Angular Fractals in Thermal CFT

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Non-perturbative methods in QFT
Kyushu IAS-iTHEMS
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Based on 2405.17562
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(See also 2306.08031:
Jaeha Lee, Hirosi Ooguri, and David Simmons-Duffin)

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In 2d, there is a universal formula for **entropy** called Cardy's formula

$$\rho^{d=2}(\Delta, j) \sim \exp\left[\sqrt{\frac{c}{3}}\pi\left(\sqrt{\Delta + j - \frac{c}{12}} + \sqrt{\Delta - j - \frac{c}{12}}\right)\right], \quad \Delta \gg c.$$

(Cardy, 1986)

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(Cardy, 1986)

Derived from modular invariance of the torus partition function

$$Z(\tau, \bar{\tau}) = Z(\gamma\tau, \gamma\bar{\tau}), \quad \gamma \in SL(2,\mathbb{Z})$$

$$Z(\beta) = Z(\beta^{-1})$$

$$\lim_{\beta \to 0} Z(\beta^{-1}) \sim e^{\frac{2\pi c}{12\beta}}$$

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Valid for **all** 2d CFTs but for holographic theories it has a beautiful interpretation as black hole entropy

(Strominger, 1997)

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Hints: black hole entropy still universal. But modular invariance not available on $S^{d-1}\times S^1$

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The gapped theory is kind of a higher dimensional "modular dual"

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Partition function of CFT on this geometry is the captured by the gapped (d-1)-dim theory coupled to (d-1)-dim background fields

$$Z_{\rm CFT}(G) = Z_{\rm gapped}(g_{ij}, A_i, \phi).$$

Lore of massive QFT: Z_{gapped} can be captured by local effective action for (d-1)-dim fields

$$Z_{\text{gapped}}(g_{ij}, A_i, \phi) \sim e^{-S_{\text{th}}[g_{ij}, A_i, \phi]}$$

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2. Weyl invariance of original theory

$$Z_{\text{CFT}}(e^{2\sigma}G) = Z_{\text{CFT}}(G)e^{-S_{\text{anom}}[G,\sigma]},$$

forces S_{th} to be a function of the gauge field and of Weyl-invariant metric $\widehat{g}_{ij} \equiv e^{-2\phi} g_{ij}$,

$$S_{\text{th}} = \int \frac{d^{d-1}\vec{x}}{\beta^{d-1}} \sqrt{\widehat{g}} \left(-f + c_1 \beta^2 \widehat{R} + c_2 \beta^2 F^2 + \dots \right) + S_{\text{anom}}$$

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Moreover f is the Casimir energy of the CFT on a circle, so in 2d f is related to the central charge c

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Now we just need to compute $S[\hat{g}, A]$ in this geometry. Put manifold in KK form, plug in $\hat{g}, A$ into thermal effective action

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$$S_{\text{th}} = \frac{\text{vol}\, S^{d-1}}{\prod_{i=1}^{n}(1+\Omega_i^2)} \left[-fT^{d-1} + (d-2)\left((d-1)c_1 + (2c_1 + \tfrac{8}{d}c_2)\sum_{i=1}^{n}\Omega_i^2 \right) T^{d-3} + \ldots \right]$$

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From this we can read off the partition function (at large T) and take an inverse Laplace transform to read off entropy as a function of Δ, J

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In 2d, this of course reproduces the usual Cardy formula

$$\rho_{d=2}^{\text{states}}(\Delta, J) \sim \exp\left[\sqrt{\frac{2c}{3}}\pi\left(\sqrt{\frac{\Delta+J}{2}-\frac{c}{24}}+\sqrt{\frac{\Delta-J}{2}-\frac{c}{24}}\right)\right]$$

For example, in 3d we get:

$$\log \rho_{d=3}^{\text{primaries}}(\Delta, J) = 3\pi^{1/3} f^{1/3} (\Delta + J)^{1/3} (\Delta - J)^{1/3} - \frac{5}{3} \log(\Delta^2 - J^2) + \log \Delta$$
$$+ \log \left(\frac{16\pi^{1/3} f^{4/3}}{\sqrt{3}} \right) - 8\pi c_1 + \frac{32 c_2 J^2 \pi}{3(\Delta^2 - J^2)} + O(\Delta^{-1/3}).$$

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d=2: $\Delta - |J| \gg c$

d>2: $\Delta - |J| \gg \sqrt{f\Delta}.$

(Aside: d>2 formula is for one fugacity turned on; for more fugacities exponent changes)

