Pinning Defects, Fusion and Factorization

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Based on 2404.05815, 2406.01550 w/ O. Diatlyk, H. Khanchandani, F. Popov
2411.16522 w/ O. Diatlyk, Z. Sun and work-to-appear w/ F. Popov

Plan of the Talk

Motivation and Background on Defects

Defect Fusion and Universal Results

Generalized Pinning Field Defects

Examples in d=2 and d=3

Factorization by RG

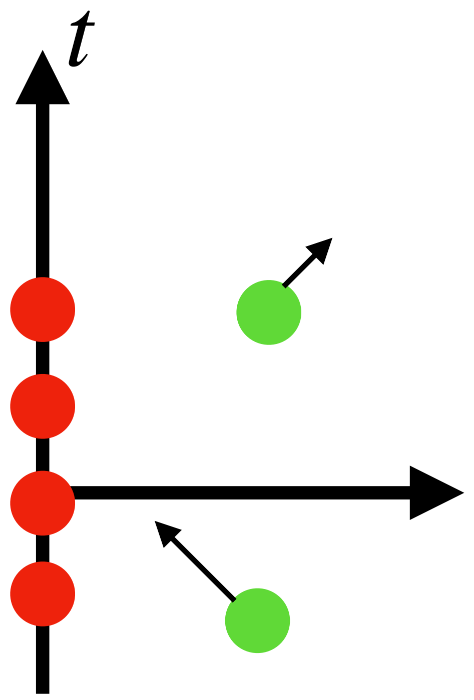
Speculations

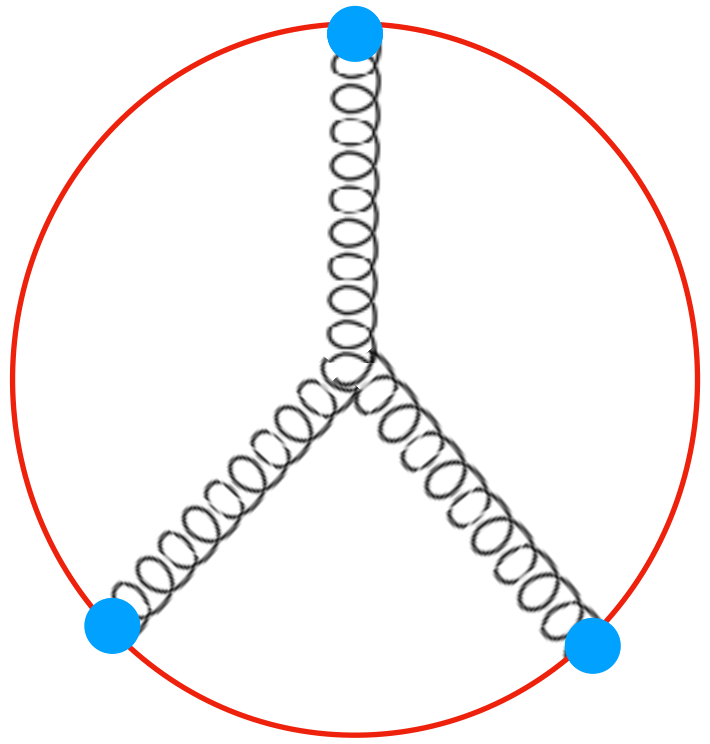
Defects are Ubiquitous

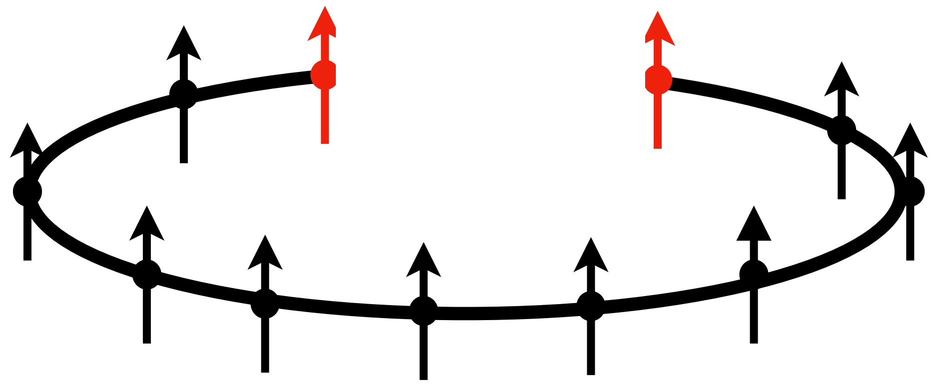
- Kondo effect: magnetic impurities in metal
- Worldlines of heavy charged particles
- Lattice systems (spin chain) with boundaries
- D-Branes in worldsheet string theory

