Bounds on σ_{tot} Reloaded

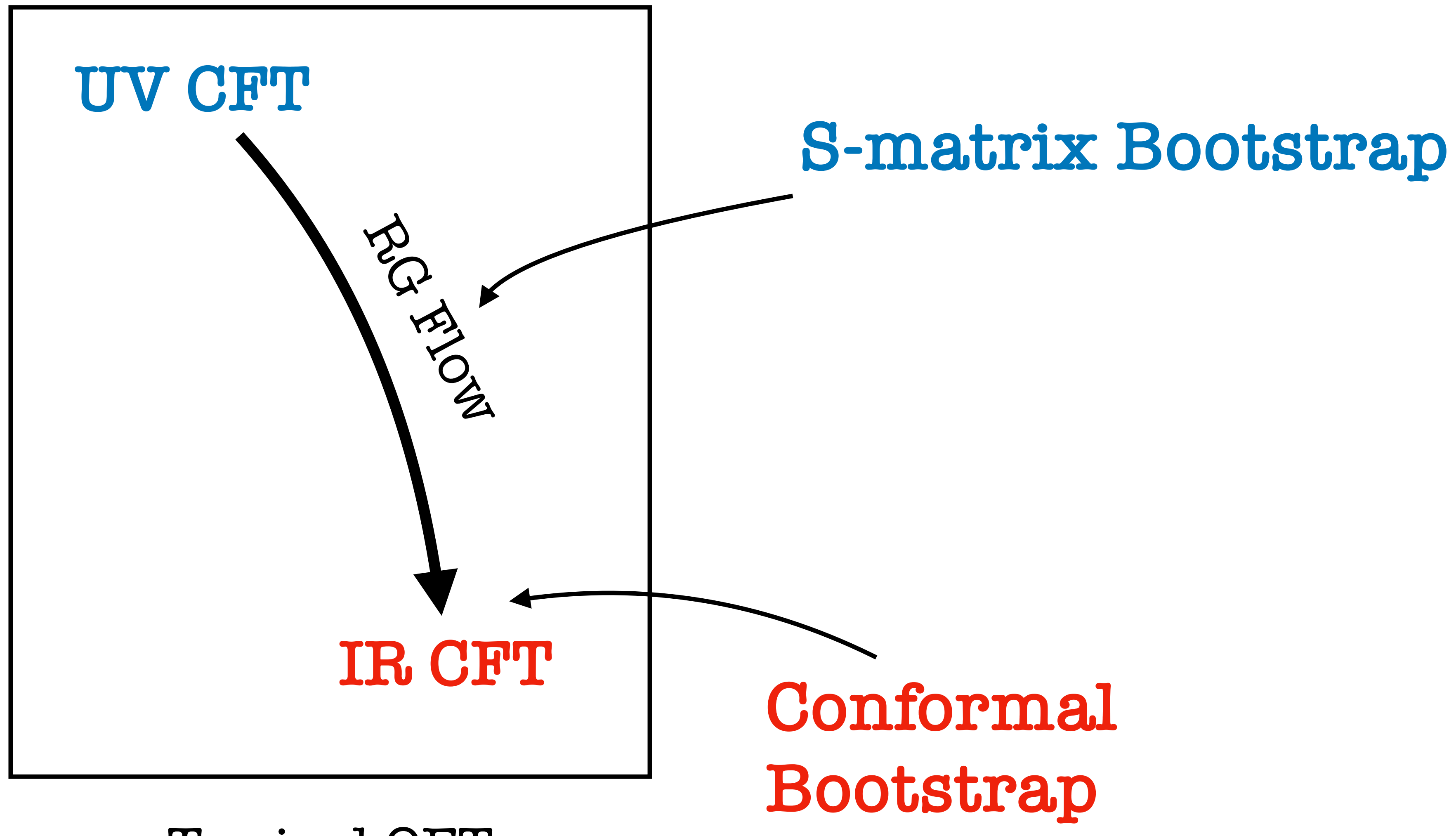
Andrea Guerrieri

March 10, 2025

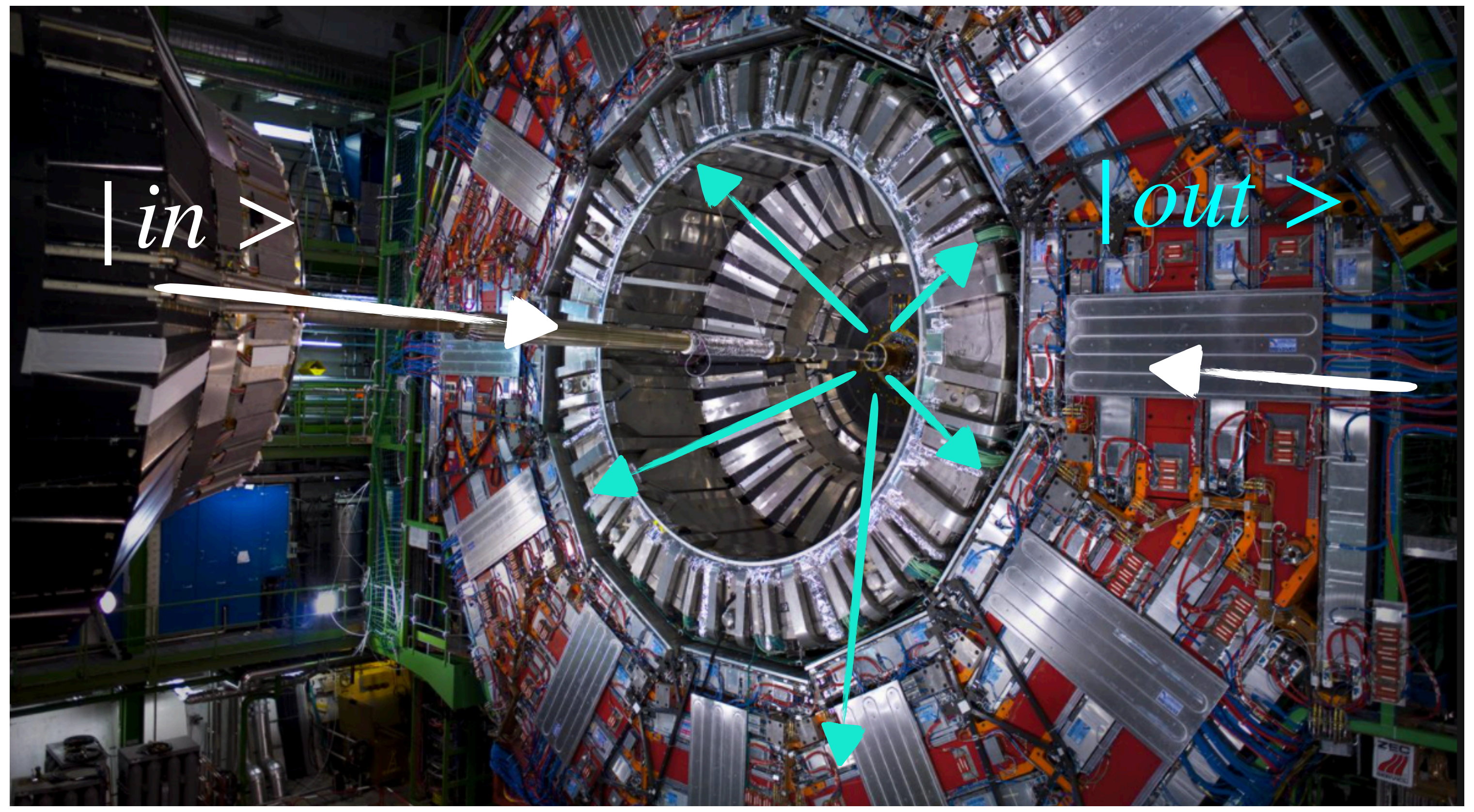
Based on M. Correia, A. Georgoudis, AG to appear soon

What is (im)possible in the Space of QFTs and Strings

Causality, symmetries, and unitarity constrain the space of physical observables



Typical QFT



What we want to measure at colliders

$$\mathcal{S}_{in \to out} \equiv < in | out >$$

Non-perturbative S-matrix Bootstrap

Relativistic QFT in flat space-time

Symmetries

Causality

Unitarity

Non-perturbative S-matrix Bootstrap

Relativistic QFT in flat space-time

- Symmetries
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→

2->2 S-matrix

- Symmetries, crossing
- Analyticity
- Probability conservation

Well-defined set of constraints

Non-perturbative S-matrix Bootstrap

Relativistic QFT in flat space-time		2->2 S-matrix
Symmetries	→	Symmetries, crossing
Causality		Analyticity
Unitarity		Probability conservation

Well defined set of constraints

1) Strong non-perturbative Bounds on scattering observables

2) Study the physical properties of the the Extremal amplitude

1) String theory is (almost) the only consistent UV completion of D $\geq$ 9 supergravity

$$A_{QG} = \int \sqrt{-g}(R + 0 \times R^2 + 0 \times R^3 + \alpha_D R^4 + \ldots)$$

Universal prediction: $\alpha_D^{\min} < \alpha_D < \infty$

ALG, Penedones, Vieira 2212.00151
ALG, Muerali, Penedones, Vieira 2210.01502

Dimension	String/M theory	Bootstrap $\alpha_D^{\min}$
9	≥ 0.2411	0.223 ± 0.002
10	≥ 0.1389	0.124 ± 0.003
11	0.1304	0.101 ± 0.005

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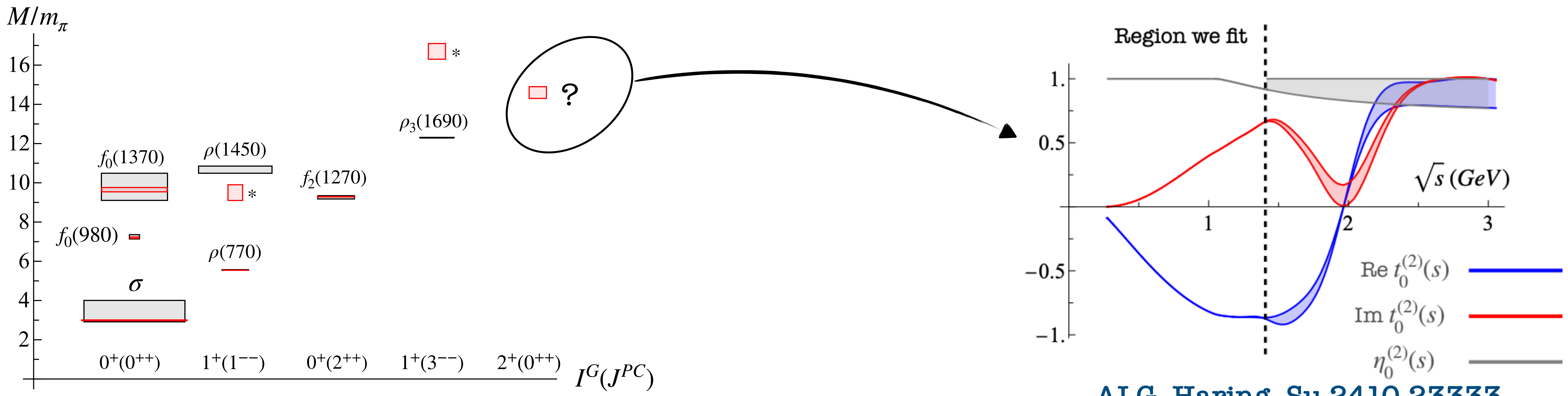
2) Combining Bootstrap and experimental data to extract the spectrum

QCD spectrum that couples to $\pi\pi$ states

We predict the measurable signal



ALG, Haring, Su 2410.23333

The total Cross-Section

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what about the total cross section?

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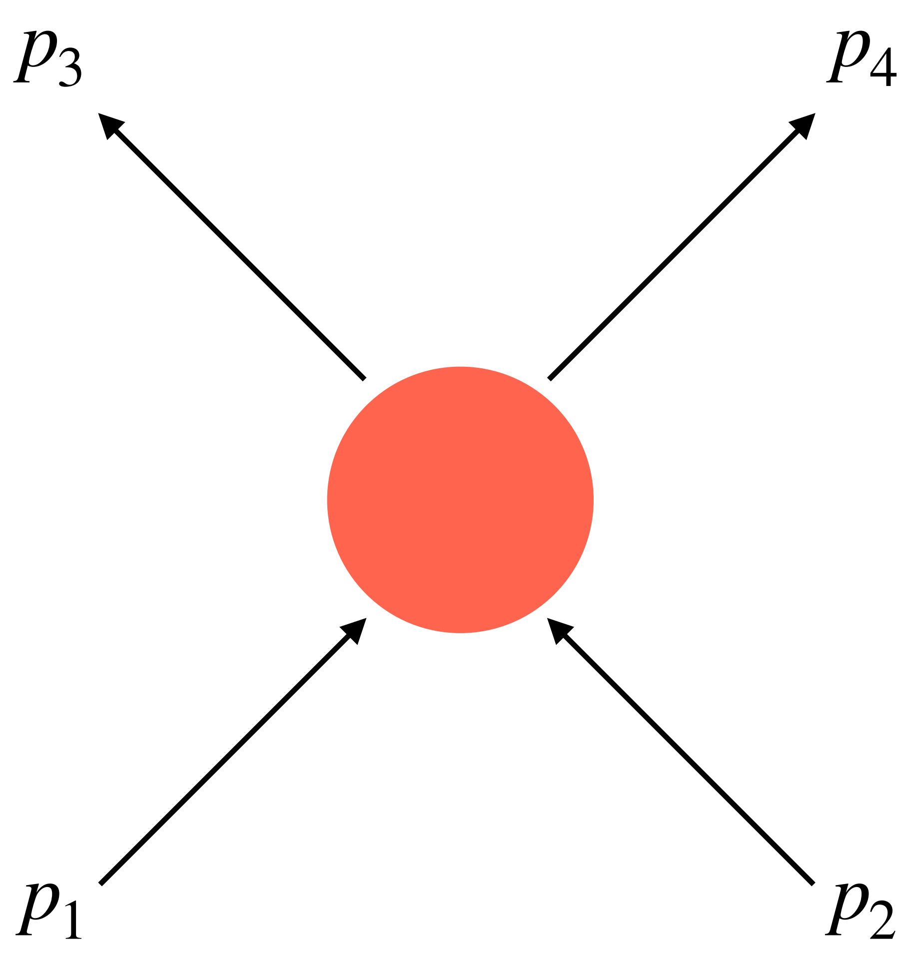

