Localization and wall-crossing of giant graviton expansions in AdS_5

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Introduction

Holography at finite N

$N \to \infty$ Gauge theory $\leftrightarrow$ Closed string theory

Finite N effects: From $N \to \infty$ to $N < \infty$

Boundary? Bulk?

Introduction

$\mathcal{N}=4$ SYM in $4d$
with $U(N)$ gauge group

$\longleftrightarrow$

Type IIB string theory on $\mathrm{AdS}_5 \times S^5$
with N units of flux on S^5

Superconformal Index. Boundary Finite N effects: Trace relations

$$I_N(q) \;=\; \mathrm{Tr}\,(-1)^F e^{-\beta\{Q,\bar{Q}\}} q^R$$

Giant Graviton Expansion. Bulk Finite N effects

$$I_N(q) \;=\; I_\infty(q) \sum_{m=0}^{\infty} q^{mN}\, \tilde{I}_m(q)$$

[Arai, Imamura ’19, …]
[Gaiotto, Lee ’21]
[Murthy ’22]

Introduction

$\frac{1}{2}$-BPS index

$$\tilde{I}_m(q) = I_m(q^{-1})$$

$$\frac{1}{(q)_N} = \frac{1}{(q)_\infty}\sum_{m=0}^{\infty}\frac{1}{(q^{-1})_m}q^{mN} = \frac{1}{(q)_\infty}\sum_{m=0}^{\infty}(-1)^m\frac{q^{m(m+1)/2}}{(q)_m}q^{mN}$$

$$(q)_N = \prod_{n=1}^{N}(1-q^n)$$

Boundary:
Trace relations

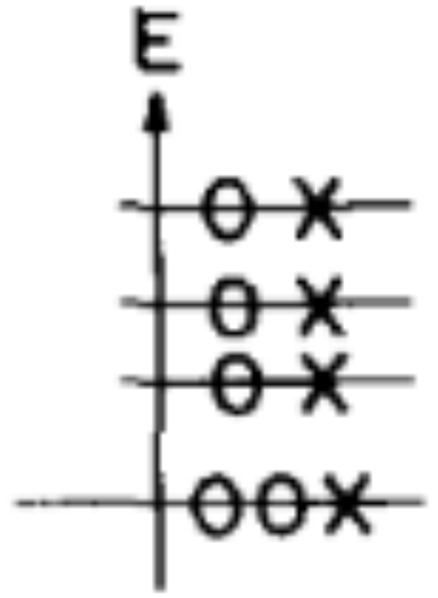
Finite N

Bulk:
Finite size effects

Gauge Theory Index

Supersymmetric index as a q series

$$I_N(q) = \mathrm{Tr}_{\mathscr{H}}(-1)^F e^{-\beta\{Q,Q^\dagger\}} q^{\mathcal{O}} = \sum_{n=0}^{\infty} d_N(n)\, q^n$$



Protected by supersymmetry: Compute it in free theory.

For $U(N)$ $\mathcal{N} = 4$ SYM

AdS/CFT $\longrightarrow$ Gravity in AdS $\longrightarrow$ Black holes $(1/16 - \mathrm{BPS})$

$$\log d_N(n) \sim S_{BH}(n)$$

[Benini, Milan ’18;
Cabo-Bizet, Cassani, Martelli, Murthy ’18;
Choi, Kim, Kim, Nahmgoong ’18; ...].

Free gauge theory combinatorics

Enumerate all gauge invariant local operators in $U(N)$ $\mathcal{N} = 4$ SYM on S^3

Take combinations of $A_\mu, \lambda^i_\alpha, X, Y, Z, \partial_\nu$

Charges $H, (R, r, r'), (j_1, j_2)$

$SU(4)_R$ $SO(4)$

For X : ordering by R-charge

$\mathrm{tr}(X)$

$\mathrm{tr}(X^2)$
$\mathrm{tr}(X)^2$

$\mathrm{tr}(X^3)$
$\mathrm{tr}(X^2)\mathrm{tr}(X)$
$\mathrm{tr}(X)^3$

$\cdots$

Free gauge theory combinatorics

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$\cdots$

$\mathrm{tr}(X^N)$
$\mathrm{tr}(X^{N-1})\mathrm{tr}(X)$
$\vdots$
$\mathrm{tr}(X)^N$

