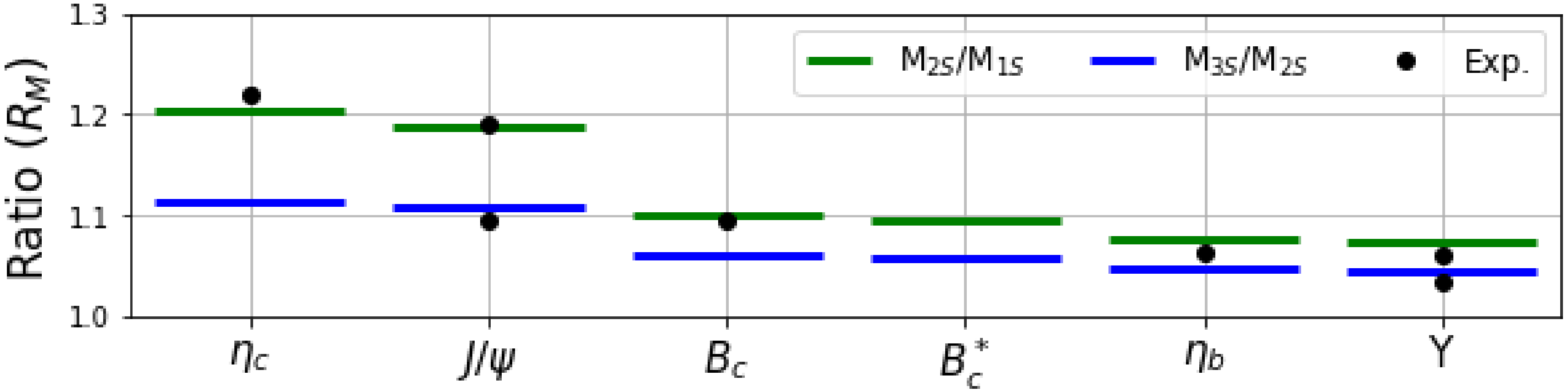


➢ All heavy mesons are in good agreement with experimental data, except $\Upsilon(3S)$.

- Mass Gap & Ratio



- $\Delta M_{2S-1S} > \Delta M_{3S-2S}$ and $\Delta M_P > \Delta M_V$.
- There are significant difference on $b\bar{b}$'s.



- $M_{2S}/M_{1S} > M_{3S}/M_{2S}$
- Overall are comparable with experimental data.

4. Result & Discussion

2) Decay Constant

- Decay Constant (in MeV)

	$\eta_c(1S)$	$J/\psi(1S)$	$B_c(1S)$	$B_c^*(1S)$	$\eta_b(1S)$	$\Upsilon(1S)$
$f_{\text{Theo.}}$	349.1	396.7	397.1	427.9	629.6	669.4
Exp.	335(75)	407(5)	...	...	...	689(5)
Lattice[24–27]	395(2.4)	405(6)	427(6)(2)	...	667(6)(2)	649(31)
RQM[28]	...	...	410(20)	...	...	...
Sum Rules[29]	387(7)	418(9)	...	...	...	...
BS[30]	292(25)	459(28)	...	...	...	496(20)
BS2[31]	385	...	519(1)	...	709	...
CCQM[32]	...	...	...	536(58)	...	...
RQM2[33]	313	411	...	...	594	718
RQM3[34]	402	393	...	...	599	665
NRQM[35]	350	326	...	...	646	647
LFQM (CJ)[36]	326	360	349	369	507	529
LFQM (CJ2)[37]	353	361	389	391	605	611

$$f_P^{\text{Exp}} = \sqrt{\frac{3M_P\Gamma_{\gamma\gamma}}{4\pi\alpha_{\text{QED}}^2 e_Q^2}}$$

$$f_V^{\text{Exp}} = \sqrt{\frac{3M_V\Gamma_{e^-e^+}}{4\pi\alpha_{\text{QED}}^2 e_Q^2}}$$