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$$\log Z\left(T,\vec{\Omega}\right) = \frac{\mathrm{vol}S^{d-1}(4\pi)^{d-1}}{4d^d G_N}\frac{\ell_{\mathrm{AdS}}^{d-1}T^{d-1}}{\prod_{i=1}^{\lfloor d/2\rfloor}\left(1+\Omega_i^2\right)}\left(1-\frac{d^2\left((d-1)+\sum_{i=1}^{\lfloor d/2\rfloor}\Omega_i^2\right)}{16\pi^2T^2}+\mathcal{O}\left(\frac{1}{T^4}\right)\right),$$

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Leading order in G_N we have:

$$f=\frac{(4\pi)^{d-1}\ell_{\mathrm{AdS}}^{d-1}}{4d^dG_N}$$

$$c_1=\frac{(4\pi)^{d-3}\ell_{\mathrm{AdS}}^{d-1}}{4(d-2)d^{d-2}G_N}$$

$$c_2=-\frac{(4\pi)^{d-3}\ell_{\mathrm{AdS}}^{d-1}}{32(d-2)d^{d-3}G_N}.$$

Kerr black holes in AdS for D>3 suffer from instability. They are only stable if (with one fugacity turned on):

$$E - J/\ell > \#\sqrt{E}\,\ell^{\frac{D-3}{2}} G_N^{-1/2}$$

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Similar analogy in $\mathrm{AdS}_3/\mathrm{CFT}_2$

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What if Ω is large, i.e. $\beta\Omega = O(1)$? (Note the 2π periodicity.)

For example this would include Tr$((-1)^J e^{-\beta(H+i\Omega J)})$

In 2d, we can compute Tr$((-1)^J e^{-\beta(H+i\Omega J)})$ by using a more complicated $SL(2,\mathbb{Z})$ modular transformation

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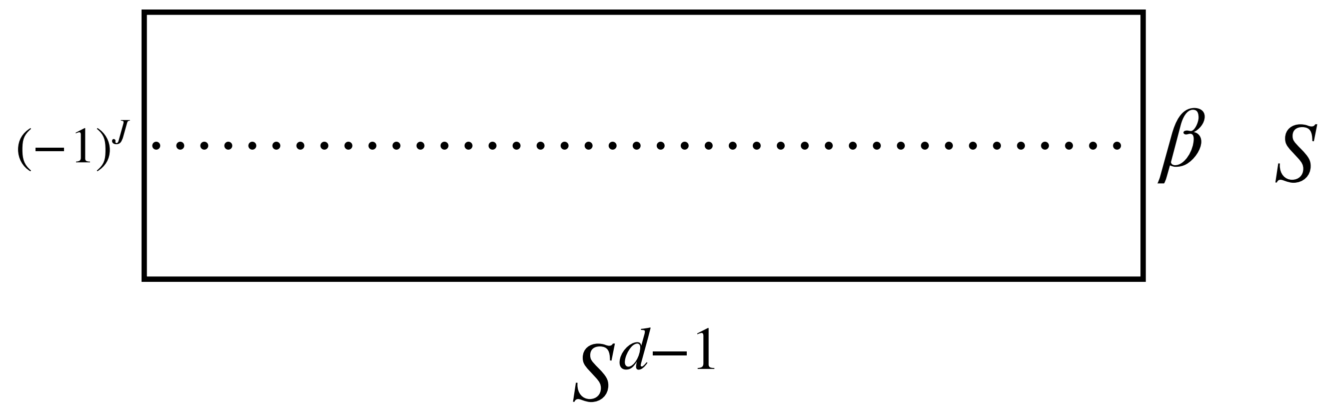
down by a factor of 4