$\mathrm{tr}(X^{N+1}) = f(\mathrm{tr}(X), \ldots, \mathrm{tr}(X^N))$

Cayley-Hamilton theorem

Trace relations

1/2 BPS index

States annihilated by choice of 16 of the 32 supercharges Q_i:

Combinations of operator X, with R-charge saturating BPS bound $H = R$

$$I_N^{1/2}(q) = \mathrm{Tr}_{\mathscr{H}_{1/2-\mathrm{BPS}}}(-1)^F e^{-\gamma(H-R)} q^R$$

Partitions of n with largest size N

$$I_N^{1/2}(q) = \sum_{n=0}^{\infty} d_N(n) q^n = \frac{1}{(q)_N}, \qquad (q)_N = \prod_{n=1}^{N}(1-q^n)$$

Trace relations

Index in bulk

$$I_N(q) \;=\; \sum_{n=0}^{\infty} d_N(n)\, q^n$$

For $n \ll N$ $\quad d_N(n) \leftrightarrow$ perturbative states (gravitons)

Around $n \sim O(N)$ starts to disagree

[Kinney, Maldacena, Minwalla, Raju '05]
[Biswas, Gaiotto, Lahiri, Minwalla '06]
[Dolan '07]
[Dutta, Gopakumar '07]
[Murthy '20]

Giant Gravitons

$$\frac{L_{\text{AdS}}^4}{\ell_s^4} \sim g_s N$$

For $n \sim O(N)$: characteristic scale of $D-$branes in $\text{AdS}_5 \times S^5$, known as giant gravitons

Giant gravitons in $\text{AdS}_5 \times S^5$ are $D3-$branes wrapping $S^3 \subset S^5$ and rotating along a transverse direction

Giant - Size given by angular momentum, up to size of S^5

Gravitons - Same quantum numbers as gravitons, point particle in AdS

[McGreevy, Susskind, Toumbas ’00]

[Grisaru, Myers, Tafjord ’00]

[Das, Jevicki, Mathur ’00]

[Hashimoto, Hirano and Itzhaki ’00]

The Giant Graviton Expansion: 1/2-BPS states

$$\frac{1}{(q)_N} = \frac{1}{(q)_\infty} \sum_{m=0}^{\infty} (-1)^m \frac{q^{m(m+1)/2}}{(q)_m} q^{mN}$$

The Giant Graviton Expansion: 1/2-BPS states

Can be derived from $q-$Pochhammer identities

$$\frac{1}{(q)_N} = \frac{1}{(q)_\infty} \sum_{m=0}^{\infty} (-1)^m \frac{q^{m(m+1)/2}}{(q)_m} q^{mN}$$

Sum over $m \leftrightarrow m =$ brane number

$(-1)^m$ grading

cf. [Lee, Stanford '24]

N only enters through q^{mN}

The Giant Graviton Expansion: 1/2-BPS states

$$\frac{1}{(q)_N} = \frac{1}{(q)_\infty} \sum_{m=0}^{\infty} (-1)^m \frac{q^{m(m+1)/2}}{(q)_m} q^{mN}$$

$$\frac{1}{(q^{-1})_m} = (-1)^m \frac{q^{m(m+1)/2}}{(q)_m}$$

[Imamura + Arai, Fujiwara, Mori ’19-’21]

Analytic continuation ⟶ Wall-crossing

The bulk

Bulk computation

Sum over all possible $\frac{1}{2}-$BPS configurations of gravitons and $D-$branes in $\mathrm{EAdS}_5 \times S^5$ with periodic b.c.

$$I_N^{\text{bulk}}(q) = \sum_{m=0}^{\infty} \int_{\mathscr{M}(m)} d\mu \int d\phi \, \exp\Big(S_{\text{brane+sugra}}(\phi; \mu, m)\Big) \, q^R$$

$$I_N^{1/2}(q) = \mathrm{Tr}_{\mathscr{H}_{1/2-\mathrm{BPS}}} (-1)^F e^{-\gamma(H-R)} q^R$$

Localization

Supersymmetric theory: $QS[\phi] = 0$

Deform partition function with $QV[\phi]$ w/ $Q^2V[\phi] = 0$

$$Z[t] = \int D\phi e^{-S[\phi] - tQV[\phi]}$$

$$\frac{\partial Z[t]}{\partial t} = 0$$

$$S[\phi] + tQV[\phi] =$$
$$S[\phi_0] + (QV)^{(2)}[\hat{\phi}] + \mathcal{O}(t^{-1/2})$$