Discrepancy between $f_{theo.}$ and $f_{exp.}$: $\eta_c(1S) \rightarrow$ 4.2% $J/\psi(1S) \rightarrow$ 2.5% $\Upsilon(1S) \rightarrow$ 2.9%

- Decay Constant (in MeV)

	$\eta_c(2S)$	$\psi(2S)$	$B_c(2S)$	$B_c^*(2S)$	$\eta_b(2S)$	$\Upsilon(2S)$
$f_{\text{Theo.}}$	224	285.6	278.9	319	456.9	510.9
Exp.	...	294(5)	...	...	...	497(5)
Lattice	...	...	...	...	...	481(39)
LFD[38]	...	288(6)	...	...	...	...
BLFQ[39]	299(68)	312(73)	...	...	524(58)	518(48)
RQM2[33]	173	261	...	...	363	459
RQM3[34]	240	293	...	...	411	475
NRQM[35]	278	257	...	...	519	519

	$\eta_c(3S)$	$\psi(3S)$	$B_c(3S)$	$B_c^*(3S)$	$\eta_b(3S)$	$\Upsilon(3S)$
$f_{\text{Theo.}}$	163.9	230.2	219.5	265.1	363.2	427.3
Exp.	...	238(5)	...	...	...	430(4)
RQM2[33]	127	206	...	...	298	385
RQM3[34]	193	258	...	...	354	418
NRQM[35]	249	230	...	...	475	475

Discrepancy:

$$\left|\frac{f_{theo.} - f_{exp.}}{f_{exp.}}\right| \times 100\%$$

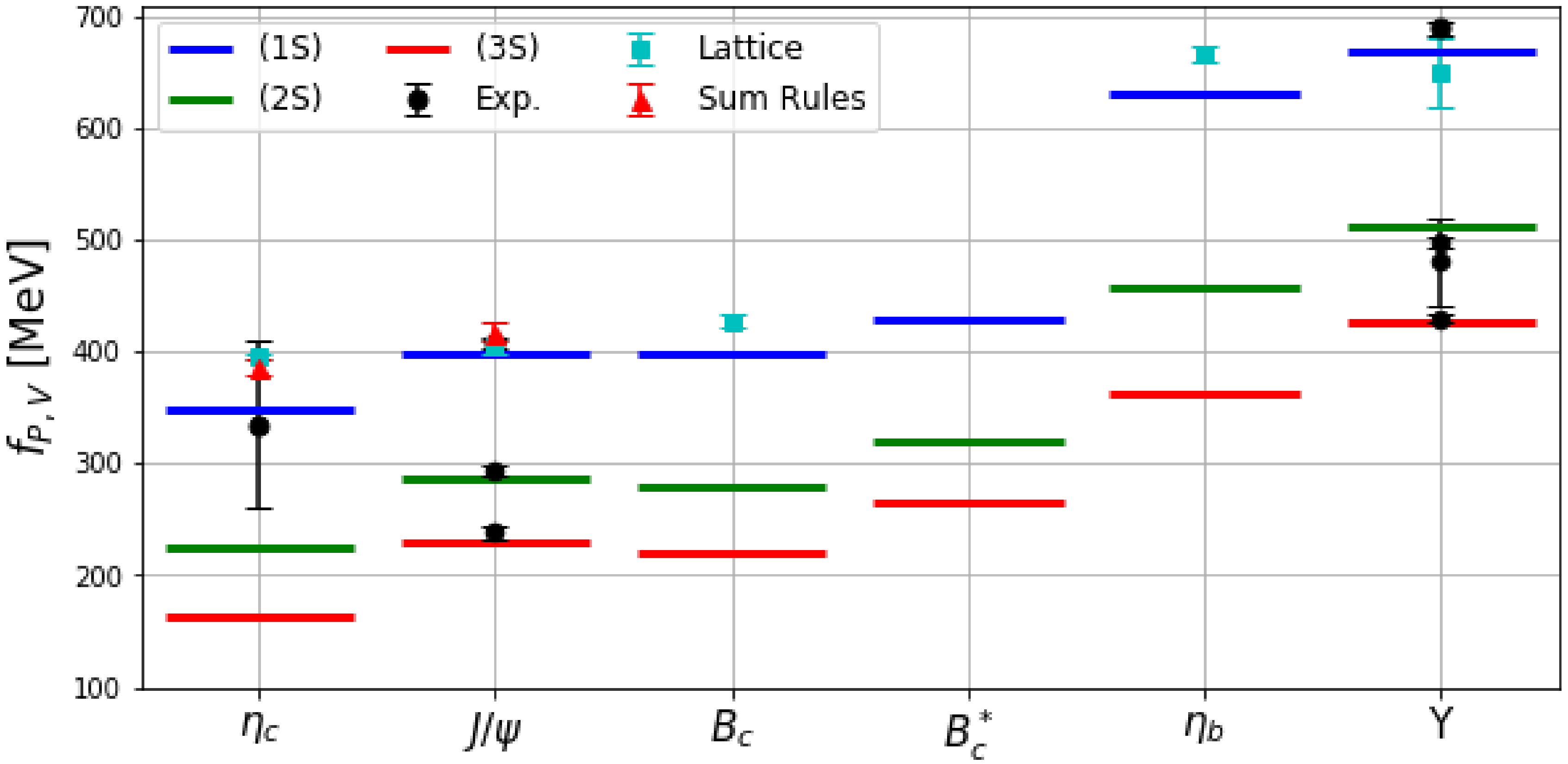
$\psi(2S) \rightarrow$ 2.9%

$\Upsilon(2S) \rightarrow$ 2.8%

$\psi(3S) \rightarrow$ 3.3%

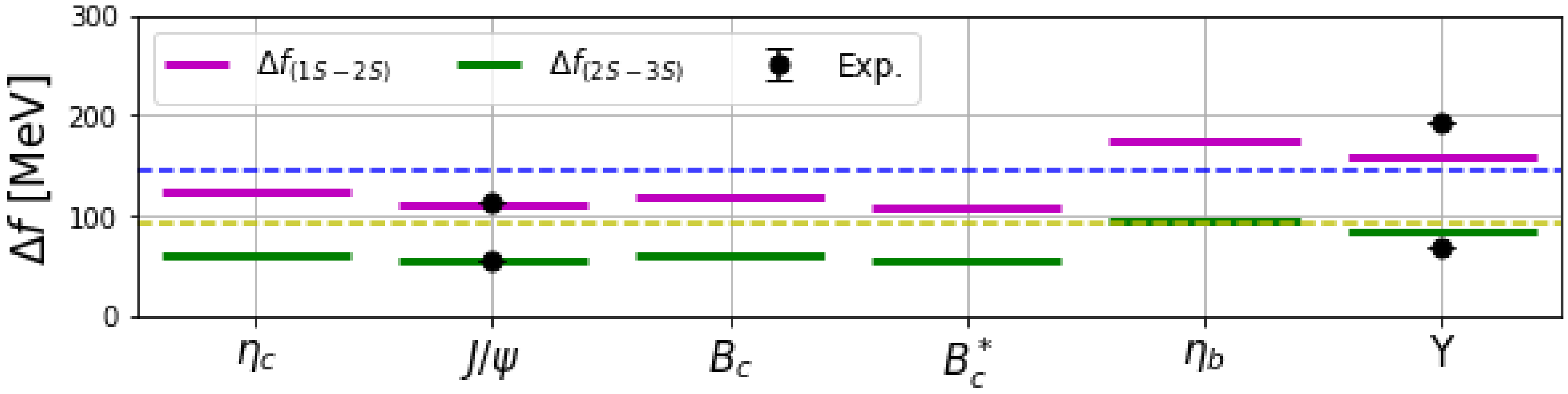
$\Upsilon(3S) \rightarrow$ 0.63%

- Decay Constant in Graph

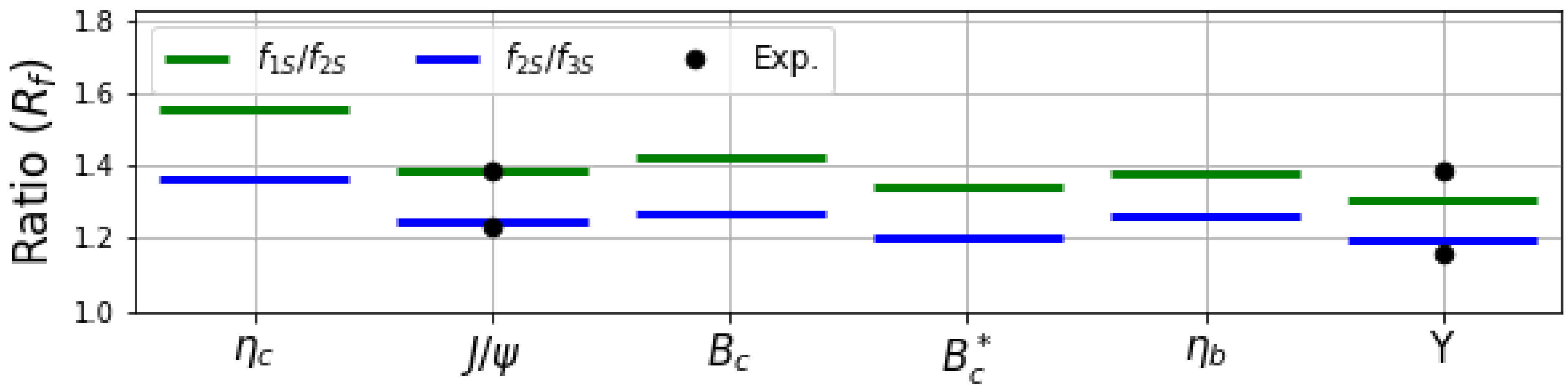


- Only Υ's which have moderately difference value between ours and experimental data.
- $f_{1S} > f_{2S} > f_{3S}$

- DC Gap & Ratio



- ➢ $\Delta f_{1S-2S} > \Delta f_{2S-3S}$ and $\Delta f_P > \Delta f_V$.
- ➢ There are significant difference on ϒ's.



- ➢ $(f/f')_P > (f/f')_V$.
- ➢ There are significant difference on ϒ's.

4. Result & Discussion

3) Radiative Decay

- Partial decay width (Γ) [KeV] for $c\bar{c}$

Transition	Result	Exp.	NRQM[41]	RQM[42]
$J/\psi(1S) \to \eta_c(1S) + \gamma$	1.94	1.57	2.72	1.05
$\psi(2S) \to \eta_c(2S) + \gamma$	0.134	0.206	1.17	0.99
$\psi(2S) \to \eta_c(1S) + \gamma$	2.27	0.99	7.51	0.95
$\psi(3S) \to \eta_c(3S)^\dagger + \gamma$	1.53×10^{-3}	...	9.93	...
$\psi(3S) \to \eta_c(2S) + \gamma$	0.475	...	...	...
$\psi(3S) \to \eta_c(1S) + \gamma$	0.22	...	...	...
$\eta_c(2S) \to J/\psi(1S) + \gamma$	3.3	...	...	...
$\eta_c(3S)^\dagger \to \psi(2S) + \gamma$	0.904	...	...	...
$\eta_c(3S)^\dagger \to J/\psi(1S) + \gamma$	0.475	...	...	...

- Few experimental data obtained from PDG.
- NRQM and RQM are used for comparison.

RQM: (Ebert et al., 2003), NRQM: (Soni et al., 2018)