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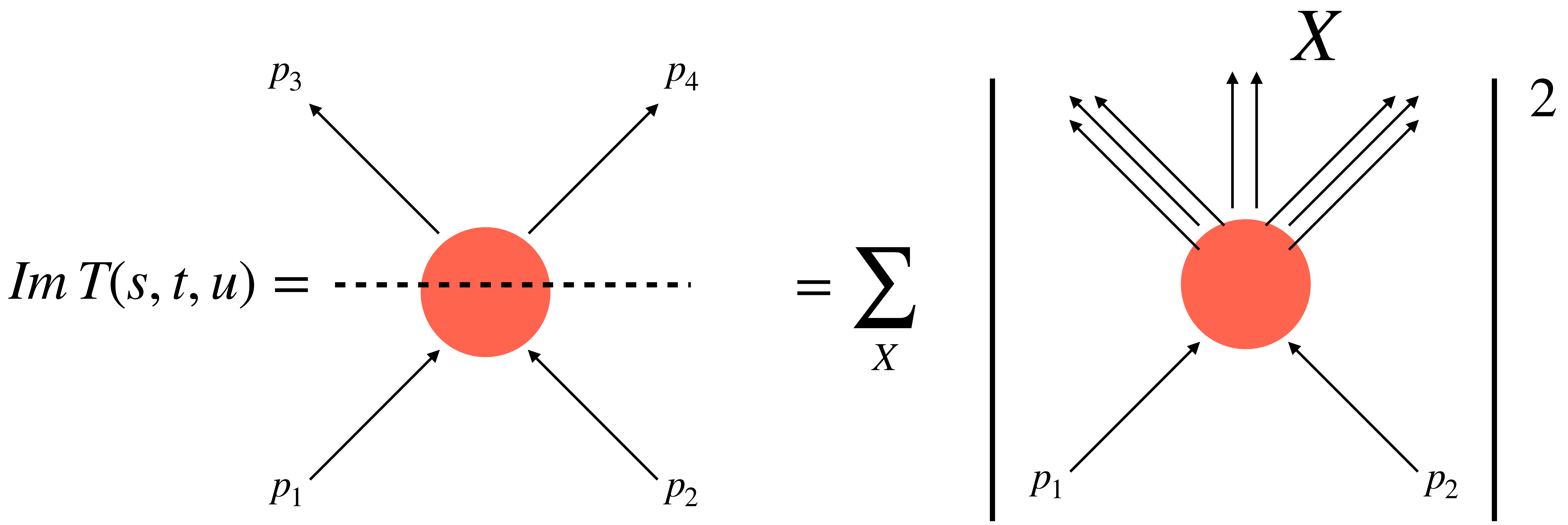

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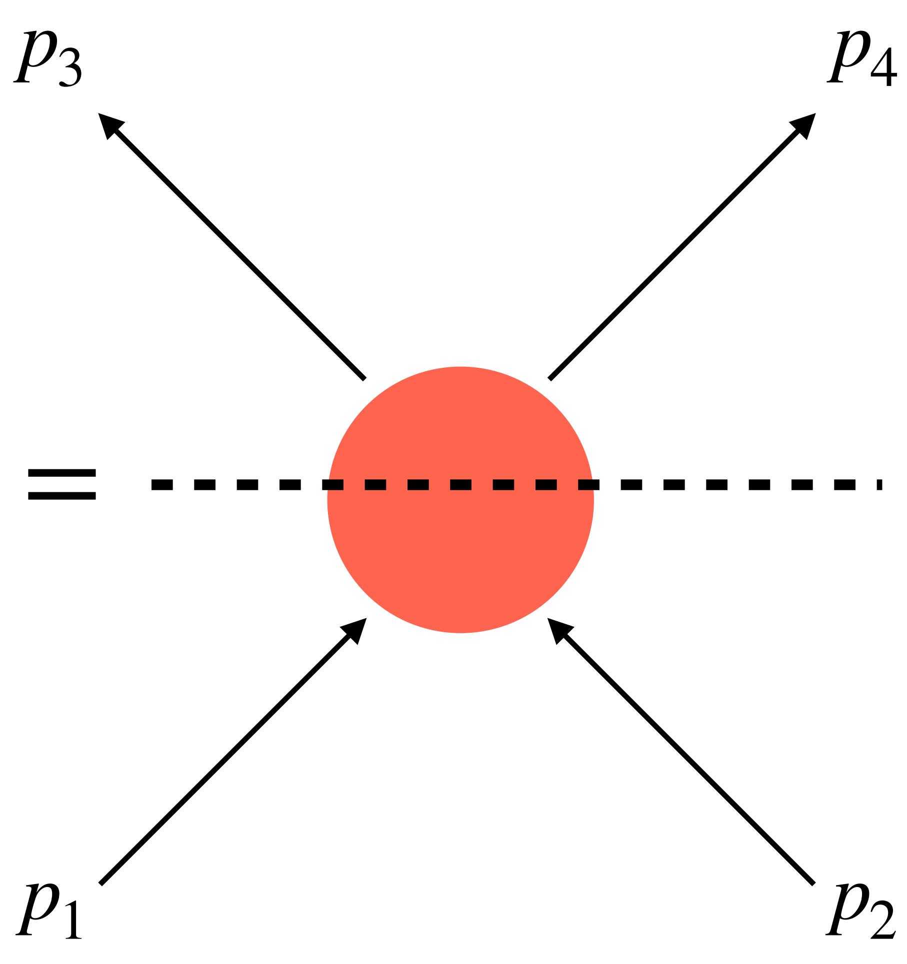
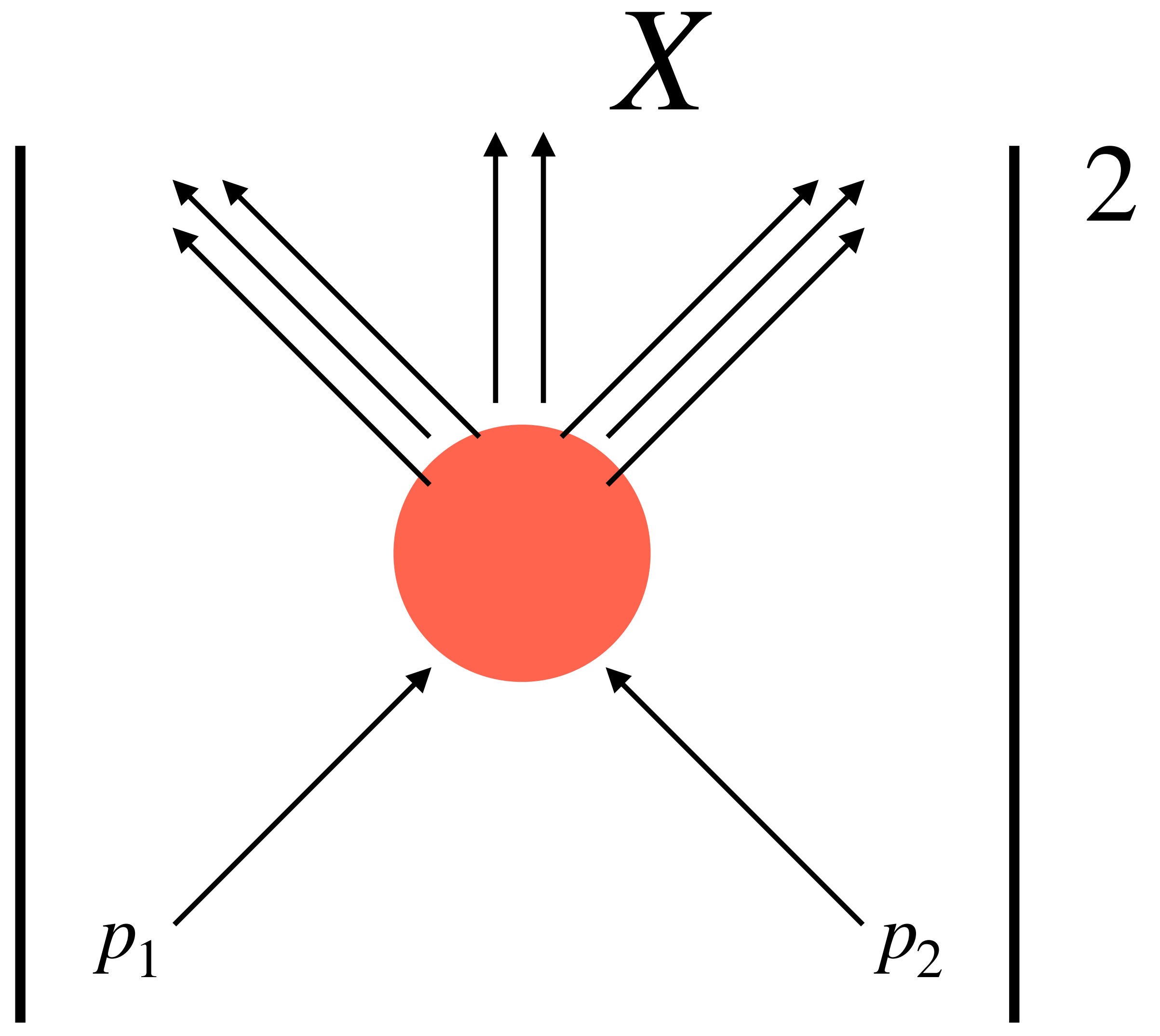
$Im\, T(s,t,u) =$



$= \sum_X$



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Total Cross Section

$t = 0 <\!\!-\!\!-\!\!> b = \infty$

$$\sigma_{tot}(s) = \frac{Im\, T(s, t=0)}{\sqrt{s(s-4)}} \lesssim \pi \log^2 s$$

Froissart bound

The Froissart-Martin Bound and its shortcomings

$$\sigma_{tot}(s) \lesssim \frac{g_s}{t_0} \log^2(s)$$

$g_s = 4\pi$

$\log^2$ because we are in D=4

t_0 imaximum momentum transfer allowed by analyticity e.g. $t_0 = 4$

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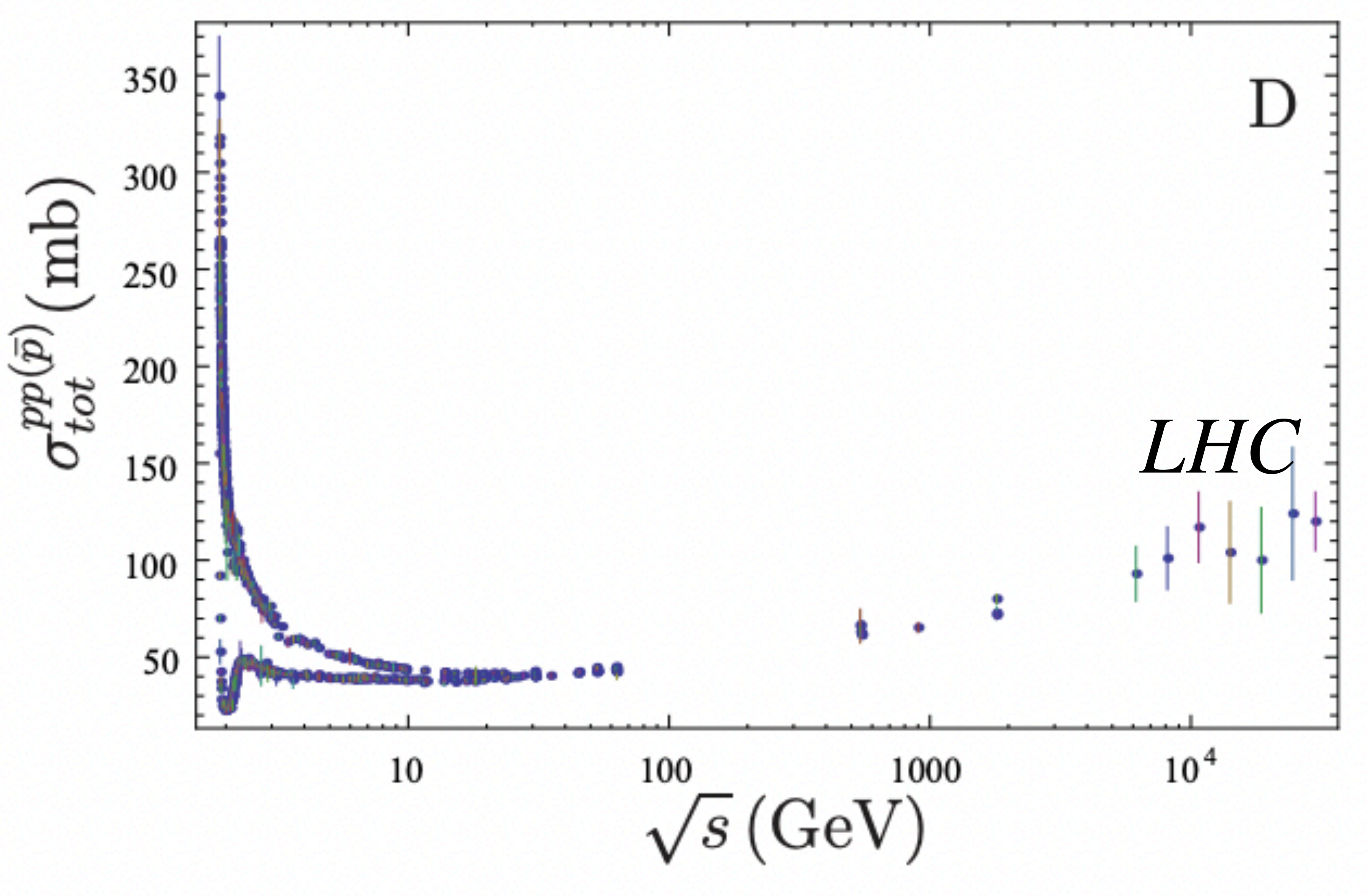
Current fit: $\sigma_{tot}(s) \sim As^{0.008} + Bs^{-0.43}$