Gets localized to fixed points of $QV[\phi]$

$$Z[t] = \int_{\mathcal{M}_{\text{BPS}}} D\phi_0 \, e^{-S[\phi_0]} \frac{1}{\text{Sdet}(QV)^{(2)}_{\phi_0}}$$

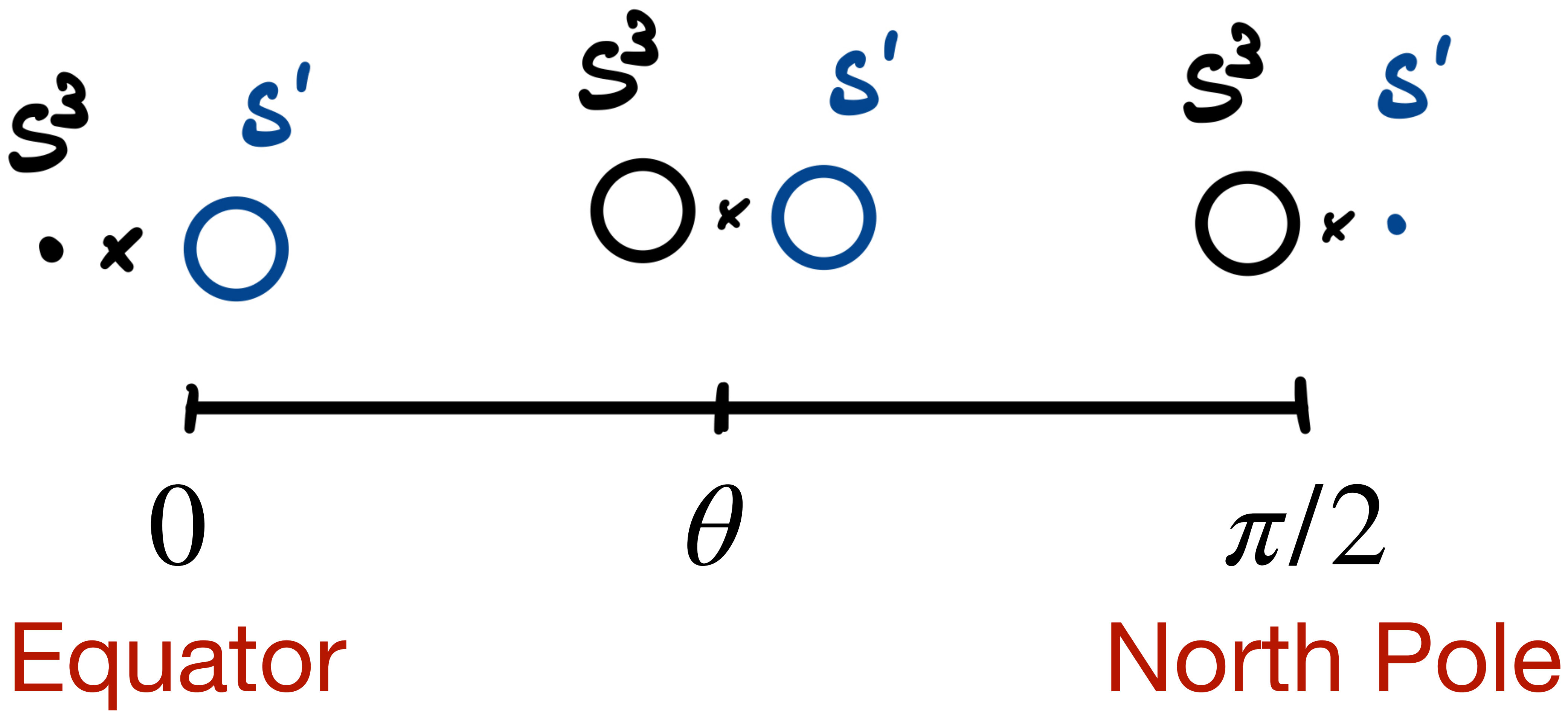
Giant graviton: The Bulk Hulk

Giant gravitons: $D3-$brane wrapping $S^3 \subset S^5$, rotate along ϕ circle with $\dot{\phi}(t) = const$.