$$\mathrm{Tr}((-1)^J e^{-\beta(H+i\Omega J)})$$



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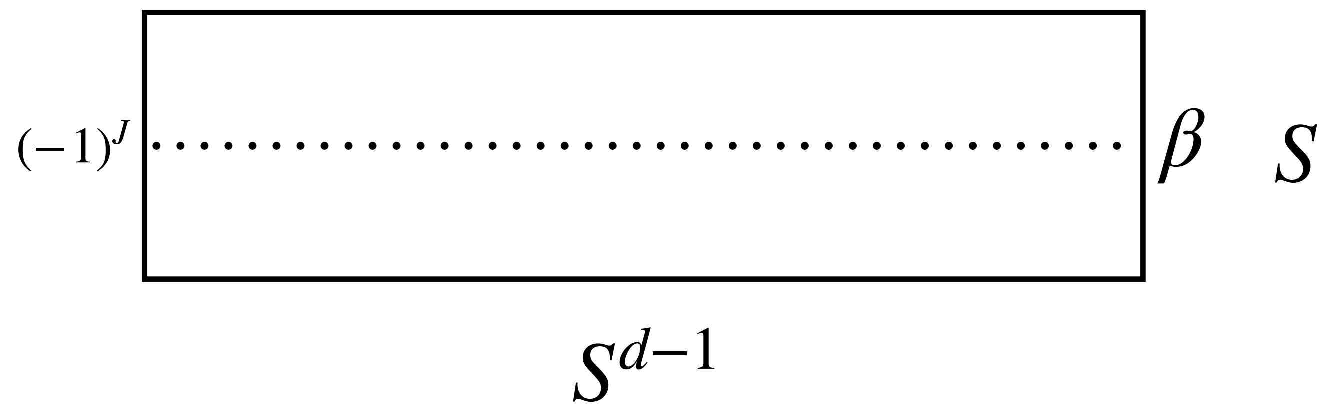