- Partial decay width (Γ) [KeV] for $b\bar{b}$

Transition	Result	Exp.	NRQM	RQM
$\Upsilon(1S) \to \eta_b(1S) + \gamma$	8.96×10^{-3}	...	3.77×10^{-4}	5.8×10^{-3}
$\Upsilon(2S) \to \eta_b(2S) + \gamma$	5.24×10^{-4}	...	5.62×10^{-3}	1.4×10^{-3}
$\Upsilon(2S) \to \eta_b(1S) + \gamma$	1.29×10^{-2}	1.76×10^{-2}	7.72×10^{-4}	6.4×10^{-3}
$\Upsilon(3S)^\dagger \to \eta_b(3S)^\dagger + \gamma$	2.34×10^{-3}	...	2.85×10^{-3}	0.8×10^{-3}
$\Upsilon(3S) \to \eta_b(2S) + \gamma$	2.77×10^{-3}	$< 1.26 \times 10^{-2}$	3.62×10^{-4}	1.5×10^{-3}
$\Upsilon(3S) \to \eta_b(1S) + \gamma$	3.02×10^{-3}	1.03×10^{-2}	7.7×10^{-4}	1.05×10^{-4}
$\eta_b(2S) \to \Upsilon(1S) + \gamma$	2.54×10^{-2}	...	...	1.18×10^{-4}
$\eta_b(3S)^\dagger \to \Upsilon(2S) + \gamma$	1.91×10^{-2}	...	...	2.8×10^{-3}
$\eta_b(3S)^\dagger \to \Upsilon(1S) + \gamma$	1.14×10^{-2}	...	...	2.4×10^{-4}

RQM: (Ebert et al., 2003), NRQM: (Soni et al., 2018)

- Partial decay width (Γ) [KeV] for $c\bar{b}$

Transition	Result	NRQM1	NRQM2	RQM
$B_c^*(1S)^\dagger \to B_c(1S) + \gamma$	4.74×10^{-3}	4.04×10^{-4}	5.31×10^{-4}	3.3×10^{-4}
$B_c^*(2S)^\dagger \to B_c(2S) + \gamma$	1.77×10^{-3}	3.3×10^{-3}	2.11×10^{-4}	1.7×10^{-4}
$B_c^*(2S)^\dagger \to B_c(1S) + \gamma$	0.575	0.56	4.82×10^{-5}	4.28×10^{-1}
$B_c^*(3S)^\dagger \to B_c(3S)^\dagger + \gamma$	8.09×10^{-3}	...	...	...
$B_c^*(3S)^\dagger \to B_c(2S) + \gamma$	0.171	...	...	...
$B_c^*(3S)^\dagger \to B_c(1S) + \gamma$	4.97×10^{-2}	...	...	...
$B_c(2S) \to B_c^*(1S)^\dagger + \gamma$	1.36	0.14	5.68×10^{-5}	4.88×10^{-1}
$B_c(3S)^\dagger \to B_c^*(2S)^\dagger + \gamma$	0.332	...	...	...
$B_c(3S)^\dagger \to B_c^*(1S)^\dagger + \gamma$	0.124	...	...	...

RQM: (Ebert et al., 2003), NRQM1: (Gao et al., 2024), NRQM2: (Soni et al., 2018)

- Branching ratio (Br) for $c\bar{c}$

Transition	Result	Exp.	NRQM	RQM
$J/\psi(1S) \to \eta_c(1S) + \gamma$	2.1×10^{-2}	$(1.7 \pm 0.4)\times10^{-2}$	2.94×10^{-2}	1.13×10^{-2}
$\psi(2S) \to \eta_c(2S) + \gamma$	4.56×10^{-4}	$(7 \pm 5)\times10^{-4}$	3.98×10^{-3}	3.37×10^{-3}
$\psi(2S) \to \eta_c(1S) + \gamma$	7.72×10^{-3}	$(3.4 \pm 0.5)\times10^{-3}$	2.55×10^{-2}	3.23×10^{-3}
$\psi(3S) \to \eta_c(3S)^{\dagger} + \gamma$	1.91×10^{-8}	...	1.24×10^{-4}	...
$\psi(3S) \to \eta_c(2S) + \gamma$	5.94×10^{-6}	...	...	...
$\psi(3S) \to \eta_c(1S) + \gamma$	2.75×10^{-6}	...	...	...
$\eta_c(2S) \to J/\psi(1S) + \gamma$	2.34×10^{-4}	1.39×10^{-2}	...	...