$$ds^2_{S^5} = L^2\left(d\theta^2 + \cos^2\theta\, d\phi^2 + \sin^2\theta\, d\Omega_3^2\right),$$

size of giant D3-brane is

parametrized by $\theta \in [0, \frac{\pi}{2}]$

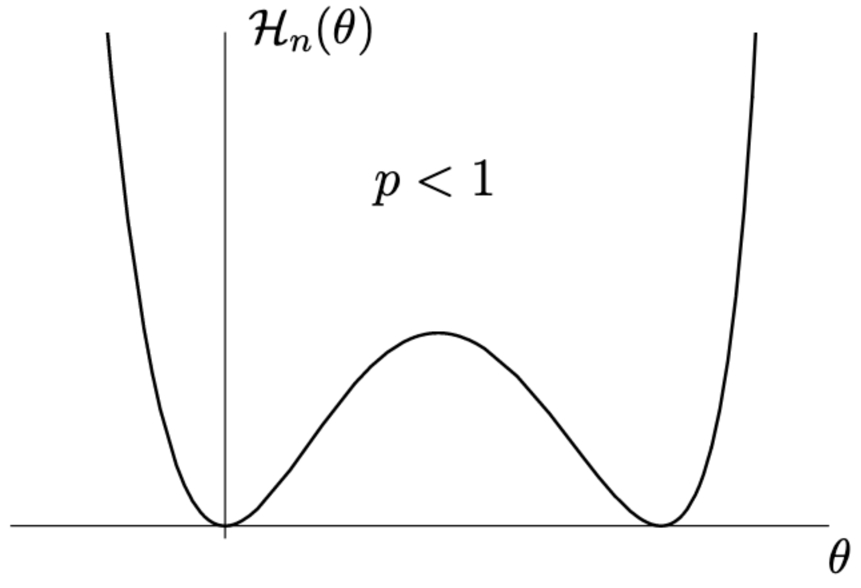


Giant gravitons

$$S_3 = - T_3 \int d^4\sigma \sqrt{-g} + T_3 \int P[A^{(4)}]$$

DBI + CS

Solution $\quad \dot{\phi} = \frac{1}{L}, \quad \theta = \theta_0, \quad \theta_0 \in [0, \frac{\pi}{2}]$



stable

Energy H - angular momentum P_ϕ relation $\quad H = \frac{P_\phi}{L}$

BPS bound

Angular momentum - size relation

$$P_\phi = N \sin^2\theta$$

$P_\phi \leq N$ bound

stringy exclusion principle

Giant graviton fluctuations

Fluctuations of a generic giant graviton are gapped [Das, Jevicki, Mathur '00]

For maximal giants $\theta_0 = \frac{\pi}{2}$, θ becomes a radial variable, not gapped

Expand around $\rho = \frac{\pi}{2} - \theta, \qquad \dot{\varphi} = \frac{1}{L} - \dot{\phi}$

$$\mathcal{L}^{(2)}_{\max} = \frac{N}{L}\left(\frac{1}{2}L^2\rho^2\dot{\varphi}^2 + \frac{1}{2}L^2\dot{\rho}^2 + L\rho^2\dot{\varphi} - L\,\dot{\varphi}\right)$$

Landau problem

Cartesian coordinates $x_1 = L\rho\cos\varphi,\ x_2 = L\rho\sin\varphi$

$$\mathscr{L}^{(2)}_{\max} = \frac{1}{2}\frac{N}{L}\left(\dot{x}_1^2 + \dot{x}_2^2\right) + \dot{\vec{x}} \cdot \vec{A}$$

Particle in 2d in a constant transverse magnetic field $B = \dfrac{2}{L}$ + solenoid flux at the origin

$$\vec{A} = \vec{A}_1 + \vec{A}_2, \quad \vec{A}_1 = \frac{N}{L^2}\left(x_1\hat{x}_2 - x_2\hat{x}_1\right), \quad \vec{A}_2 = -\frac{N}{x_1^2 + x_2^2}\left(x_1\hat{x}_2 - x_2\hat{x}_1\right)$$

Landau problem: solution

Ignoring solenoid term

$$H_{\text{Lan}} = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\left(\frac{B}{2}\right)^2\left(x_1^2 + x_2^2\right) - \frac{B}{2}\left(x_1 p_2 - x_2 p_1\right)$$

Canonically quantized

$$H_{\text{Lan}} = B\left(a^\dagger a + \frac{1}{2}\right), \qquad E_n = B\left(n + \frac{1}{2}\right)$$