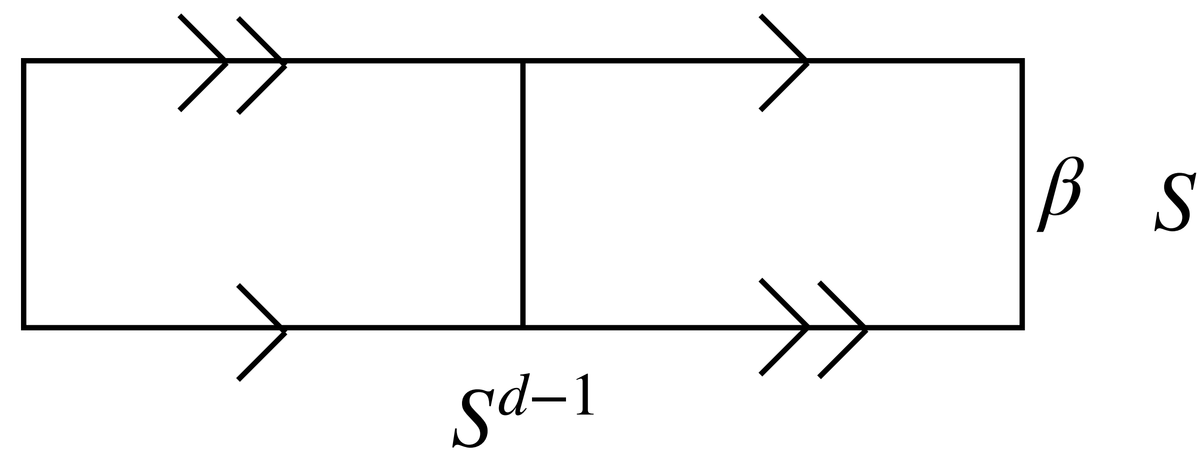

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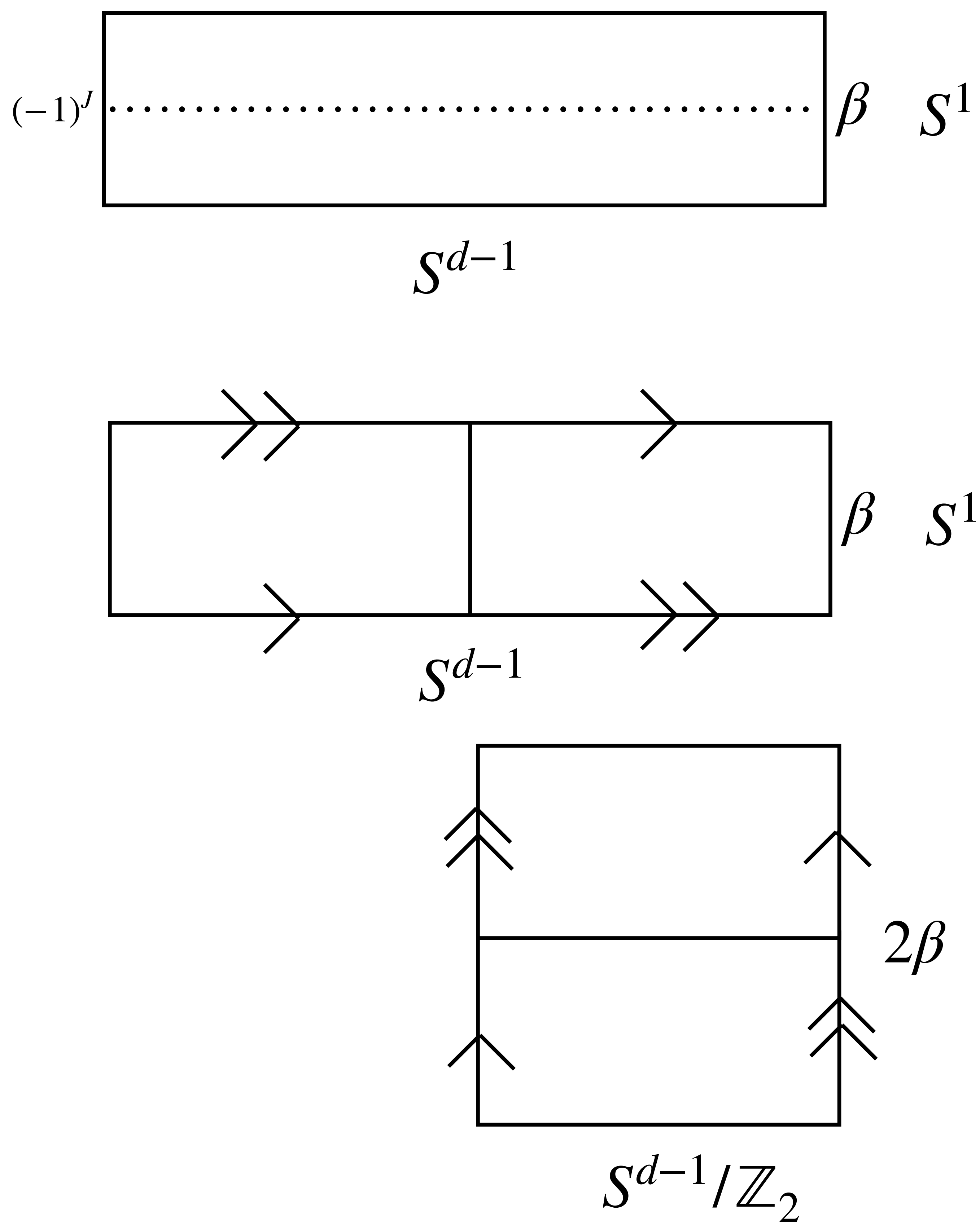