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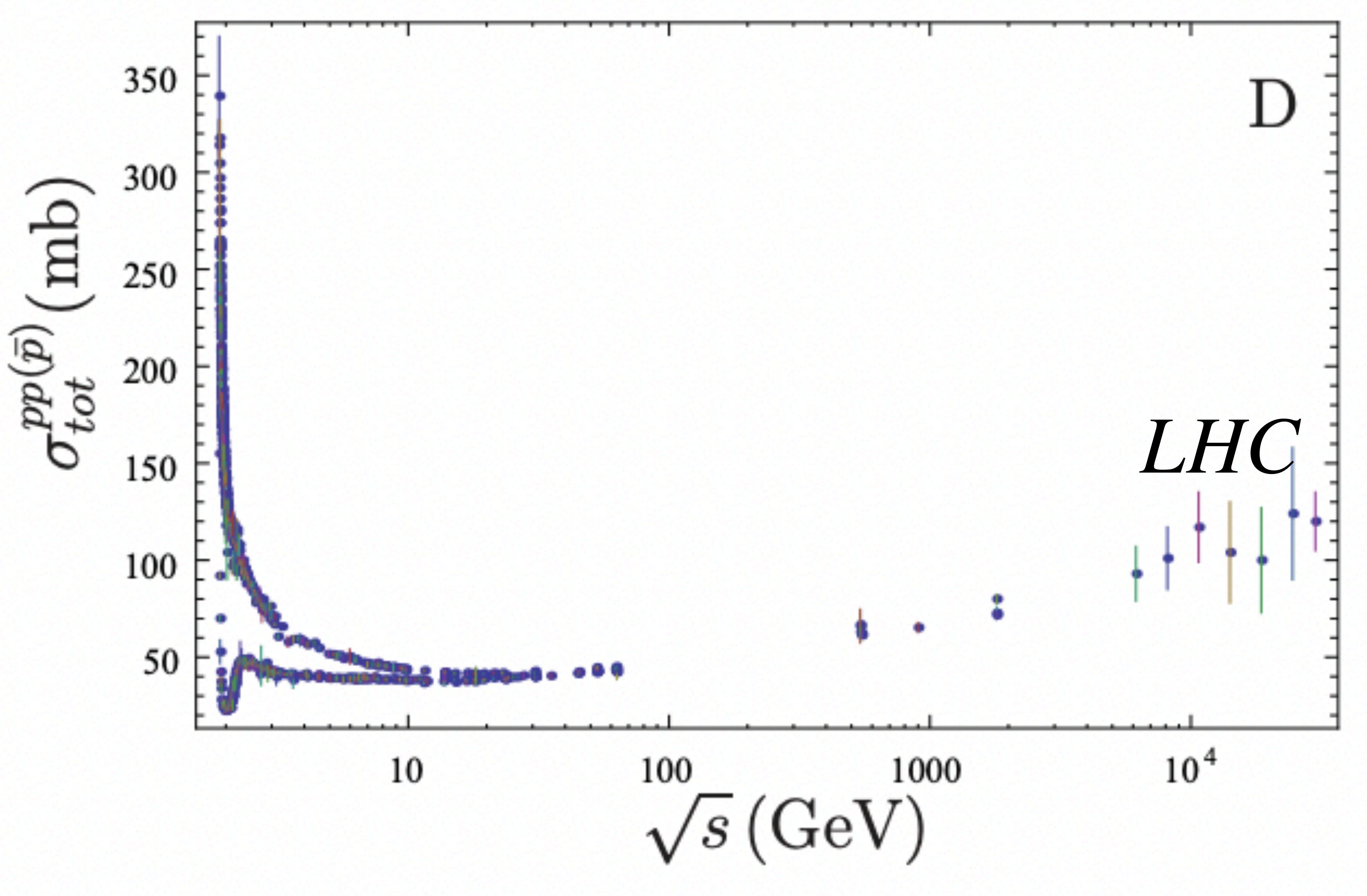

2) Affected by the Martin Pathology: a spin-2 at threshold is not forbidden by unitarity

$$T \supset \frac{P_2((u-t)/(u+t))}{\sqrt{s-4}} + \ldots$$

For $\pi\pi$ scattering is ok, but not in general!

Non-perturbative S-matrix Bootstrap

Maximize your favorite observable following the procedure below

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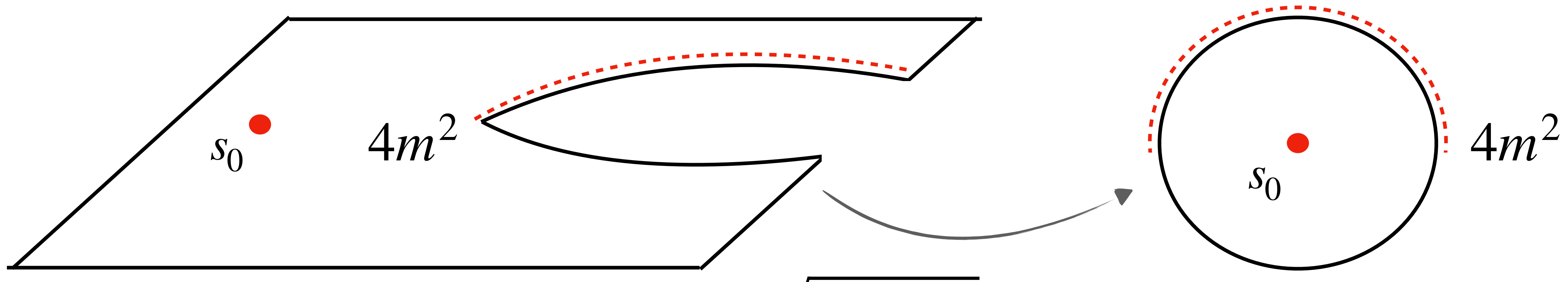
CROSSING SYMMETRY + ANALYTICITY

$$T(s,t,u) = \sum_{a,b,c}^{N_{max}} \alpha_{(abc)} \rho_s^a \rho_t^b \rho_u^c$$



$$\rho_s = \frac{\sqrt{4m^2 - s_0} - \sqrt{4m^2 - s}}{\sqrt{4m^2 - s_0} + \sqrt{4m^2 - s}}$$

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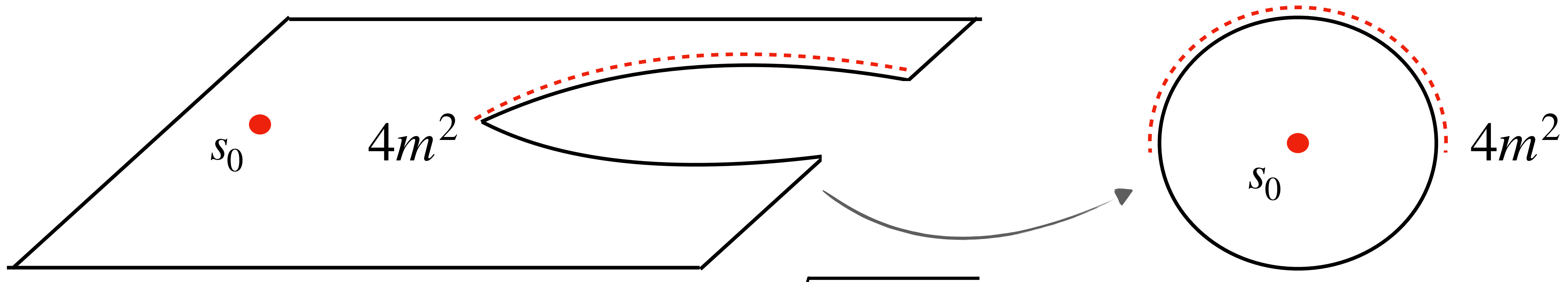

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UNITARITY $$S_\ell(s) = 1 + i\frac{\sqrt{s-4m^2}}{32\pi\sqrt{s}} \int_{-1}^{1} dz\, P_\ell(z)\, T(s,t(z)) \qquad z = \cos\theta$$

Truncated set of semidefinite-positive constraints $\quad |S_\ell|^2 \leq 1 \qquad s_{grid} > 4m^2, \quad \ell = 0,\ldots,L_{max}$

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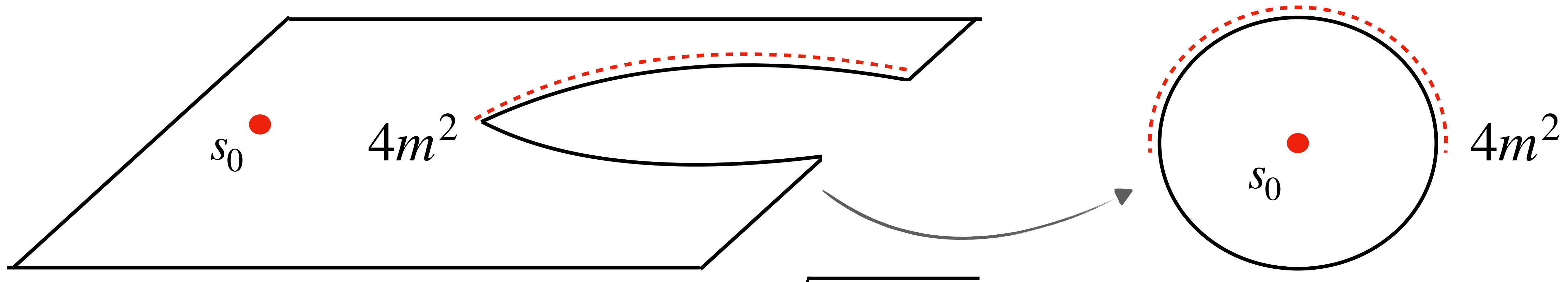

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Triple Extrapolation

$$N_{max} \to \infty, L_{max} \to \infty, s_{grid} \to s$$

Paulos, Penedones, Toledo, van Rees, Vieira '17

Bounds on the cross-section at finite energies I

$$\bar{\sigma}_{\text{tot}}(s) \equiv \frac{1}{16\pi} \int_{4m^2}^{s} \frac{s' - 4m^2}{s - 4m^2} \, \sigma_{\text{tot}}(s') \, ds'$$

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1) We decompose the cross section in partial waves

$$T(s,t) = \frac{16\pi i \sqrt{s}}{\sqrt{s-4}} \sum_{\ell=0}^{\infty} (2l+1) \, P_l(z) \left(1 - S_\ell(s)\right)$$

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2) Divide et impera: split the sum and bound separately low and high spins

$$\bar{\sigma}_{\text{tot}}(s) = \int_4^s ds' \left(\sum_{\ell=0}^{L-2} + \sum_{\ell=L}^{\infty} \right) \frac{(2\ell+1)(1 - \text{Re}S_l(s'))}{s-4}$$

$$\leq L(L-1) + \boxed{\int_4^s ds' \sum_{\ell=L}^{\infty} \frac{(2\ell+1)(1 - \text{Re}S_l(s'))}{s-4}} \quad (8)$$

Bounds on the cross-section at finite energies II

$$\bar{\sigma}_{\rm tot}(s) \equiv \frac{1}{16\pi}\int_{4m^2}^{s}\frac{s'-4m^2}{s-4m^2}\,\sigma_{\rm tot}(s')\,ds' \;\leq L(L-1) + \int_4^s ds' \sum_{\ell=L}^{\infty}\frac{(2\ell+1)\big(1-{\rm Re}S_l(s')\big)}{s-4} \quad (8)$$

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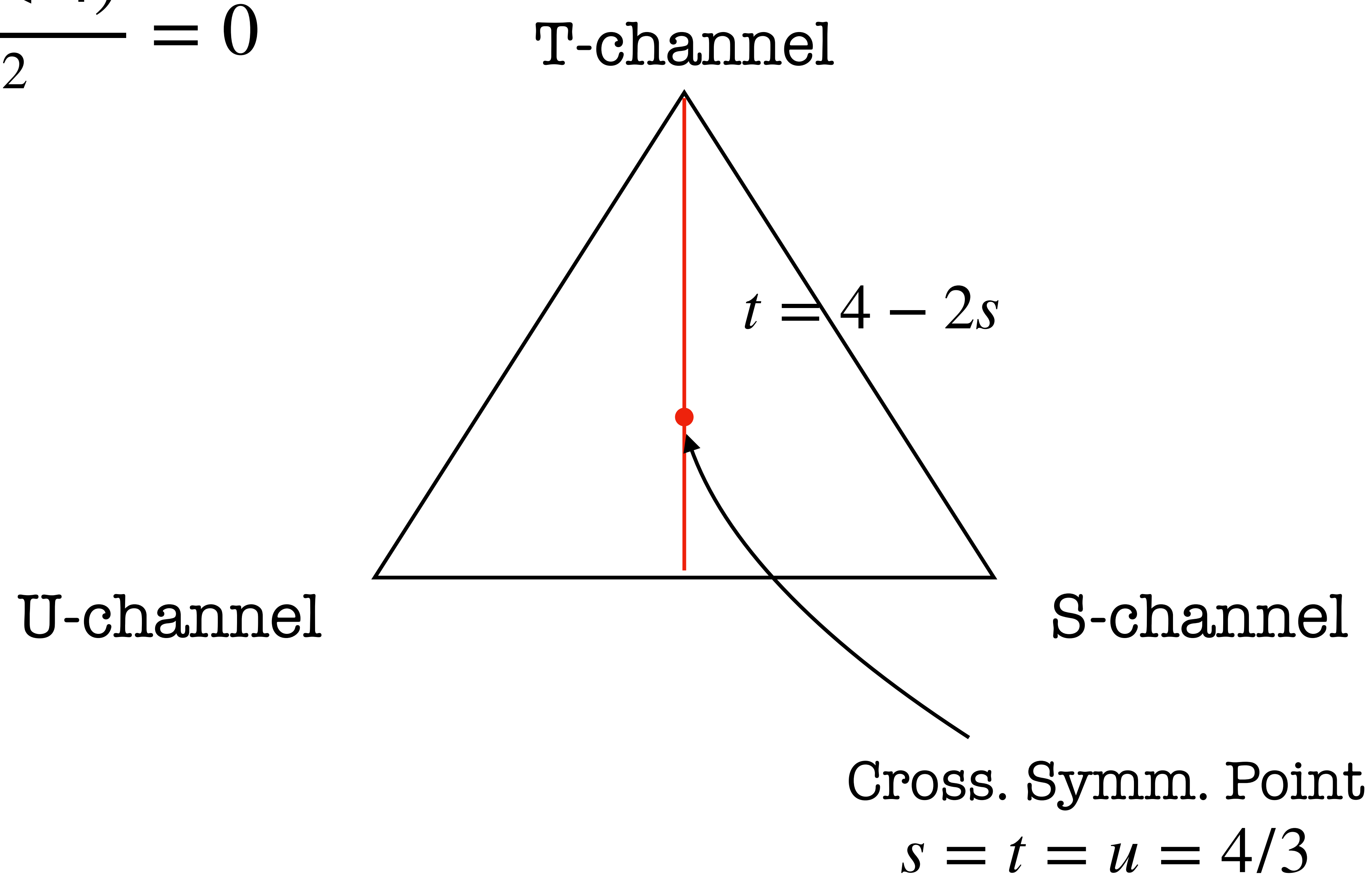