1d Harmonic oscillator - Each energy level is infinitely degenerate (Landau level)

Lowest Landau Level

$$H_{\mathrm{Lan}} = B\left(a^{\dagger}a + \frac{1}{2}\right), \qquad E_n = B\left(n+\tfrac{1}{2}\right)$$

Using rotational symmetry of problem, label states by angular momentum

$$\hat{L} = x_1 p_2 - x_2 p_1$$

$$H_{\mathrm{Lan}}\,|n,\ell\rangle = B\left(n + \frac{1}{2}\right)|n,\ell\rangle, \qquad \hat{L}\,|n,\ell\rangle = (-n+\ell)\,|n,\ell\rangle$$

Groundstates $|0,\ell\rangle, \ell = 0,1,2,\ldots,$ form LLL

Supersymmetric Landau Problem

Adding Majorana fermions λ_1, λ_2

$$H_{\text{SLan}} = H_{\text{Lan}} + \frac{i}{2} B \left[\lambda_1, \lambda_2\right]$$

$$[H_{\text{SLan}}, \hat{L}] = 0$$

$$[Q, \hat{L}] = 0$$

there is Q s.t. $Q^2 = H_{\text{SLan}}\,,\;\; [H_{\text{SLan}}, Q] = 0$

$$H_{\text{SLan}} \,|\, n, \ell, s\rangle = \left(|B|\left(n + \tfrac{1}{2}\right) + B\left(n_F - \tfrac{1}{2}\right)\right)|\, n, \ell, n_F\rangle$$

$$\hat{L}\,|\, n, \ell, s\rangle = \left(\text{sign}(B)\left(-n + \ell\right) - n_F\right)|\, n, \ell, n_F\rangle$$

Supersymmetric index

Since $[Q, \widehat{L}] = 0$, substituting $R = -i\partial_\phi = N - \widehat{L}$

$$\mathrm{Tr}_{\mathscr{H}_{\mathrm{SLan}}}(-1)^F e^{-\gamma H_{\mathrm{SLan}}} q^R = \begin{cases} q^N \sum_{\ell=0}^{\infty} q^{-\ell} = \dfrac{q^N}{1-q^{-1}}, & B > 0 \\ q^N \sum_{\ell=0}^{\infty} q^{\ell+1} = -\dfrac{q^{N+1}}{1-q}, & B < 0 \end{cases}$$

Wall-crossing $Q\,\psi(z,\bar{z}) = 0 \implies \psi(z,\bar{z}) = \begin{pmatrix} \psi_1(z,\bar{z}) \\ \psi_2(z,\bar{z}) \end{pmatrix} = \begin{pmatrix} f(z)\, e^{-\frac{B}{4}z\bar{z}} \\ g(\bar{z})\, e^{\frac{B}{4}z\bar{z}} \end{pmatrix}$

Lagrangian index

Determinant over $\frac{1}{2}$ −BPS fluctuations divergent [Gautason, van Muiden '24]

(c.f. SLandau partition function $Z = \mathcal{N}\, V_2\, B\gamma \prod_{n=1}^{\infty} \frac{1}{4\pi^2 n^2}$)

Refine index by q^R: deform susy algebra s.t. Q−cohomology $\delta_\alpha^2 = -\,\partial_{t_E} - \alpha L$

Defining $V_1 = -\frac{1}{2}\int_0^\gamma dt_E \left(\lambda_1\, \delta_\alpha \lambda_1 + \lambda_2\, \delta_\alpha \lambda_2\right)$, $\qquad V_2 = i\int_0^\gamma dt_E \left(x_2\, \delta_\alpha x_1 - x_1\, \delta_\alpha x_2\right)$

$$\mathcal{S}^E_{\text{SLan}}(\alpha) = \int_0^\gamma dt_E\, \mathcal{L}^E_{\text{SLan}}(\alpha) = \delta_\alpha\left(V_1 + \frac{B}{2} V_2\right), \quad H_{\text{SLan}}(\alpha) = H_{\text{SLan}}(0) - i\alpha\left(\hat{L} + \frac{1}{2}\right)$$