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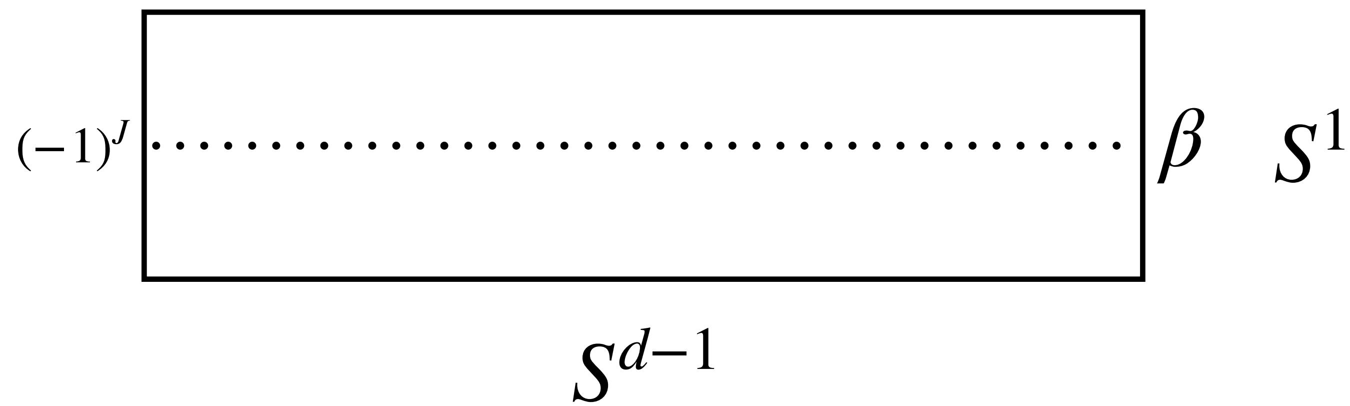
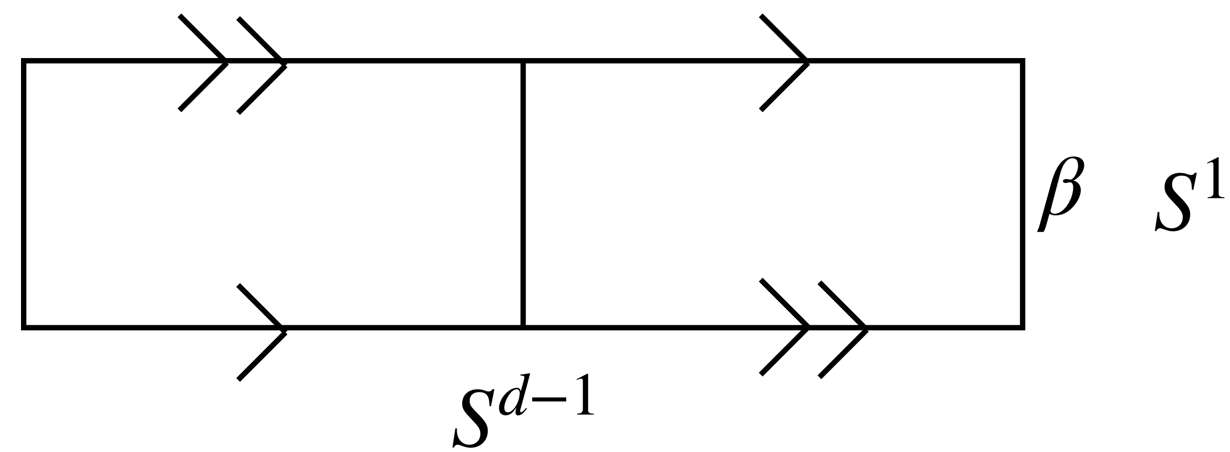
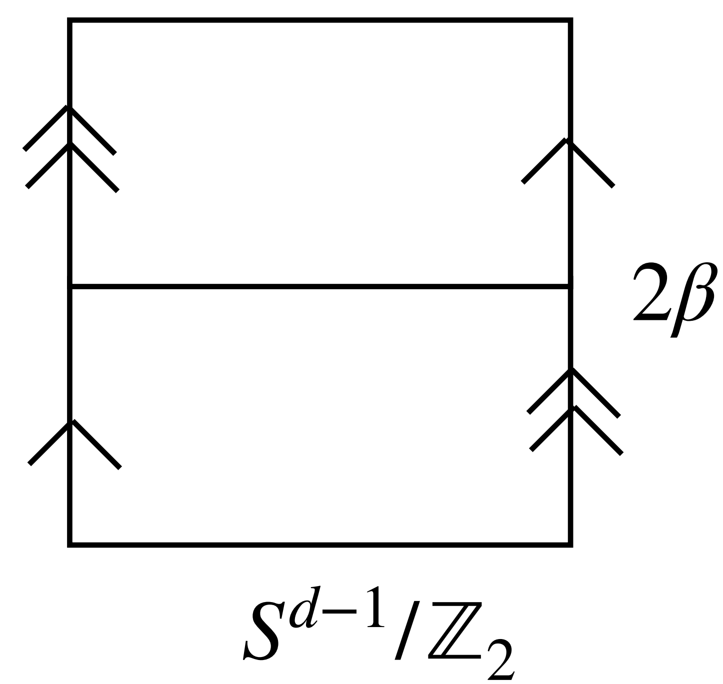




Use thermal EFT, assuming $2\beta \ll 1$



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In general if we add a phase of $e^{2\pi i(p/q)J}$, get factor of q^d

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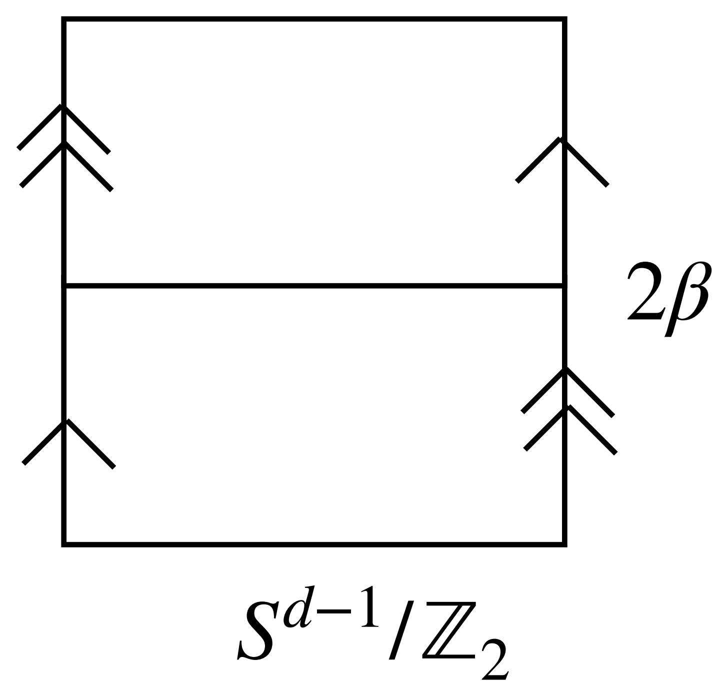