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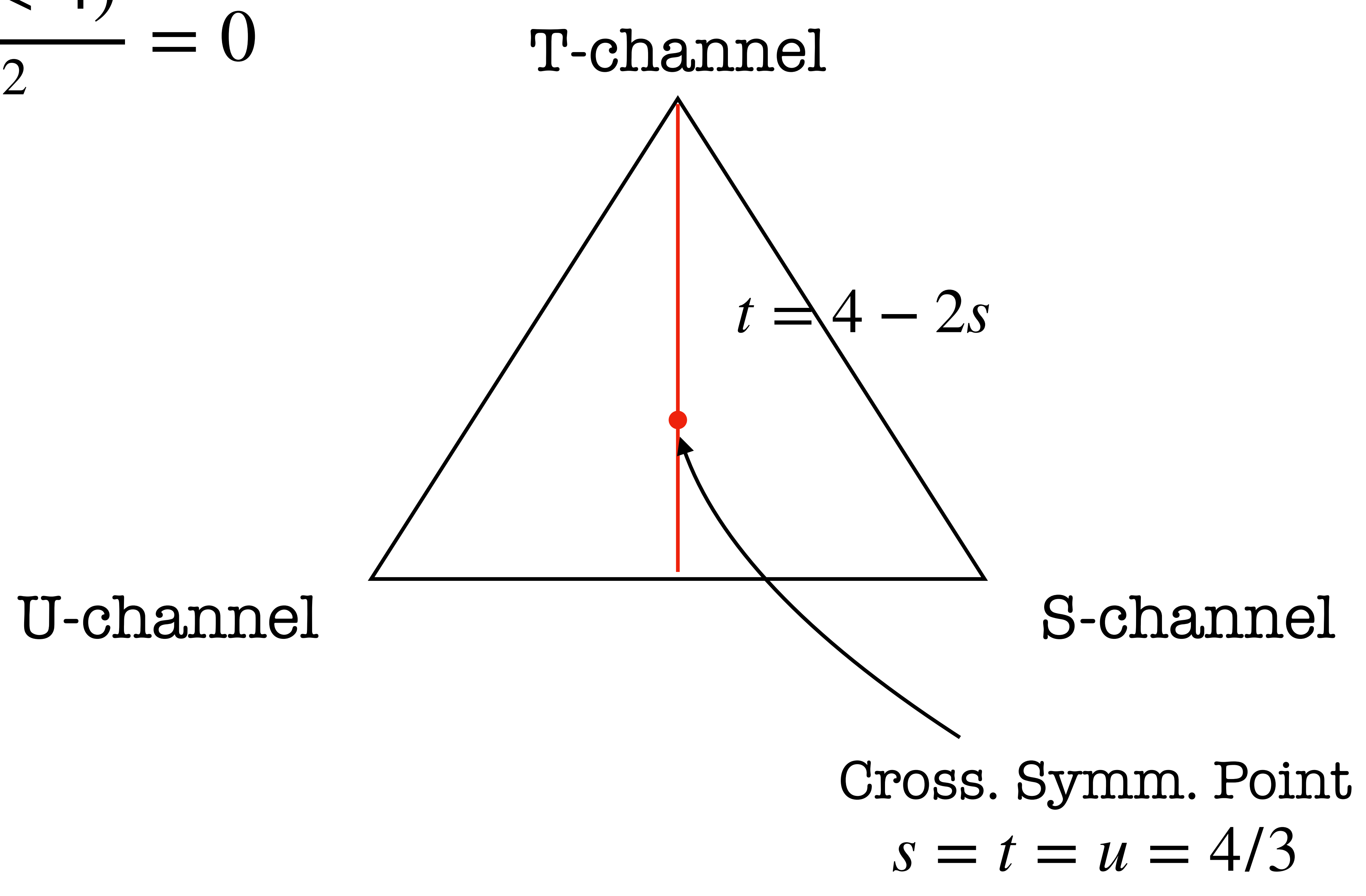

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Lowest dimensional dispersive coefficient

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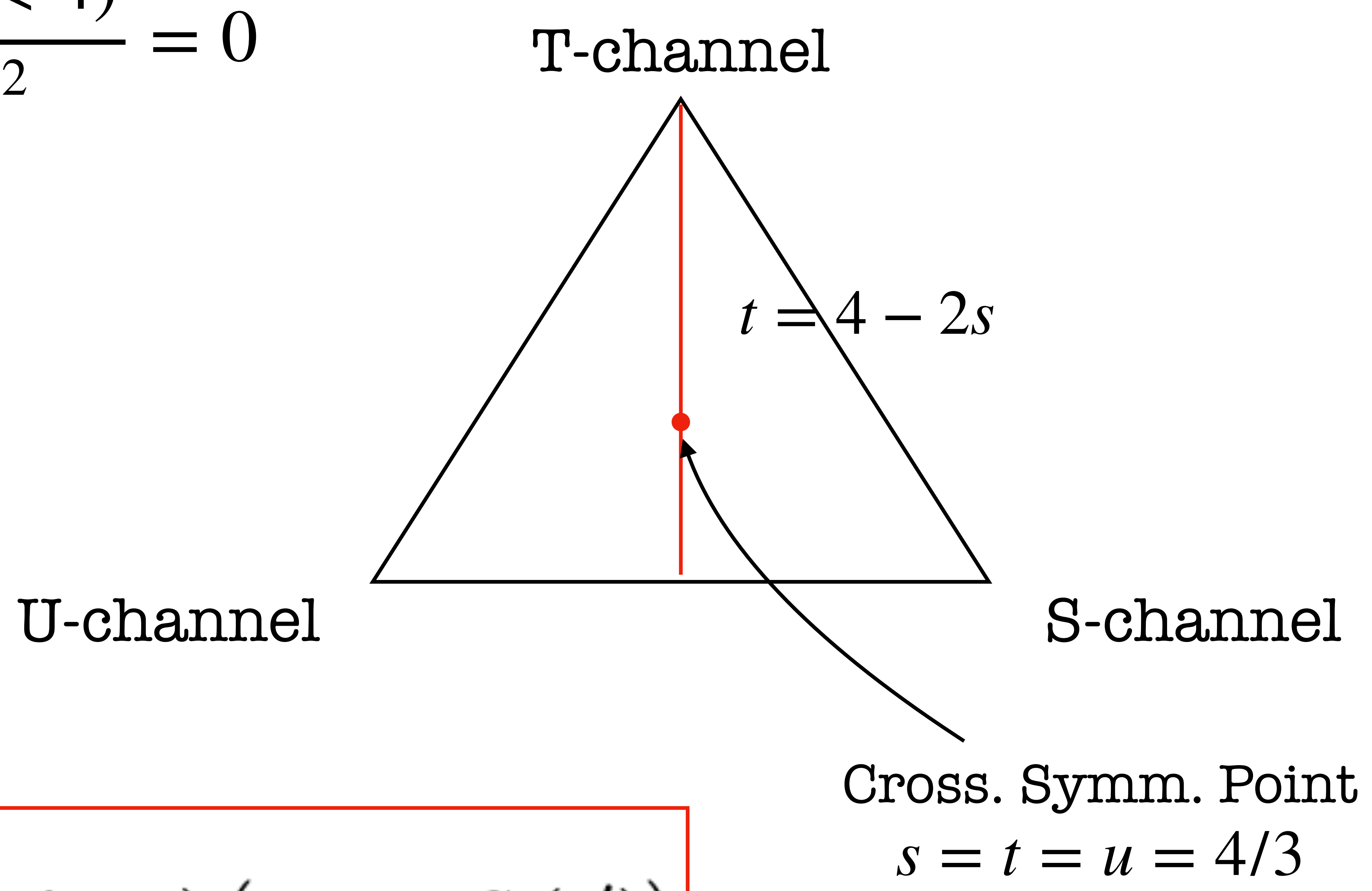

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$$\mathcal{K}_\ell(s,t_0) = \frac{1}{(s-2+\frac{t_0}{2})^3}\frac{\sqrt{s}}{\sqrt{s-4}} P_\ell\left(1+\frac{2t_0}{s-4}\right) > 0$$

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$$\bar{\sigma}_{\text{tot}}(s) \leq L(L-1) + \frac{2\pi\, c_2(t_0)}{(s-4)\,\mathcal{K}_L(s,t_0)}$$

True bound for any s, L, t_0 given $c_2(t_0)$

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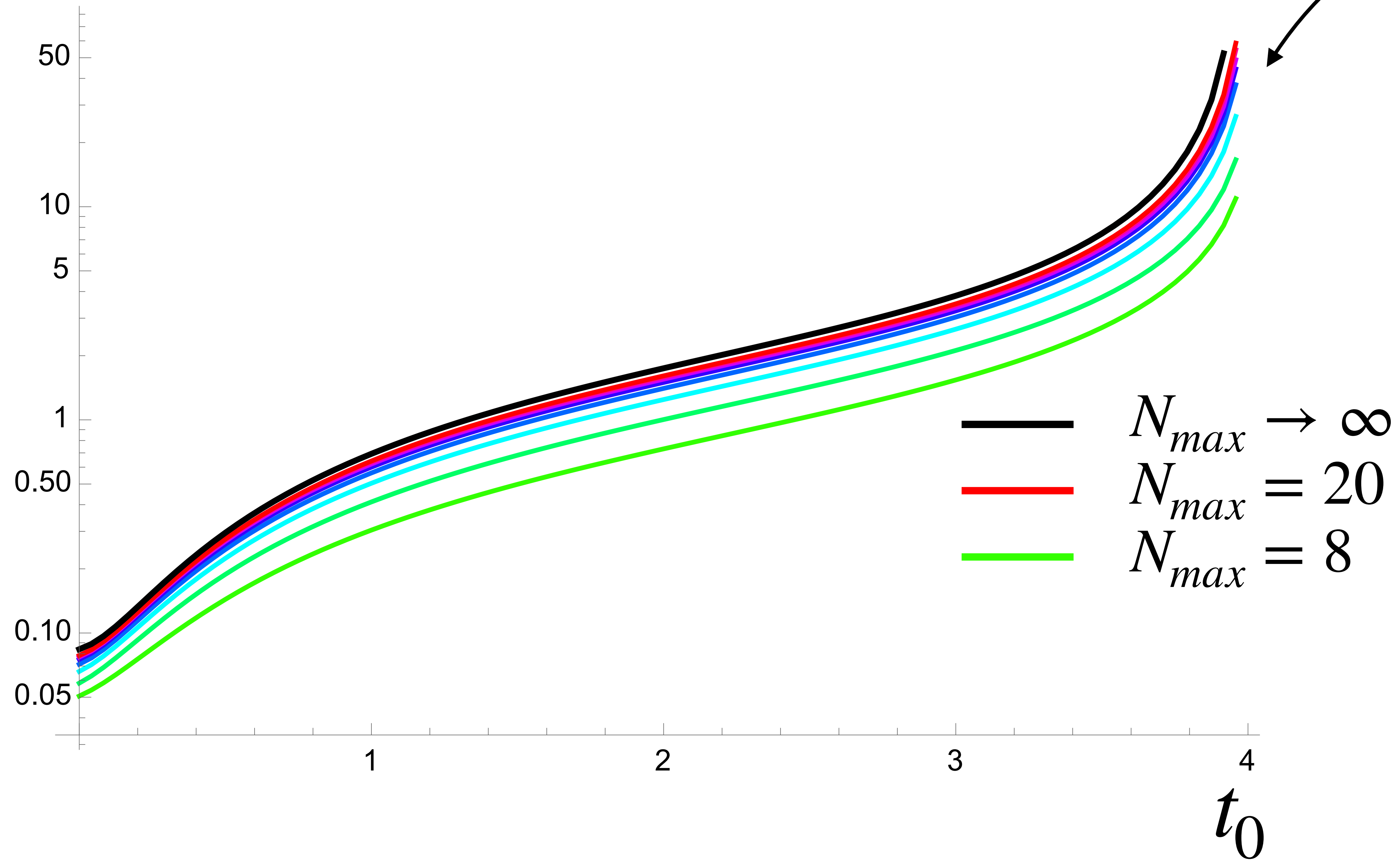