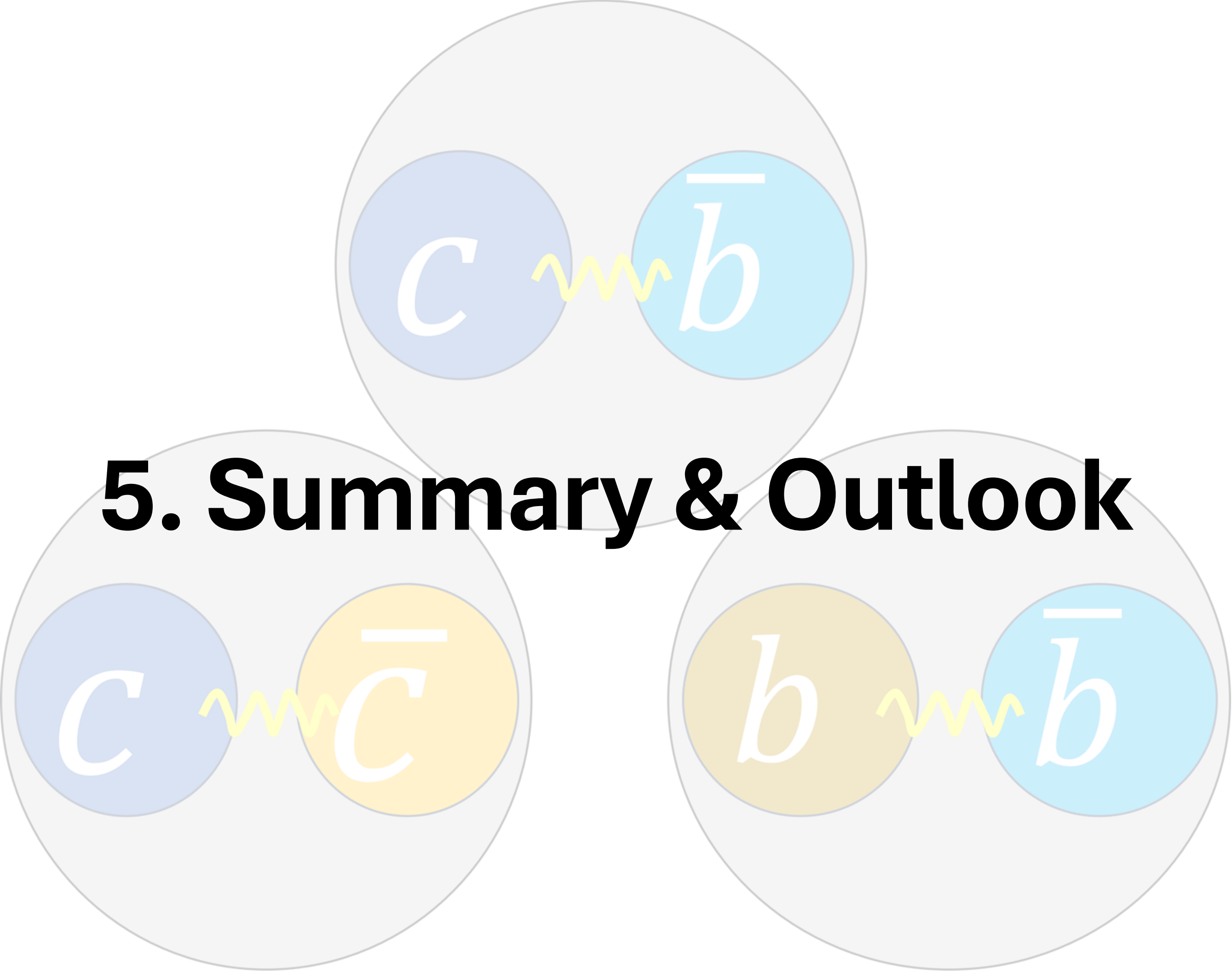
- Few experimental data obtained from PDG.
- Our result and exp. are quite comparable.