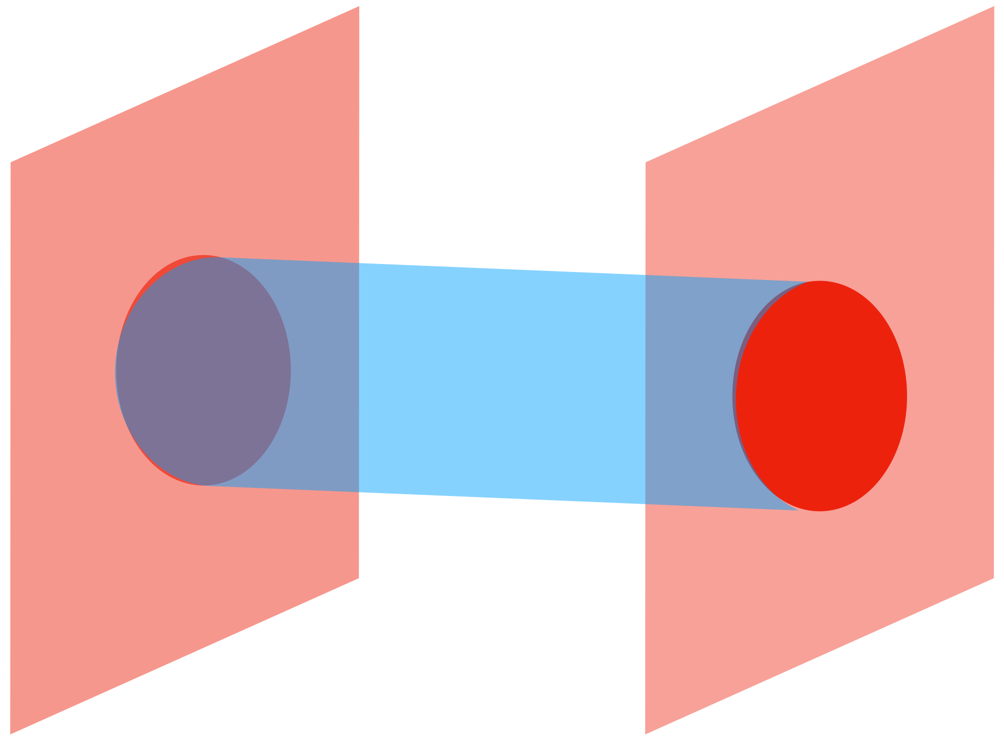
Descriptions in QFT →

- Defect lines in SU(2) WZW [Affleck-Ludwig,…]
- Wilson/’t Hooft loops [Wilson, t’ Hooft, Kapustin,…]
- Conformal boundaries [Cardy, Diehl,…]
- 2d Cardy states [Cardy,…]









Monodromy (twist) defects ⟶ **Renyi and Entanglement Entropy**

[Calabrese Cardy 04, Hung Myers Smolkin 14,…]

Defects with isotopy invariance ⟶ **Topological Defects and Generalized Symmetries**

[Gaiotto Kapustin Seiberg Willet 14, …]