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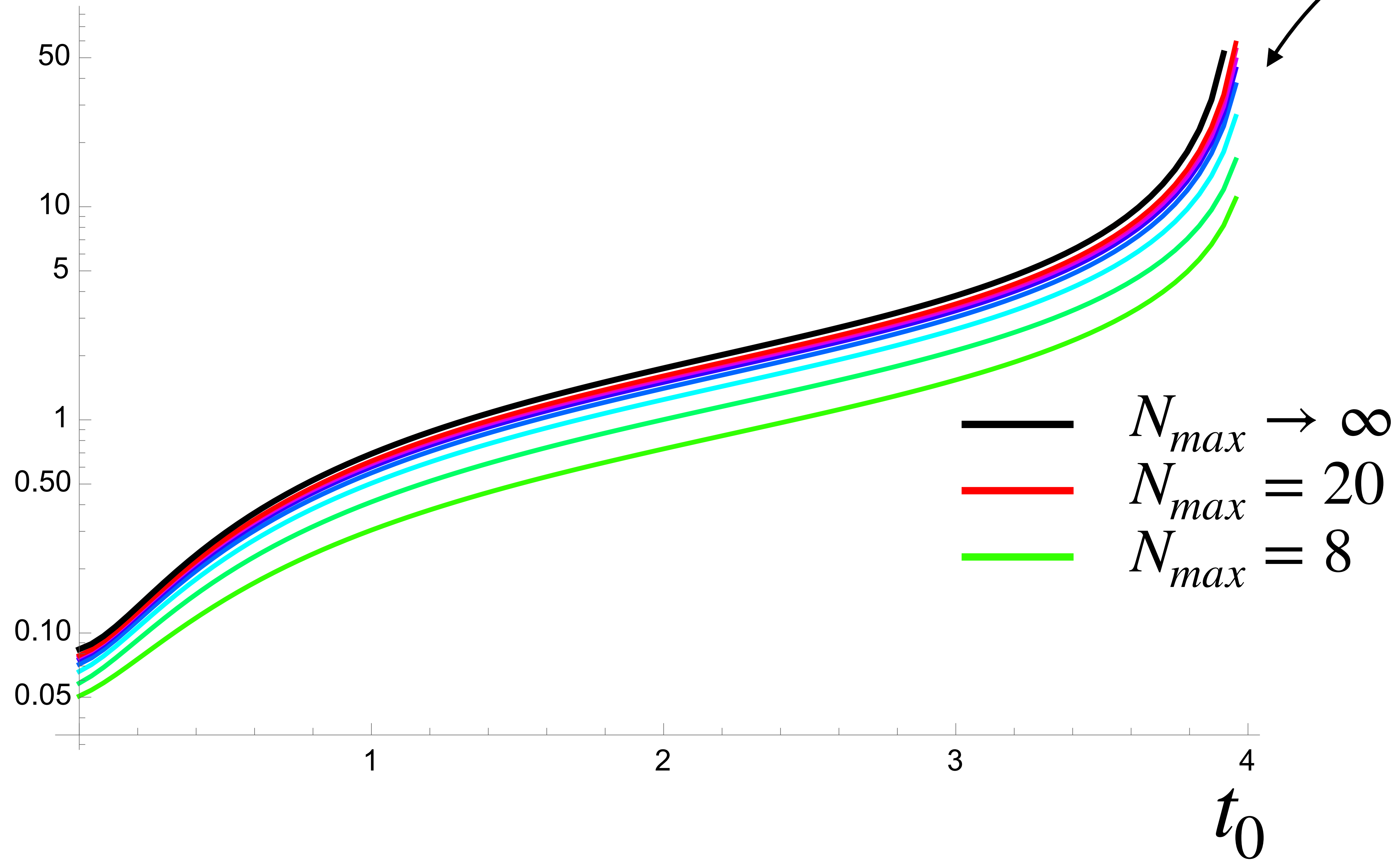
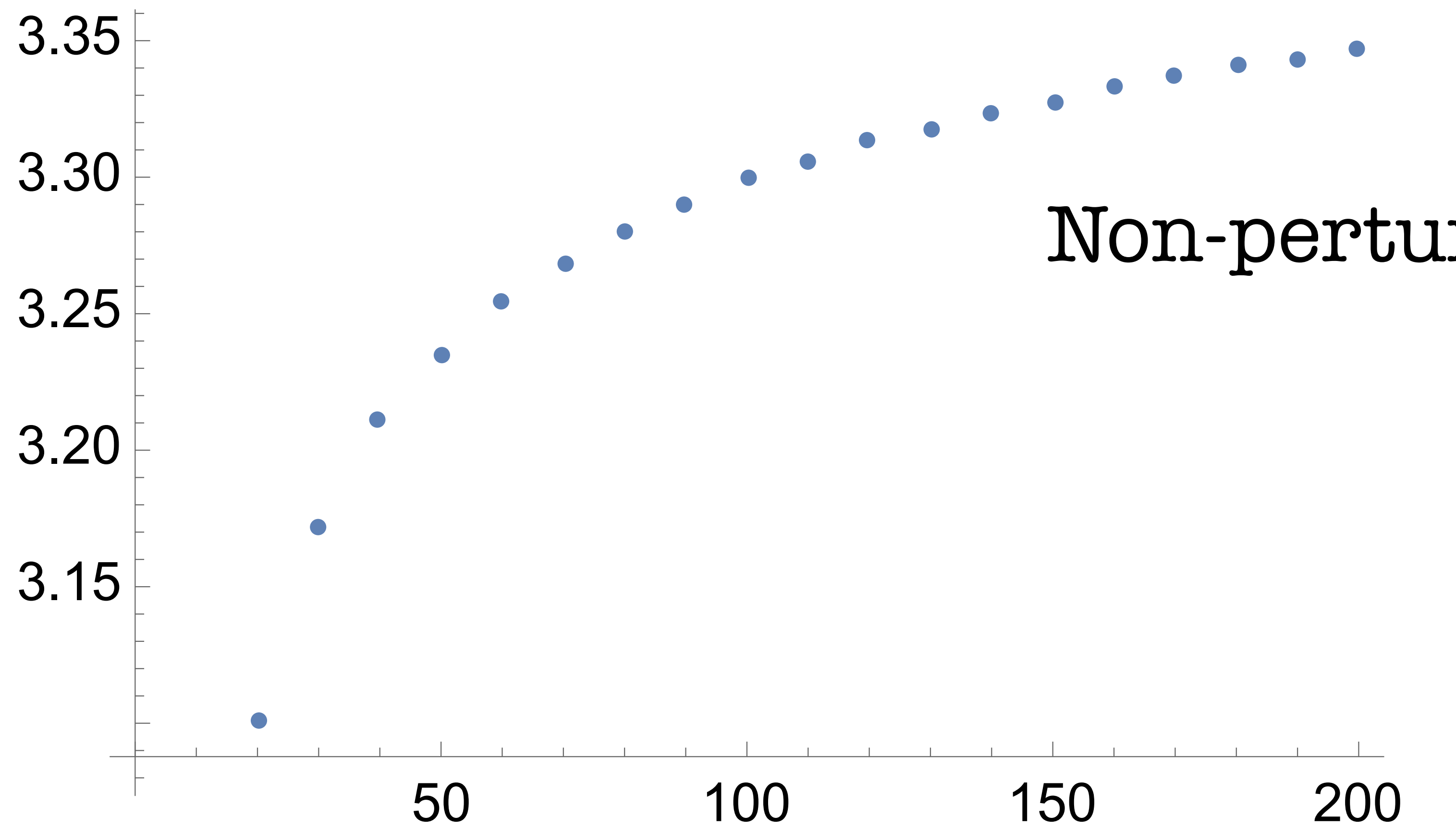


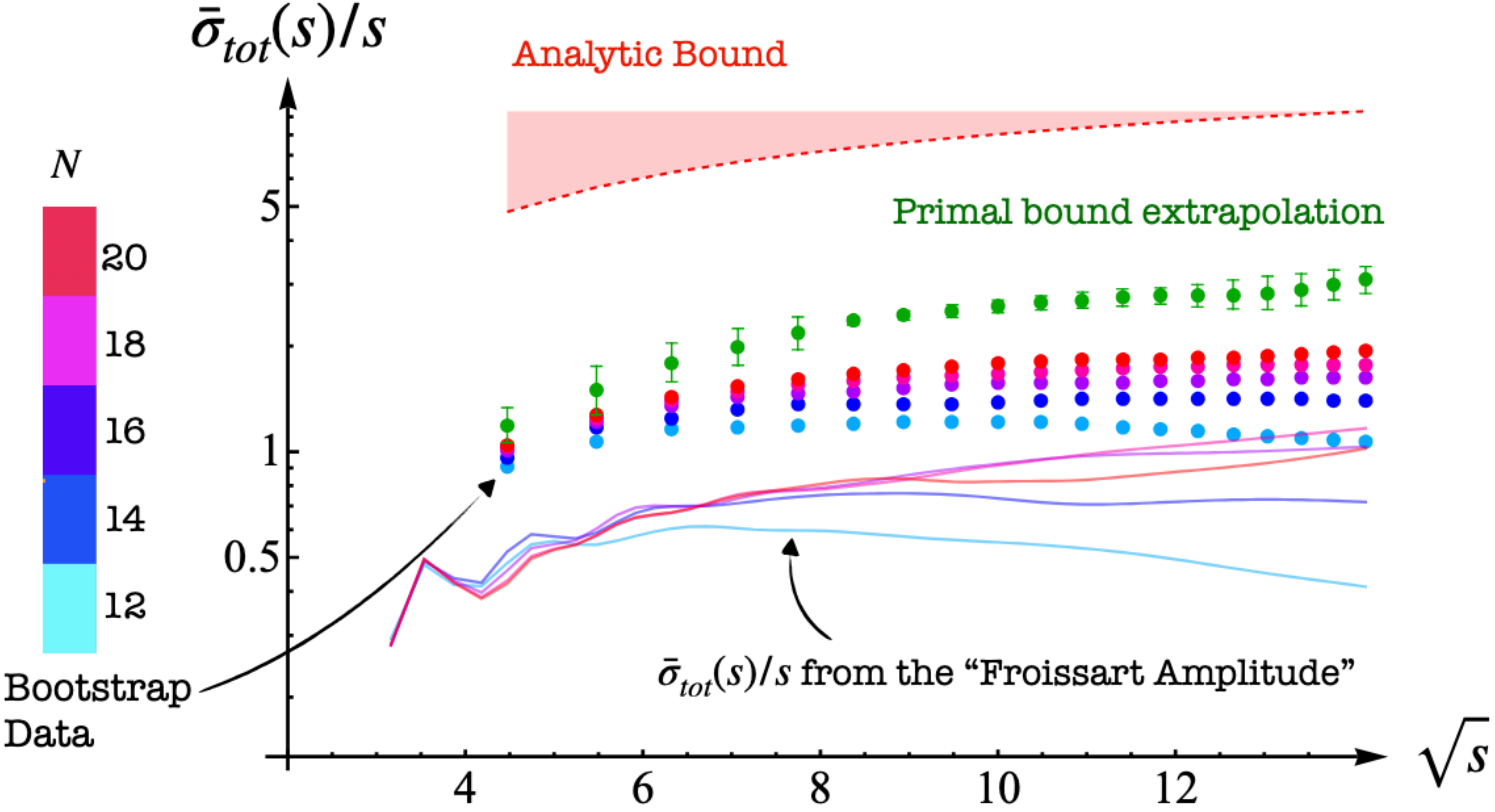
We cannot set $t_0 = 4$, and then take the limit!!

Optimal t_0



Non-perturbative saddle

Bounds on the cross-section at finite energies IV



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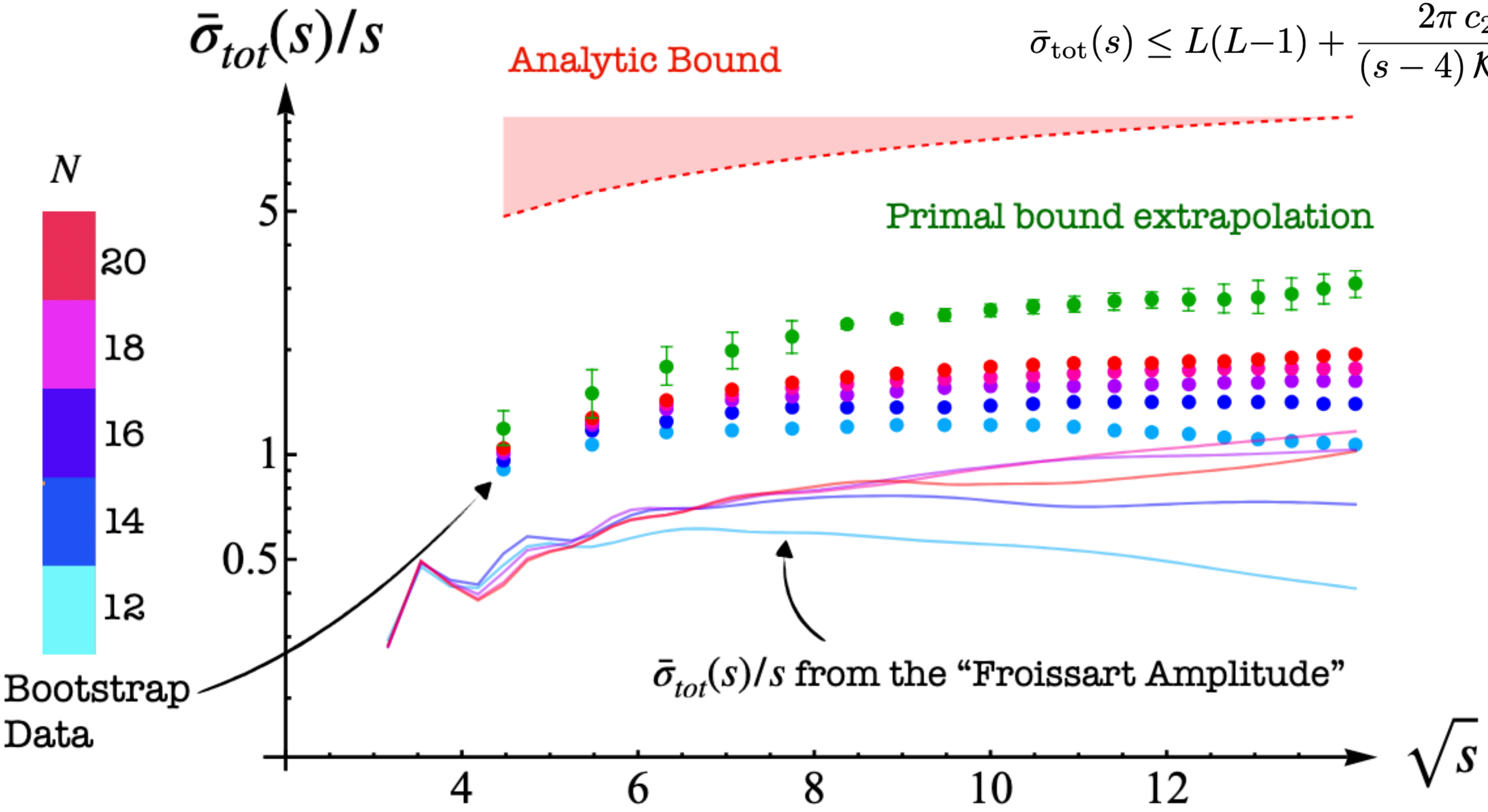