$(\text{Im}(\alpha) \leftrightarrow B)$

Localization for giants

Deform action by

$$\delta_\alpha\left(V_1 + b\,V_2\right)\Big|_{\alpha=a+ib}^{\text{bos.}} = -\int_0^\gamma dt_E \left(\frac{1}{2}\left(\dot{x}_1 - a x_2\right)^2 + \frac{1}{2}\left(\dot{x}_2 + a x_1\right)^2 + \frac{b^2}{2}\left(x_1^2 + x_2^2\right)\right)$$

Negative definite

Critical points of $\delta_\alpha\left(V_1 + b\,V_2\right)|_{\alpha=a+ib}$ are the maximal giants $x_1 = x_2 = 0$

Answer factorized into fluctuations of background $I_{\text{sugra}}(q) = I_\infty(q)$ times fluctuations of brane

$$I_N^{\text{bulk}}(q) \;=\; I_{\text{sugra}}(q) \sum_{m=0}^{\infty} \int d\phi \,\exp\left(S_{\text{brane}}(\phi; m)\right) q^R$$

Wall-crossing for giants

Left with computing determinant of quadratic fluctuations of maximal giant

Deformed superdeterminant gets simplified to

$$\mathrm{SDet}(\delta_\alpha V)^{-1} = \mathcal{N}\frac{1}{\alpha^2\gamma^2}(i\alpha\gamma)\prod_{n=1}^{\infty}\frac{1}{4\pi^2n^2}\frac{1}{\left(1-\alpha^2\gamma^2/4\pi^2n^2\right)} = \mathcal{N}\frac{1}{\alpha\gamma}\frac{i\alpha\gamma/2}{\sin(\alpha\gamma/2)}$$

with $q = e^{-i\gamma\alpha}$

c.f. [Witten '83]
c.f. [Lee, Stanford '24]

$\mathrm{Im}(\alpha)$

$$\mathrm{SDet}(\delta_\alpha V)^{-1} = \frac{1}{1-q^{-1}} \overset{|q|>1}{=} 1 + q^{-1} + q^{-2} + \ldots = -\frac{q}{1-q} \overset{|q|<1}{=} -q - q^2 - q^3 - \ldots$$

$$\mathrm{SDet}(\delta_\alpha V)^{-1}_{+\,\mathrm{flux}} = \frac{q^N}{1-q^{-1}} = -\frac{q^{N+1}}{1-q}$$

Multiple giants

Spectrum of m coincident maximal giant gravitons

One giant in complex coordinates $\mathscr{L}_{\text{Lan}} = \dot{z}\dot{\bar{z}} - i\frac{B}{2}\left(\dot{z}\bar{z} - \dot{\bar{z}}z\right)$

Matrix version

$$\mathscr{L}_m = \text{Tr}\left(\dot{Z}\dot{Z}^\dagger - i\frac{B}{2}(\dot{Z}Z^\dagger - \dot{Z}^\dagger Z)\right)$$

with $U(m)$ gauge symmetry

Multiple giants

$$\mathscr{L}_m = \mathrm{Tr}\left(\dot{Z}\dot{Z}^\dagger - i\frac{B}{2}(\dot{Z}Z^\dagger - \dot{Z}^\dagger Z)\right)$$

Complex matrix $2m^2$ d.o.f.

$$\Pi_{ij} = -i\left(\frac{\partial}{\partial \bar{Z}_{ij}} + \frac{B}{2}Z_{ij}\right)$$

Hamiltonian $H^m_{\mathrm{Lan}} = \mathrm{Tr}\,\Pi^\dagger\,\Pi + \frac{1}{2}m^2 B,\ \ [\Pi_{ij}, \overline{\Pi}_{ij}] = B$

Reduced to m^2 H.O. + $U(m)$ gauge symmetry → ground states are holomorphic/antiholomorphic symmetric functions of m eigenvalues

Groundstate degeneracies

Different side of wall $B > 0$, $B < 0$ (bosonic/fermionic)

$$\Psi_{\text{LLL}}(Z, Z^\dagger) = \begin{cases} f(z)\, e^{-\frac{B}{2}\text{Tr}\,(ZZ^\dagger)}\,, & B > 0 \\ \bar{f}(\bar{z})\,\Delta(\bar{z})\, e^{\frac{B}{2}\text{Tr}\,(ZZ^\dagger)}\,, & B < 0 \end{cases}$$

yields,

$$\text{Tr}\,(-1)^F q^R \Big|_{m\,\text{giants}} = \begin{cases} q^{mN} \displaystyle\prod_{r=1}^{m} \frac{1}{1 - q^{-r}}\,, & B > 0 \\ (-1)^m q^{mN} q^{m(m+1)/2} \displaystyle\prod_{r=1}^{m} \frac{1}{1 - q^{r}}\,, & B < 0 \end{cases}$$