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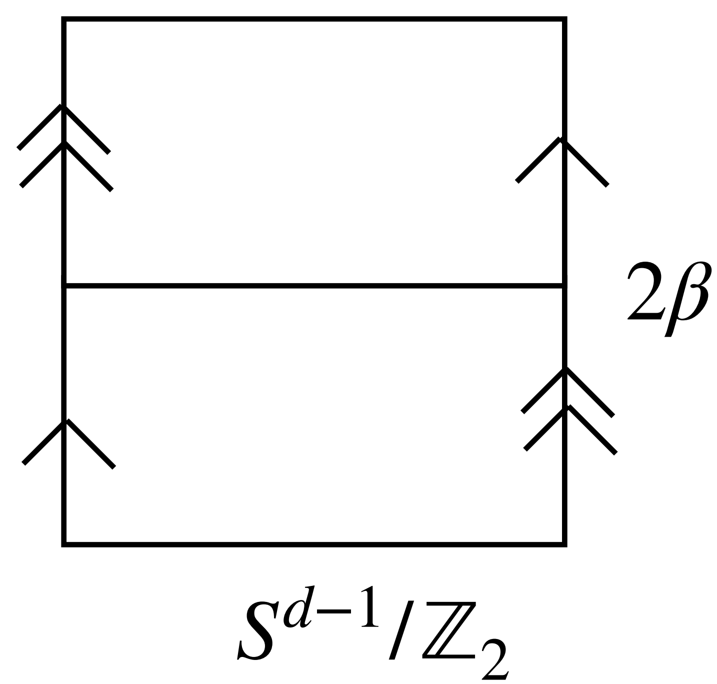

For example, if d=3, $S^{3-1}/\mathbb{Z}_2$ has fixed points on the north and south poles

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If $R^q = 1$,

$$\log \text{Tr}(e^{-\beta H} R) \sim \frac{1}{q} \log \text{Tr}(e^{-q\beta H}) + S_D$$

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$$\mathrm{Tr}\left[(-1)^F e^{-\beta(H-i\Omega J)}\right] \sim e^{-\tilde{S}_{\mathrm{th}}}.$$

$$S_{\mathrm{th}} = \int \frac{d^{d-1}\vec{x}}{\beta^{d-1}}\sqrt{\widehat{g}}\left(-f + c_1\beta^2\widehat{R} + c_2\beta^2F^2 + \dots\right)$$
$$\tilde{S}_{\mathrm{th}} = \int \frac{d^{d-1}\vec{x}}{\beta^{d-1}}\sqrt{\widehat{g}}\left(-\tilde{f} + \tilde{c_1}\beta^2\widehat{R} + \tilde{c_2}\beta^2F^2 + \dots\right)$$

$$\log\left[\mathrm{Tr}(e^{-\beta(H-i\Omega J)}e^{2\pi i\frac{p}{q}J})\right]\sim\begin{cases}\frac{1}{q^d}\frac{f\ \text{vol}\ S^{d-1}}{\beta^{d-1}(1+\Omega^2)}+\ldots, & p+q\ \text{odd},\\ \frac{1}{q^d}\frac{\tilde{f}\ \text{vol}\ S^{d-1}}{\beta^{d-1}(1+\Omega^2)}+\ldots, & p+q\ \text{even}.\end{cases}$$

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This is exactly like what happens in a 2d CFT from modular invariance — depending on p+q, we get either NS or Ramond sector

$$S_{\mathrm{th}} = \int \frac{d^{d-1}\vec{x}}{\beta^{d-1}}\sqrt{\widehat{g}}\left(-f + c_1\beta^2\widehat{R} + c_2\beta^2 F^2 + \ldots\right)$$

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↑ ↑ ↑

Differences in Wilson coefficients are the higher d analog of Ramond ground state energy

Summary

We described a technique called the thermal effective action to systematically study CFT data at large dimension

This encodes the spectrum of local CFT operators as a function of dimension and spin at large dimension

Also obtained large chemical potential formulas using similar techniques