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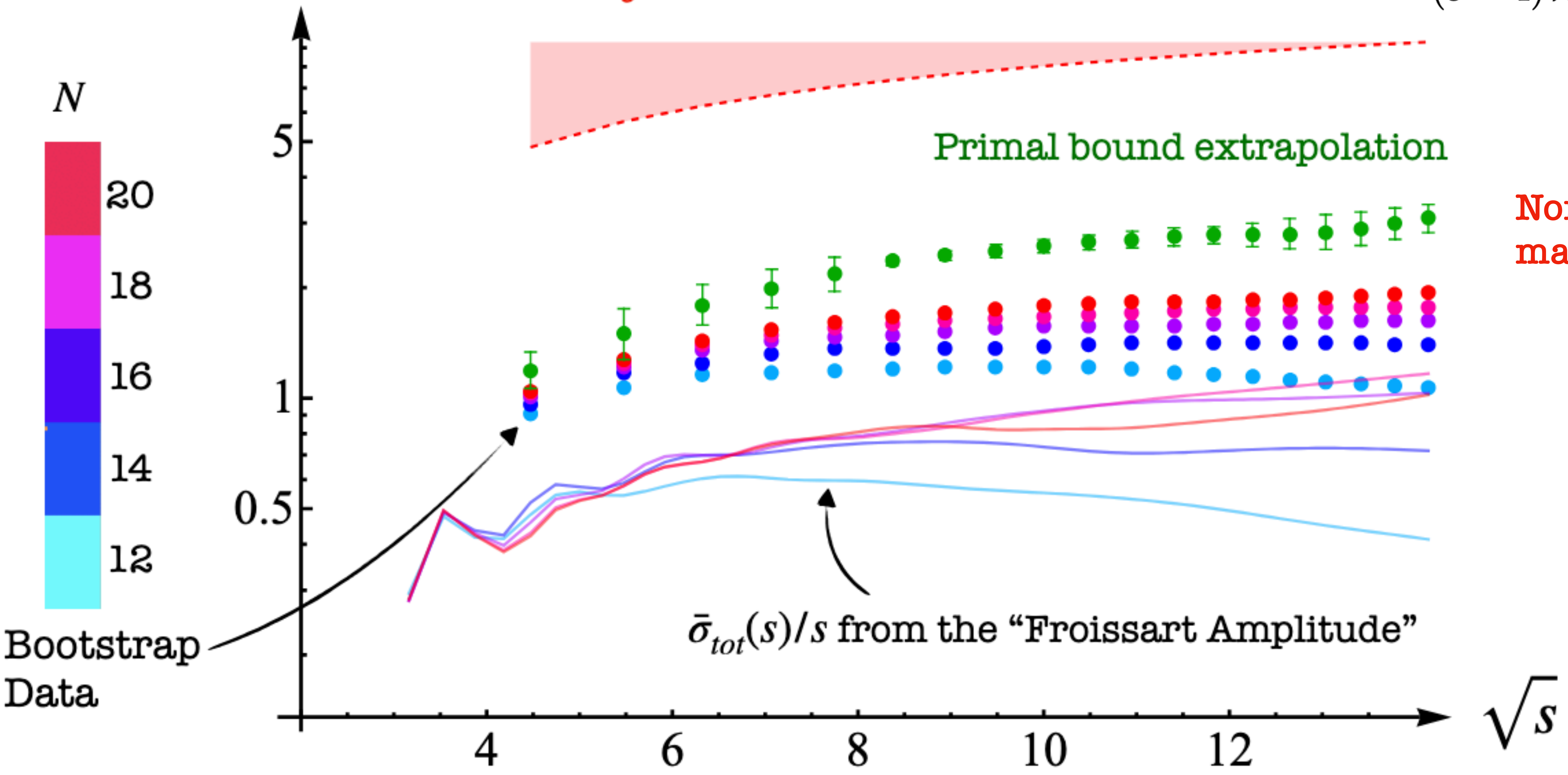
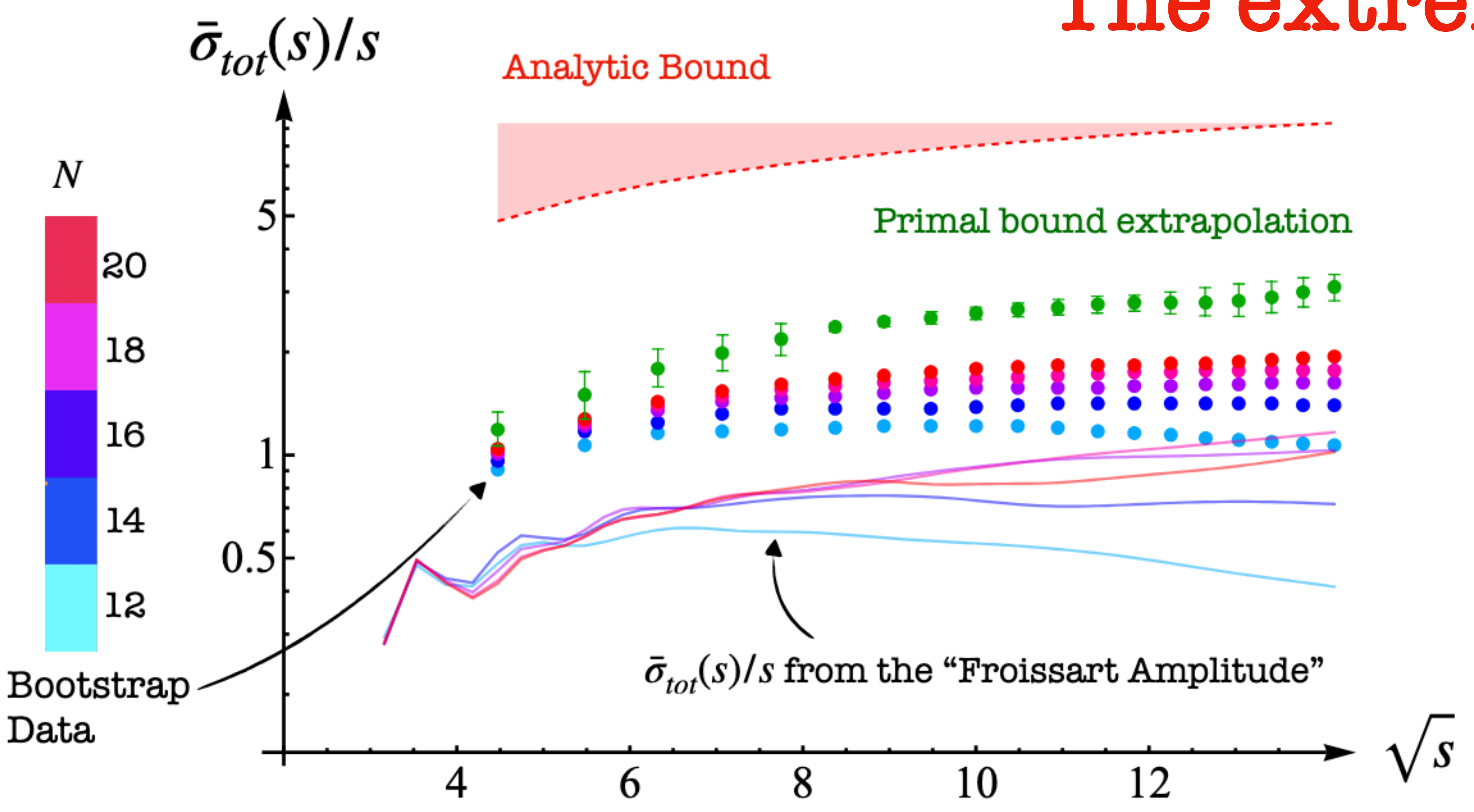


Non-perturbative S-matrix Bootstrap

$\max \bar{\sigma}_{tot}(s)$

The extremal amplitude at finite s



Factor 5 discrepancy:

$1 - ReS_\ell = 2$ in the analytic for low ℓ?

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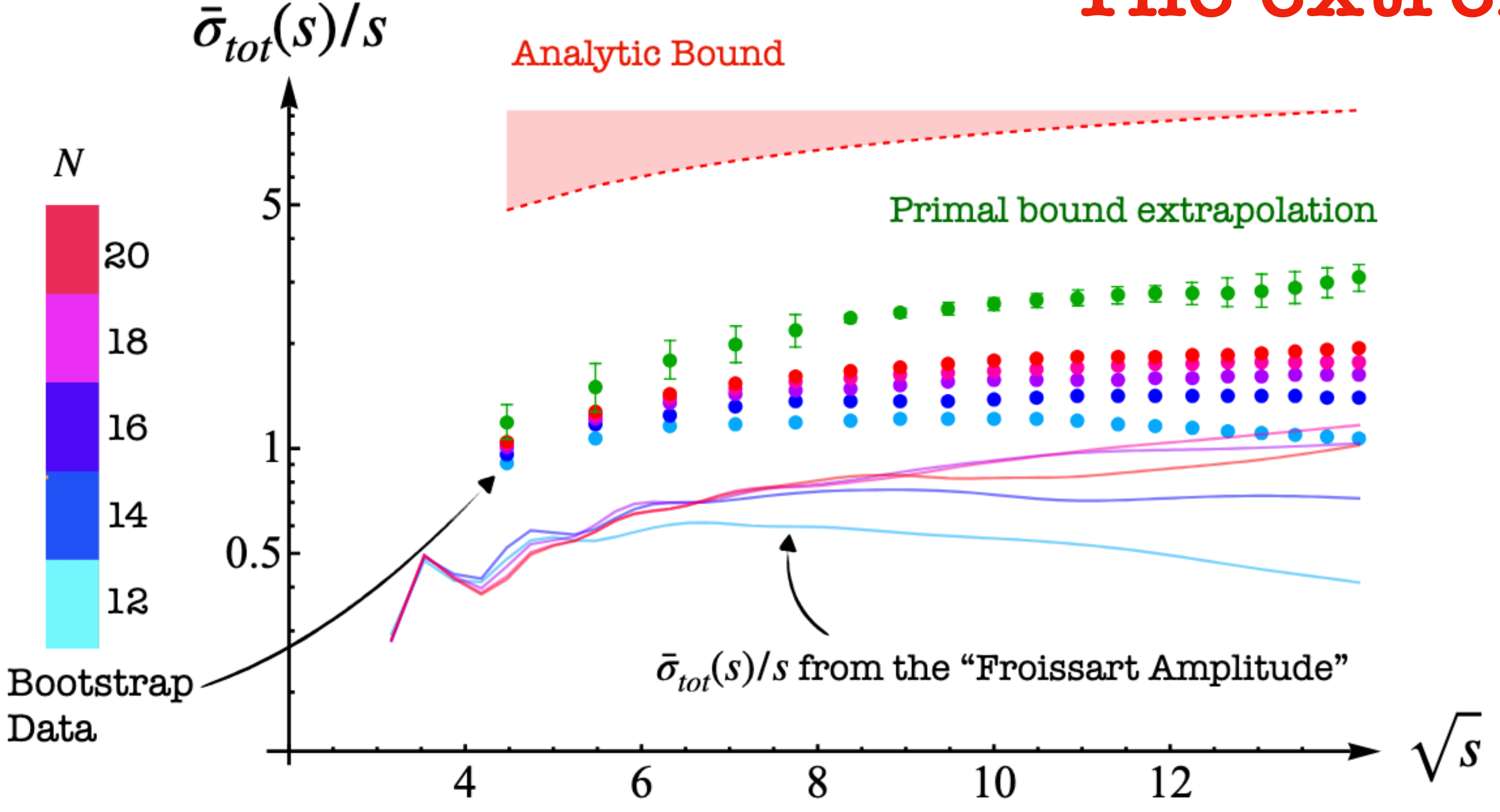