Final answer

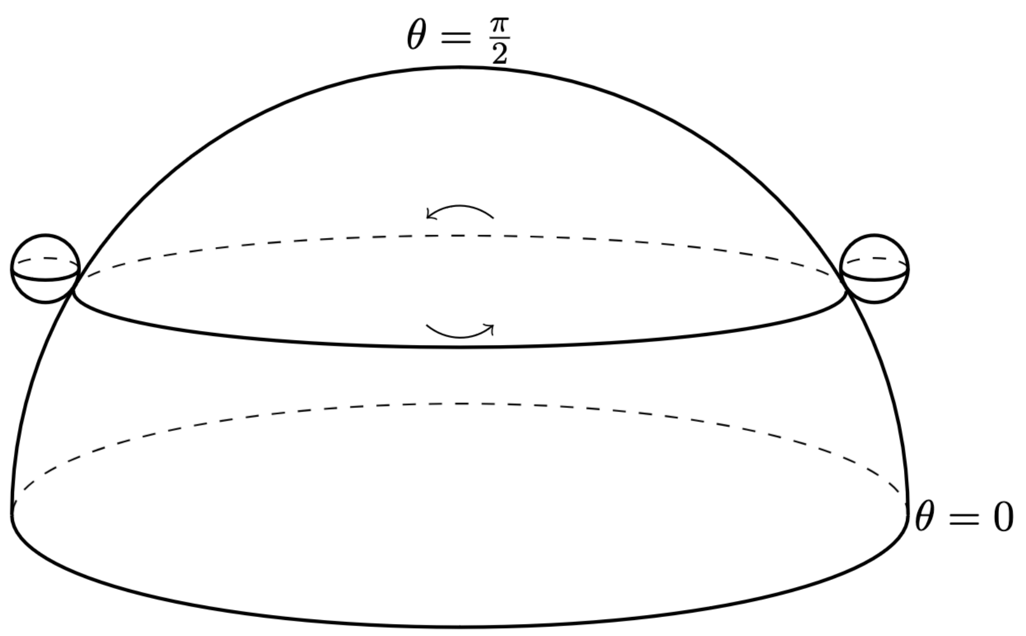
$$I_N^{\text{bulk}}(q) = \sum_{m=0}^{\infty} \int_{\mathscr{M}(m)} d\mu \int d\phi \, \exp\big(S_{\text{brane+sugra}}(\phi; \mu, m)\big) \, q^R$$

after localization and summing all contributions (with flux) becomes

$$I_N^{\text{bulk}}(q) = I_\infty(q) \sum_{m=0}^{\infty} (-1)^m \frac{q^{m(m+1)/2}}{(q)_m} q^{mN} \qquad |q| < 1$$

$SO(N)$ and $Sp(N)$

Boundary Orientifold 3-plane $O3$ parallel to N $D3$-branes,



[Witten '98]

Orientifold projection $U(N) \to SO(N)$ or $Sp(N/2)$

Bulk dual $AdS_5 \times \mathbb{RP}^5$

Compute index for BPS brane configurations invariant under projection and project to invariant states

$$I_{Sp(k)}(q) = I_{SO(2k+1)}(q) = \frac{1}{(q^2)_\infty} \sum_{m=0}^{\infty} (-1)^m \frac{q^{2\binom{m+1}{2}}}{(q^2)_m} q^{2mk}$$

cf. [Fujiwara, Imamura, Mori, Murayama, Yokoyama '23]

$SO(2k)$ subtlety from invariant single $D3$ brane related to Pfaffian operator which works nicely

Summary and Outlook

Bulk interpretation of the Giant Graviton Expansion for the $1/2-$BPS index as a result of the localization of the path integral in $\mathrm{AdS}_5 \times S^5$

Localization explains factorization

$\delta_\alpha V$ deformation explains analytic continuation as wall-crossing

Can be generalized to orthogonal and symplectic gauge theories

Other indices? $1/16-$BPS index? BHs from branes? Closed form expression?

Instance of Open-Closed-Open triality $(m \leftrightarrow N)$ [Gopakumar '10]

ありがとうございます!