- Branching ratio (Br) for $b\bar{b}$

Transition	Result	Exp.	NRQM	RQM
$\Upsilon(1S) \to \eta_b(1S) + \gamma$	1.66×10^{-4}	...	6.98×10^{-6}	1.07×10^{-4}
$\Upsilon(2S) \to \eta_b(2S) + \gamma$	1.64×10^{-5}	...	1.76×10^{-4}	4.38×10^{-5}
$\Upsilon(2S) \to \eta_b(1S) + \gamma$	4.03×10^{-4}	$5.5^{+1.1}_{-0.9} \times 10^{-4}$	2.41×10^{-5}	2×10^{-4}
$\Upsilon(3S)^\dagger \to \eta_b(3S)^\dagger + \gamma$	1.15×10^{-4}	...	1.4×10^{-4}	3.94×10^{-5}
$\Upsilon(3S) \to \eta_b(2S) + \gamma$	4.54×10^{-4}	$<6.7 \times 10^{-4}$	1.78×10^{-4}	7.38×10^{-5}
$\Upsilon(3S) \to \eta_b(1S) + \gamma$	2.42×10^{-2}	$(5.1 \pm 0.7) \times 10^{-4}$	3.79×10^{-4}	5.17×10^{-5}
$\eta_b(2S) \to \Upsilon(1S) + \gamma$	1.06×10^{-6}	...	...	1.18×10^{-8}
$\eta_b(3S)^\dagger \to \Upsilon(2S) + \gamma$	...	...	...	1.17×10^{-7}
$\eta_b(3S)^\dagger \to \Upsilon(1S) + \gamma$	...	...	...	1×10^{-8}

➢ No Γ_{tot} for $\eta_c(3S)$ and $c\bar{b}$ mesons.

5. Summary & Outlook



Summary

- We have studied and obtained mass spectra, decay constant, and radiative decay of $c\bar{c}$, $b\bar{b}$, and $c\bar{b}$ mesons in the LFQM.
- Mass spectra: with using screened effect, our calculation is overall in agreement with experimental result where $\Delta M_{2S-1S} > \Delta M_{3S-2S}$ and $\Delta M_P > \Delta M_V$.
- Decay constant: We have obtained the hierarchy $f_{1S} > f_{2S} > f_{3S}$.
- Radiative Decay: We have obtained g_{VP} for good and transverse current with the same value before we proceed to seek Γ and Br.

Outlook

- In the future, we would consider GEM (Gaussian Expansion Method) ansatz to be used as the realistic wave function.
- We would also consider to obtain the one-loop integral for bad current I^- for the sake of completeness.

$$\phi_{1S}^{\mathrm{HO}}(\boldsymbol{k}) = \frac{\sqrt{2(2\pi)^3}}{\pi^{3/4}\beta^{3/2}} e^{-k^2/2\beta^2},$$

$$\phi_{2S}^{\mathrm{HO}}(\boldsymbol{k}) = \frac{\sqrt{2(2\pi)^3}(2k^2 - 3\beta^2)}{\sqrt{6}\pi^{3/4}\beta^{7/2}} e^{-k^2/2\beta^2},$$

$$\phi_{3S}^{\mathrm{HO}}(\boldsymbol{k}) = \frac{\sqrt{2(2\pi)^3}(15\beta^4 - 20\beta^2 k^2 + 4k^4)}{2\sqrt{30}\pi^{3/4}\beta^{11/2}} e^{-k^2/2\beta^2},$$

For comparison to Li's work for Integrand vs $k_{\perp}$(PR D 98, 2018):