Factor 5 discrepancy:
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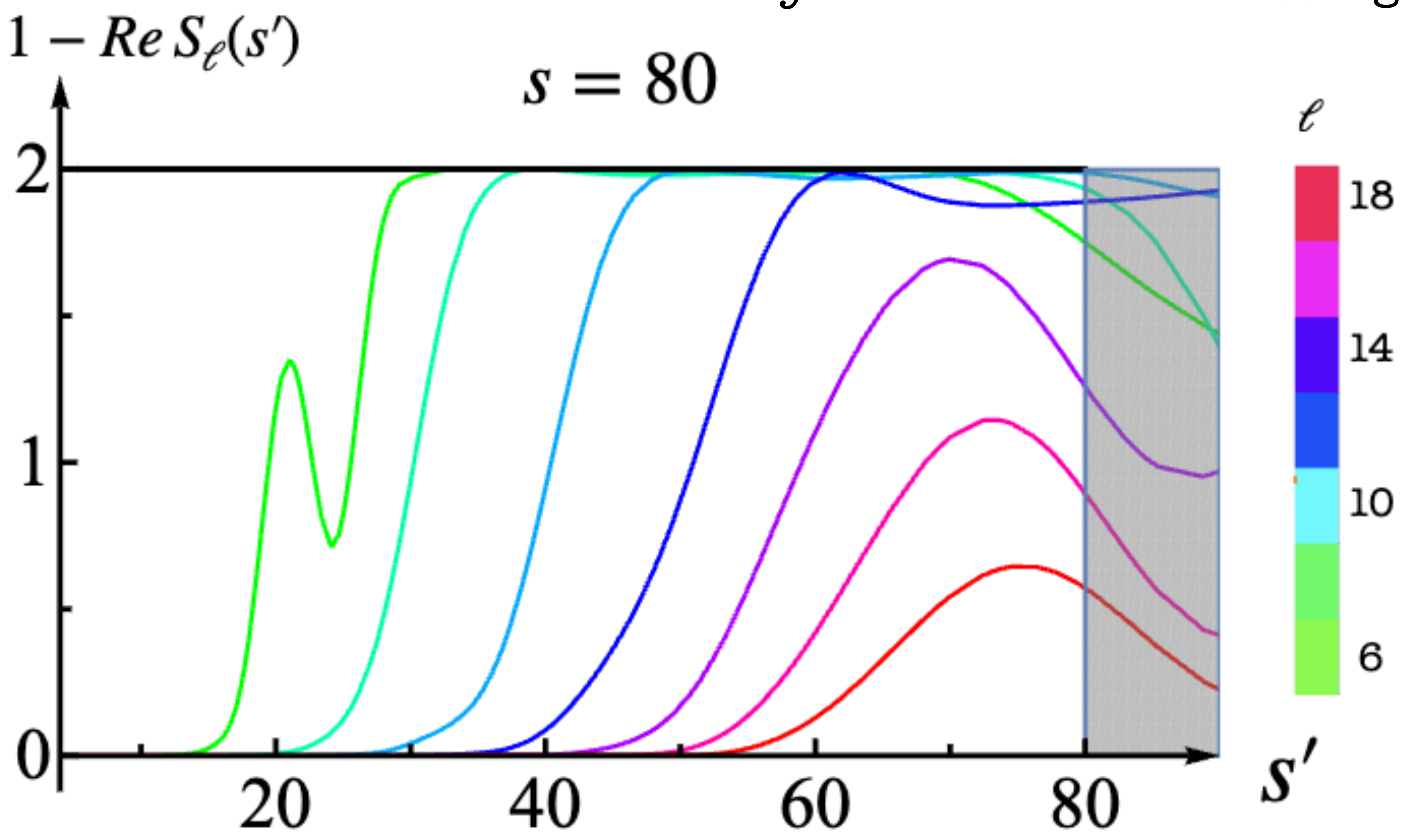
In the analytic bound we are missing unitarity in t-channel



Regge trajectory shoots up, similar to a Coon amplitude!

The Froissart Amplitude I

What happens when $s \to \infty$?

What is the amplitude that maximizes the interaction at all scales?

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What is the amplitude that maximizes the interaction at all scales?

We look for it in the space of **S-matrix Data**

$$T(s,t) = \sum_{n,k=0}^{\infty} c_{n,k}\, \sigma^n\, \tau^k, \quad \sigma = \bar{s}^2 + \bar{t}^2 + \bar{u}^2,\ \tau = \bar{s}\bar{t}\bar{u}$$

$\bar{s} = s - 4/3$

Shifted Mandesltam at the crossing symmetric point s=t=u=4/3

The Froissart Amplitude I

What happens when $s \to \infty$?

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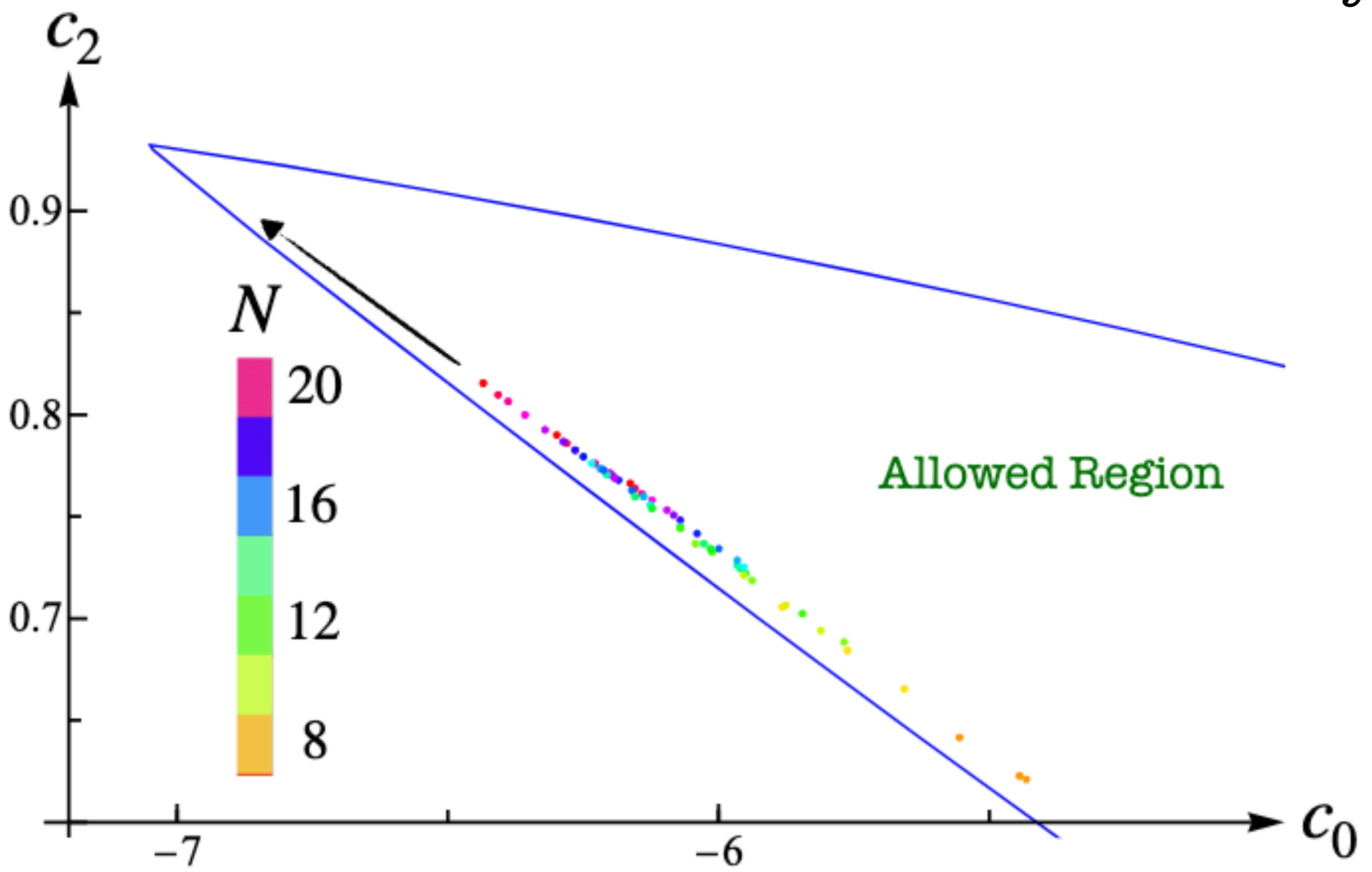
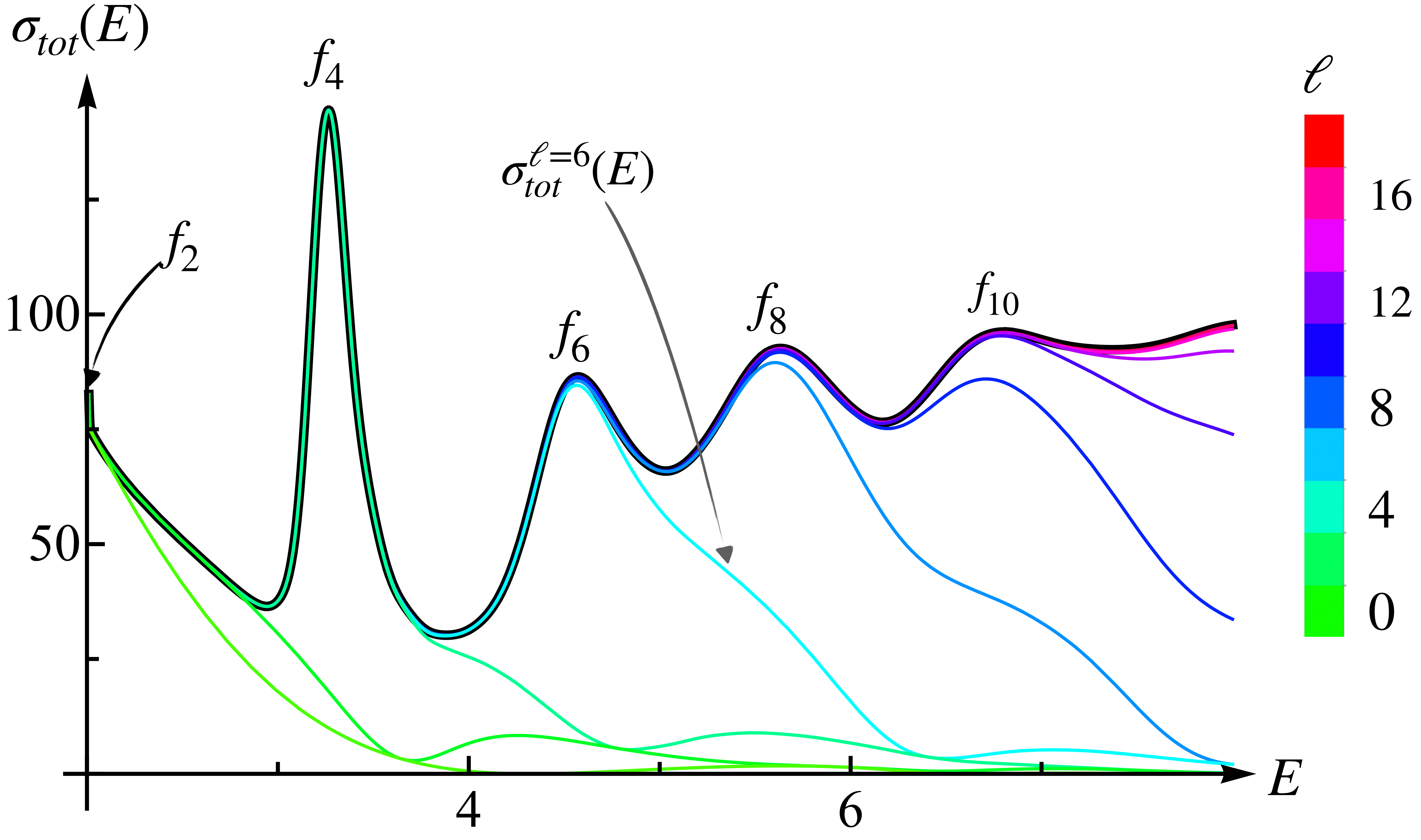


$c_{0,0} := c_0$

$c_{2,0} := c_2 \equiv c_2(4/3)$

The Froissart Amplitude II

Non-perturbative S-matrix Bootstrap: max c_2



We can extract the spectrum looking at complex zeros of $S_\ell(s) = 0$ and compare with the peaks

The Froissart Amplitude III

We can also continue in spin using Froissart-Gribov and analytically continue the spectrum

$$S_\ell(s) = 1 + \frac{8\pi i}{\sqrt{s(s-4)}} \int_4^\infty Q_\ell(1 + \tfrac{2t}{s-4}) T_t^{\text{ans}}(s,t) dt$$

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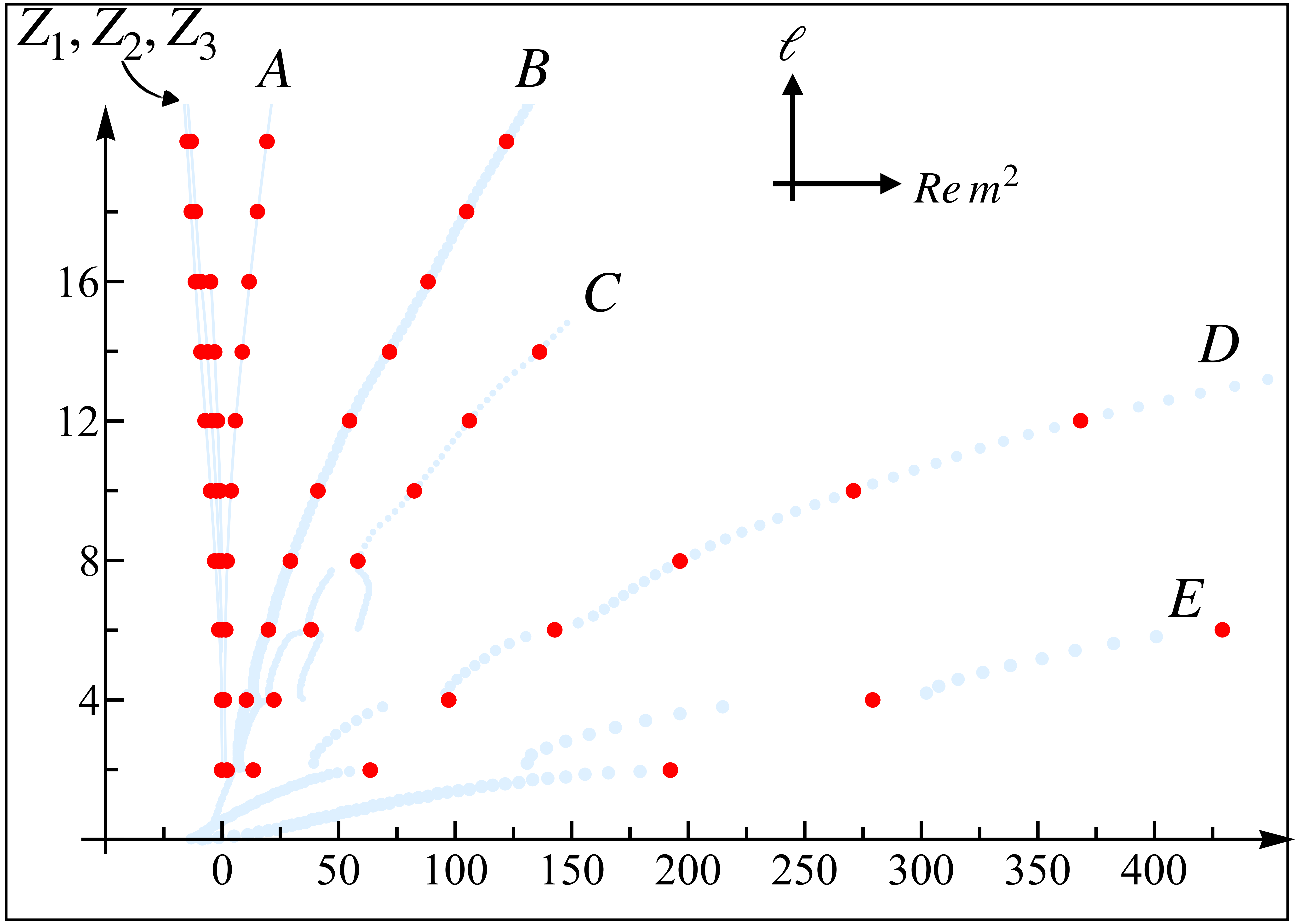