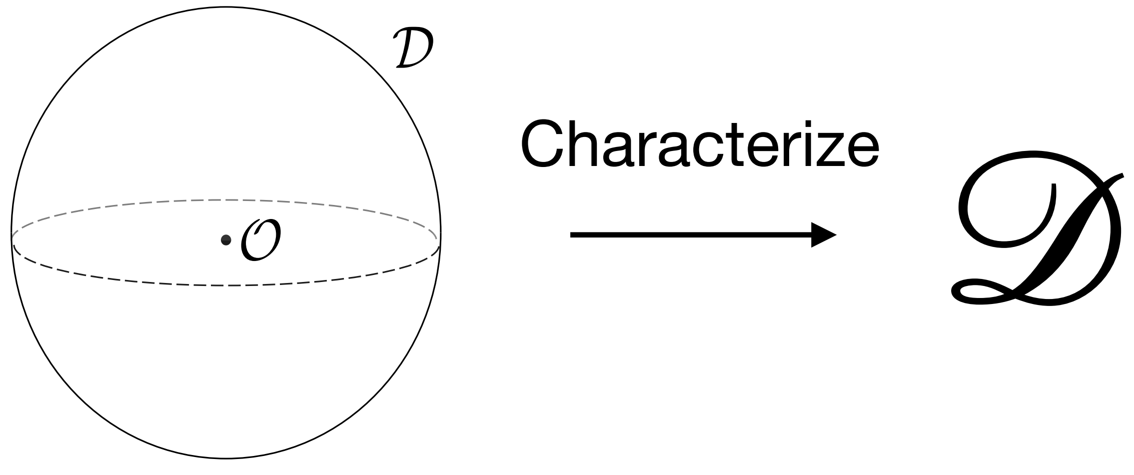
Conformal Defects (DCFT)

- **Critical phase** in the presence of boundaries/defects
- **Universality classes** of defect RG flows (monotonicity theorems)
- **No local p-dimensional** stress tensor or currents (generically)
- **New critical exponents and OPE data**
  (e.g. defect local ops $\mathscr{S}$ and bulk local op. 1pf $\langle \mathscr{O} \rangle_{\mathscr{D}}$)
- Constrained by **defect bootstrap** equations
  (e.g. residual conf symmetry, crossing and unitarity)

Residual conformal symmetry of p dimensional conformal defect

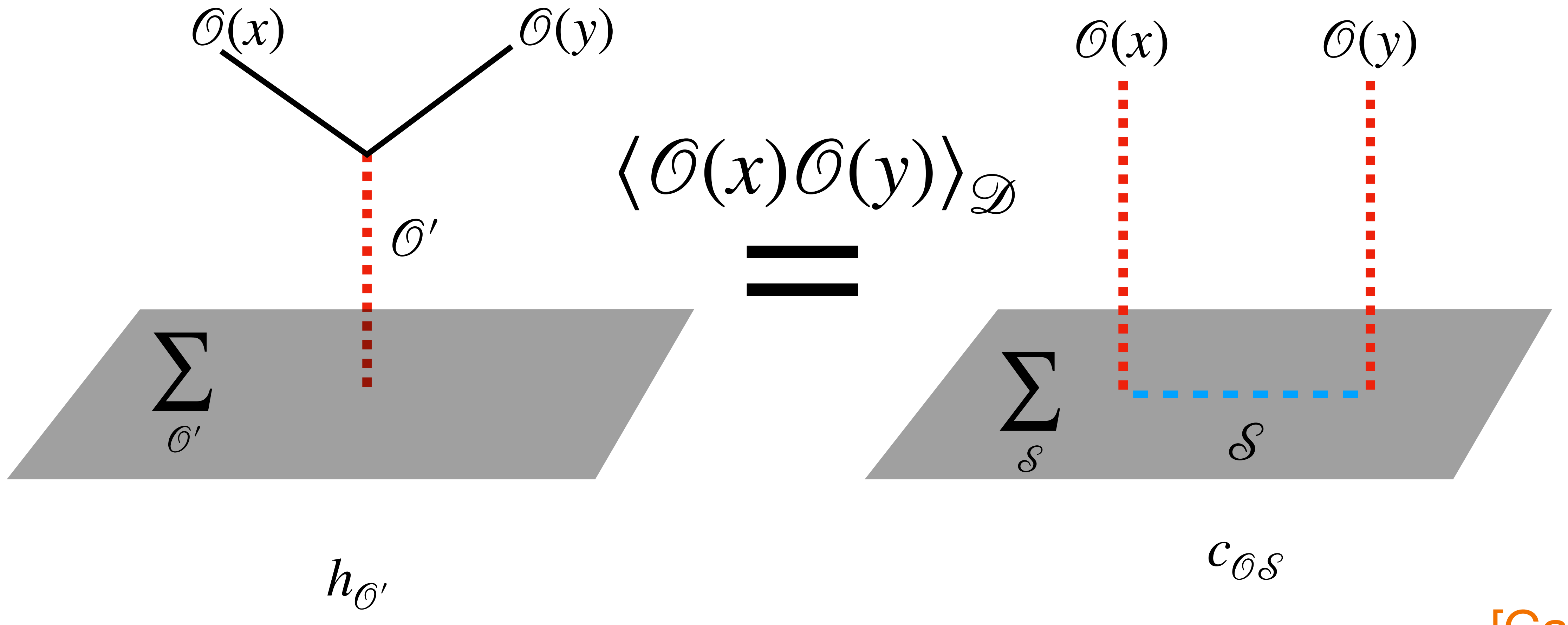
$$SO(p,2) \times SO(d-p) \subset SO(d,2)$$