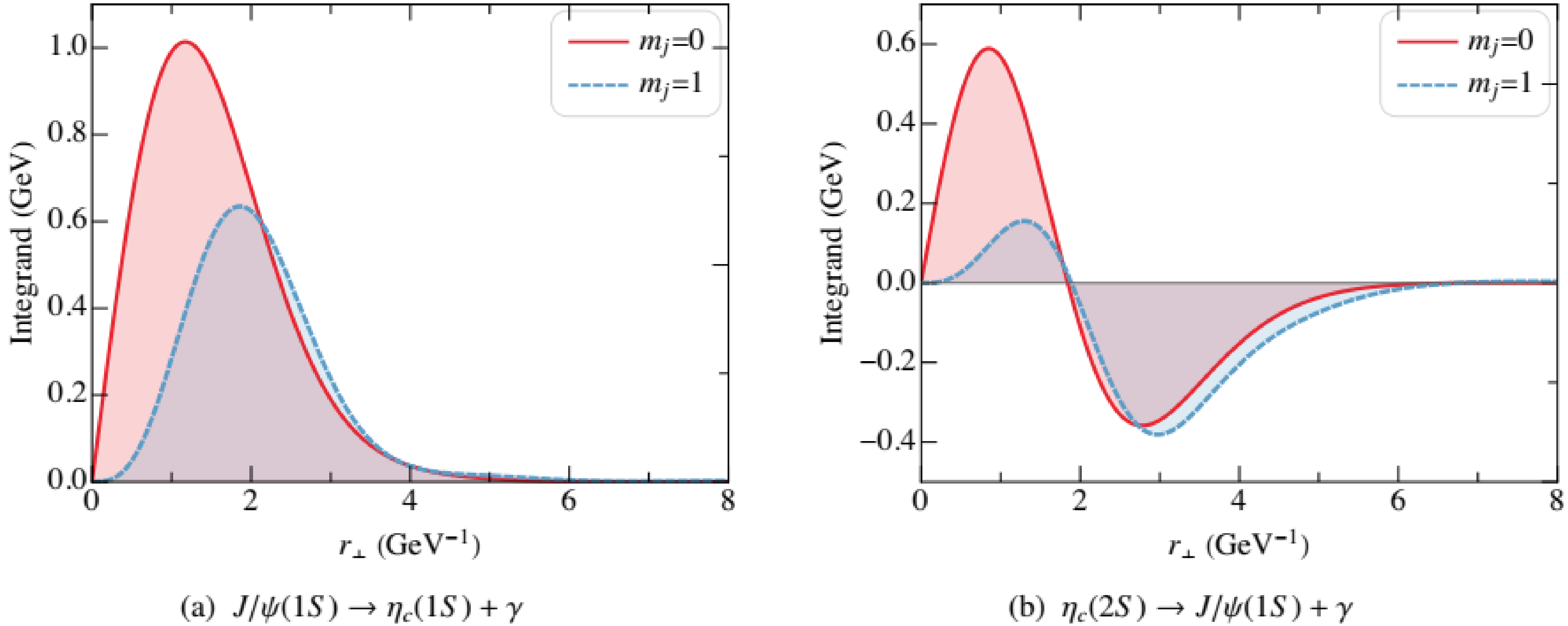


FIG. 3. Integrands of $\hat{V}(0)$ according to Eqs. (15) ($m_j = 0$) and (16) ($m_j = 1$). As a representative of the allowed ($nS \rightarrow nS + \gamma$) transitions, the integrand in (a) has the same sign in the entire $r_{\perp}$ region. On the other hand, (b) involves a transition with radial excitation, which is sensitive to small changes in the cancellations between positive and negative contributions.

cb mesons

Transition	Result	BLFQ	GI
$B_c^*(1S)^\dagger \to B_c(1S) + \gamma$	0.299	...	...
$B_c^*(2S)^\dagger \to B_c(2S) + \gamma$	0.281	...	...
$B_c^*(2S)^\dagger \to B_c(1S) + \gamma$	-0.0339	...	...
$B_c^*(3S)^\dagger \to B_c(3S)^\dagger + \gamma$	0.268	...	...
$B_c^*(3S)^\dagger \to B_c(2S) + \gamma$	-0.0326	...	...
$B_c^*(3S)^\dagger \to B_c(1S) + \gamma$	-0.00494	...	...
$B_c(2S) \to B_c^*(1S)^\dagger + \gamma$	-0.00339	...	...
$B_c(3S)^\dagger \to B_c^*(2S)^\dagger + \gamma$	-0.0326	...	...
$B_c(3S)^\dagger \to B_c^*(1S)^\dagger + \gamma$	-0.00494	...	...