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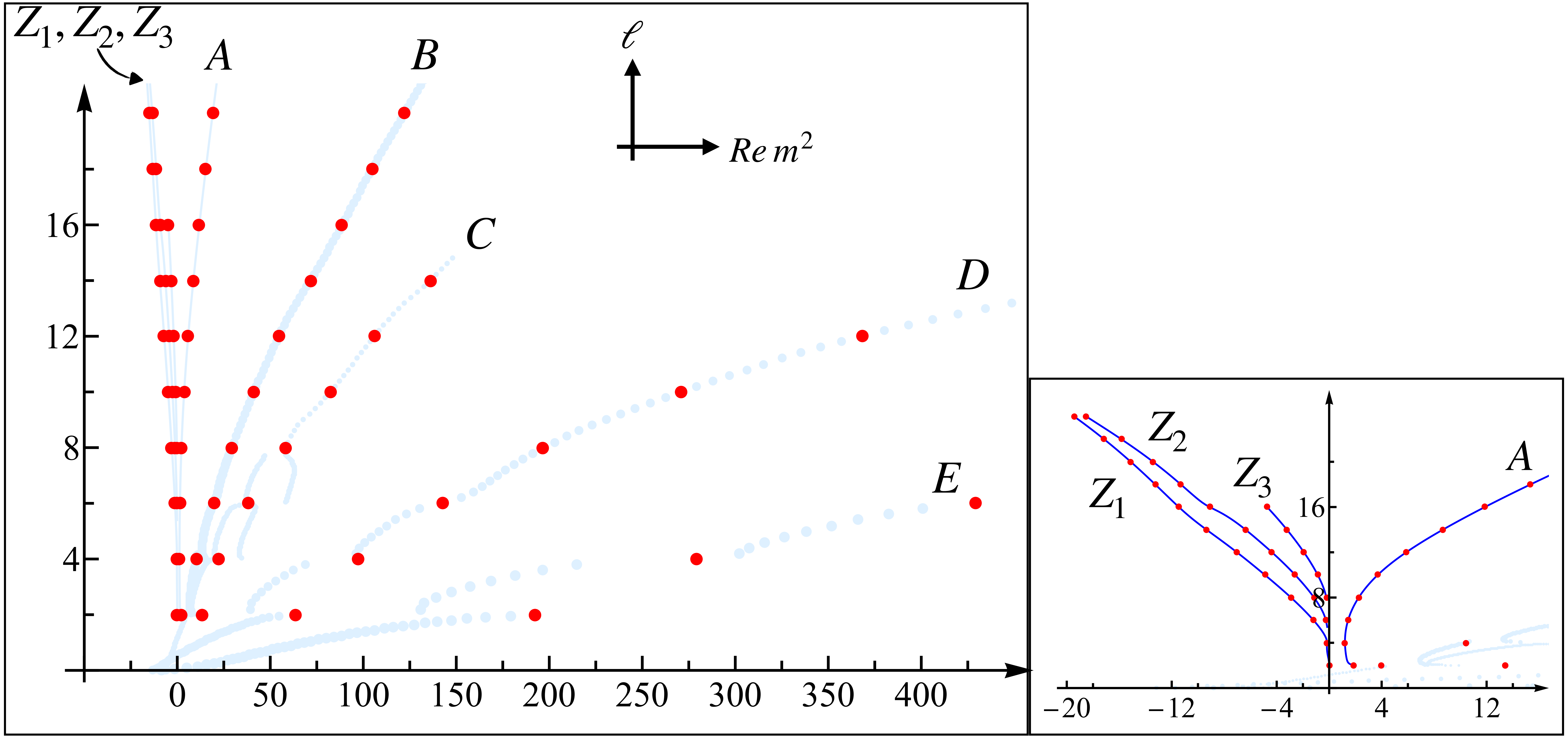

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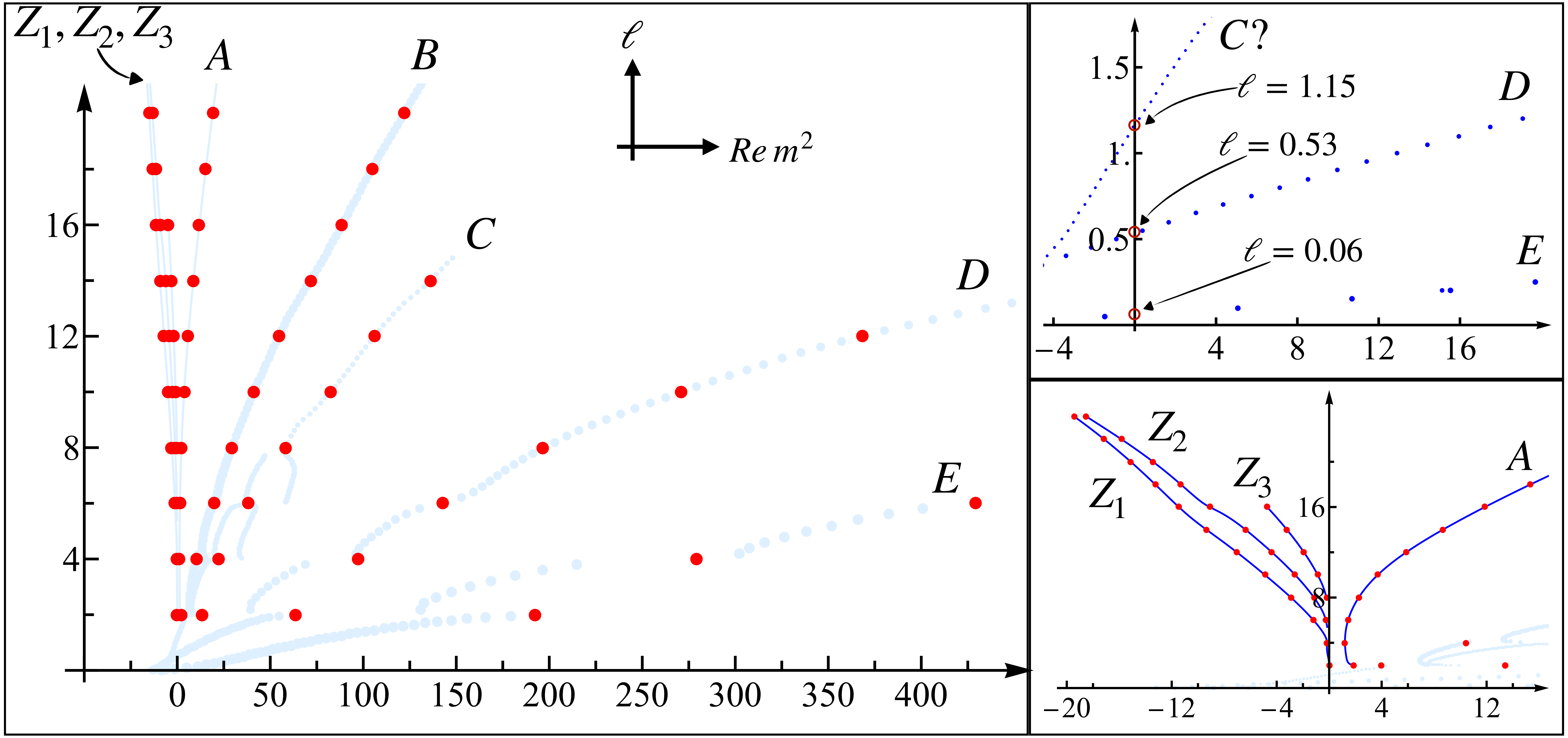

Regge theory interpretation

For $s \to \infty$, and fixed-t, Regge theory implies $T(s,t) \sim f(t) s^{\alpha(t)}$

Leading intercept should be $s^{1.15}$, but the ρ ansatz goes to a constant!

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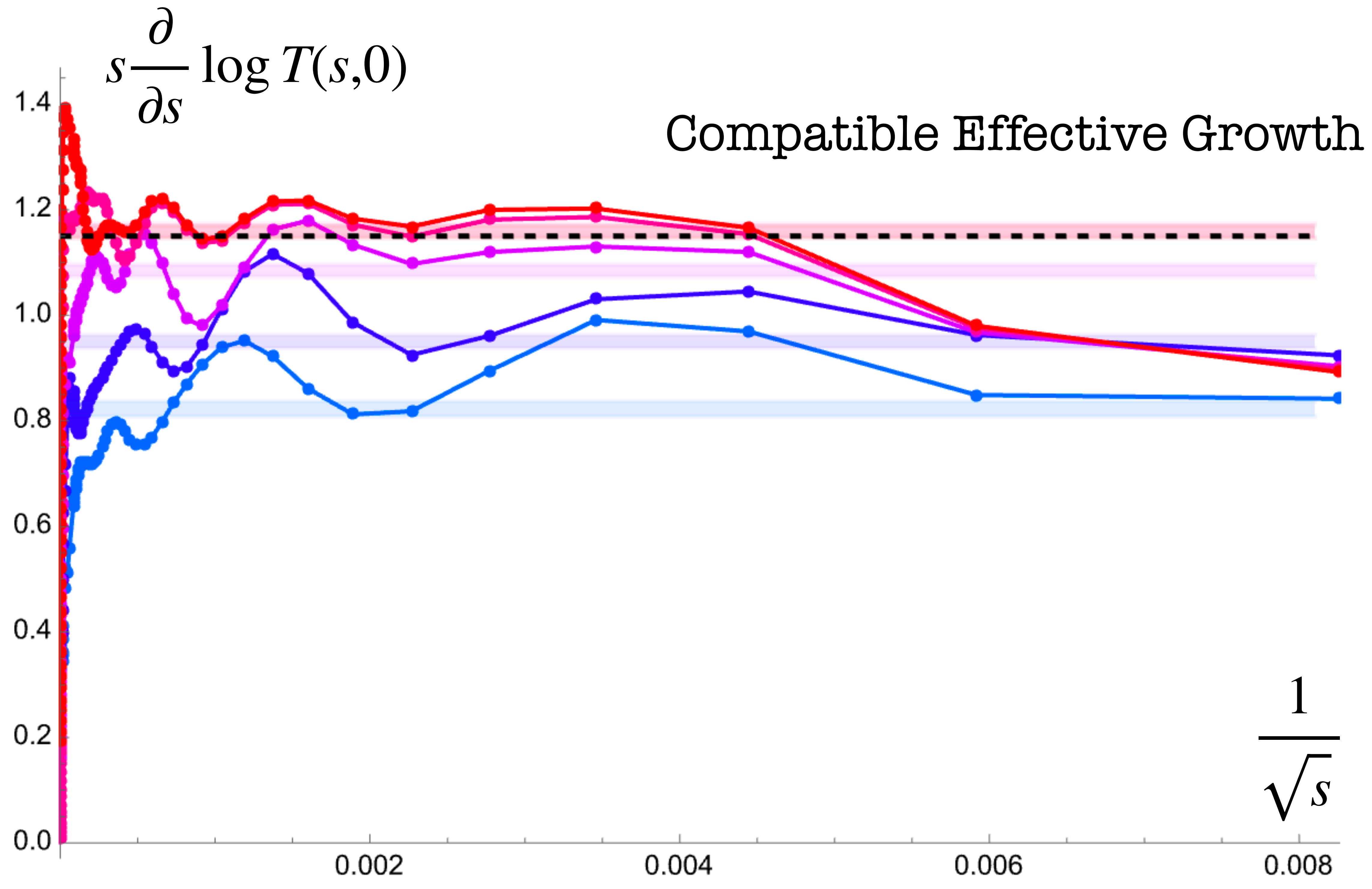

Conclusions

We cleaned the dust around the Froissart bound (1964)

We conjectured the amplitude that maximizes asymptotically the cross-section is at a cusp in the S-matrix Data

Still to do: diffractive peak analysis, impact parameter representation

For the future: the realistic case of proton-proton scattering

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"As far as I'm aware, Chew and I originally drew straight-line trajectories purely for simplicity. In the early days of Regge poles, speakers who had calculated the trajectories for some nonrelativistic potentials could always get a good laugh by contrasting our straight lines with their non-linear, crooked, cusp-ridden results." Frautschi (1985)