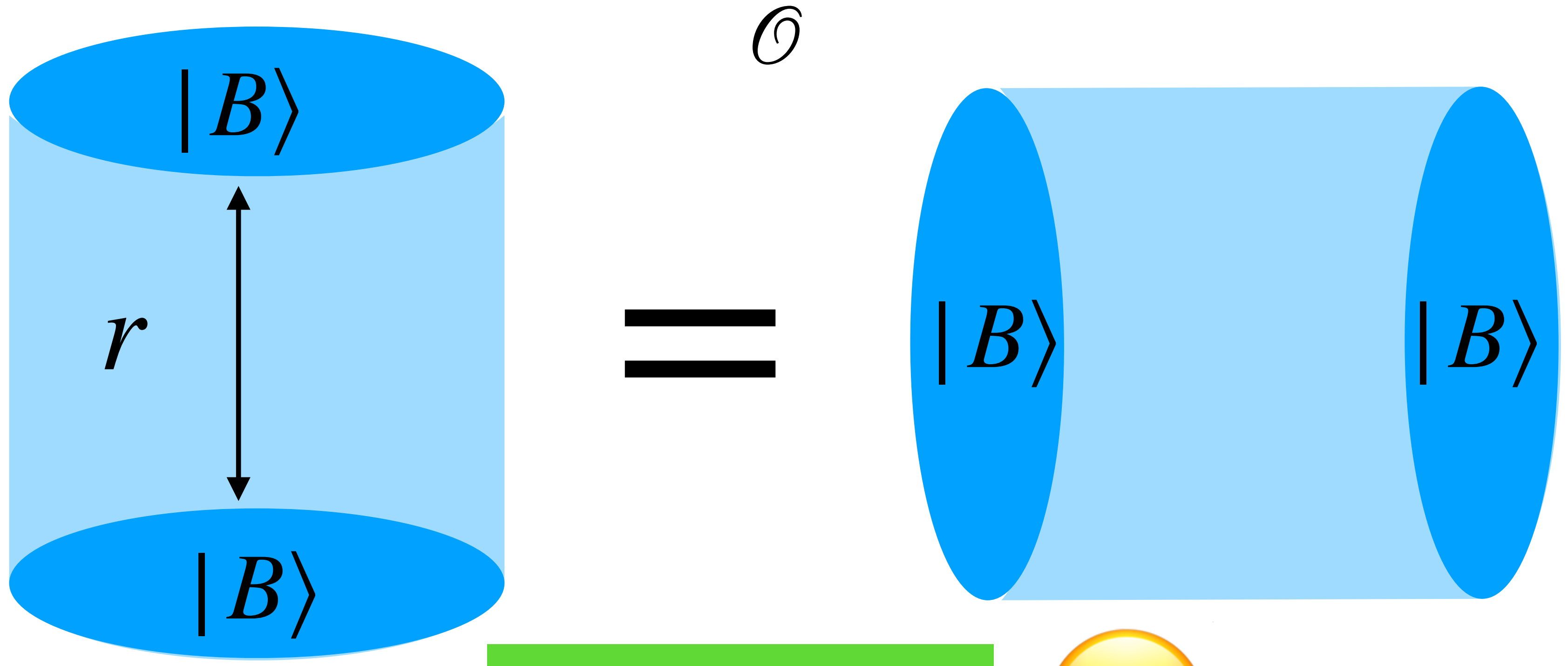
topological defect



All bulk op $\mathscr{O}$

$$|B\rangle = \sum_{\mathscr{O}} \langle \mathscr{O} \rangle_B |\mathscr{O}\rangle\rangle$$





Positive! 🙂

[Cardy, Cardy Lewellen, Liendo Rastelli van Rees, Gaiotto Mazac Paulos, Liendo Meneghelli, Billo Goncalves Lauria Meineri,…]

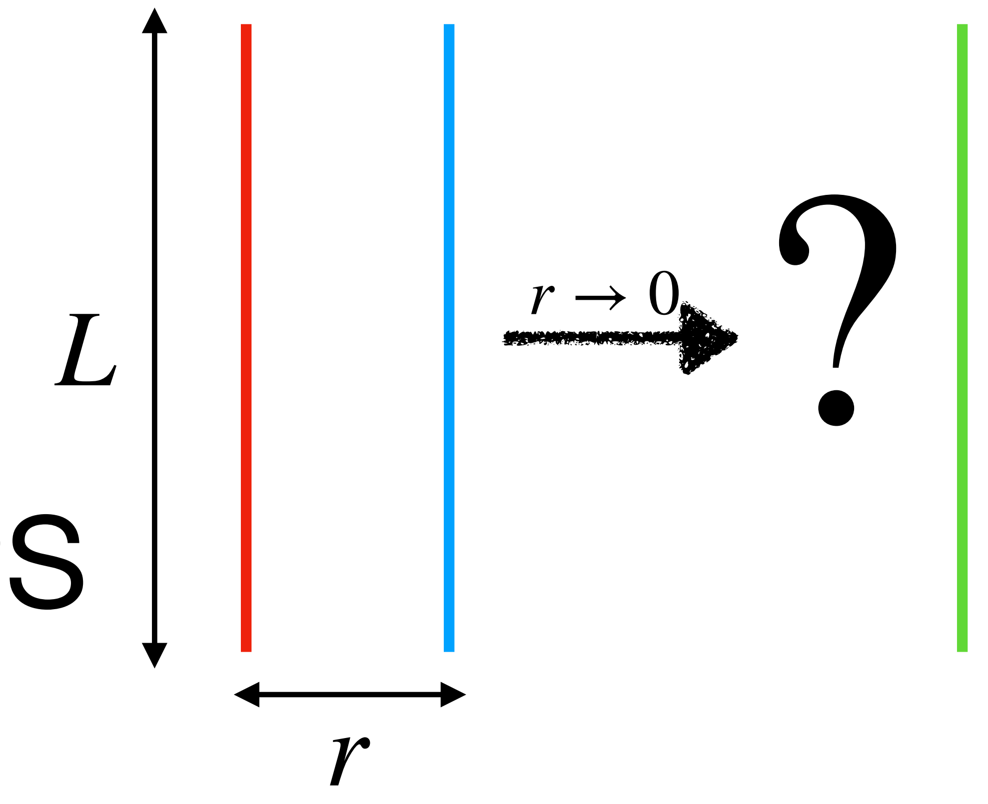
Defect Fusion

What is it and What is it good for?

Defect Fusion (topological and non-topological)

- Defect Operator Product Expansion: **collective effect** of multiple defects at long distance/combined operation of defect operators on Hilbert space

- Basic binary operation: **Defect Fusion** $D_1 \circ D_2 = \oplus_i D_i$



- Well-understood for **topological** defects/interfaces (also mutually BPS defects).

- Encode important generalized symmetry structures, generalized anomalies, generalized gaugings

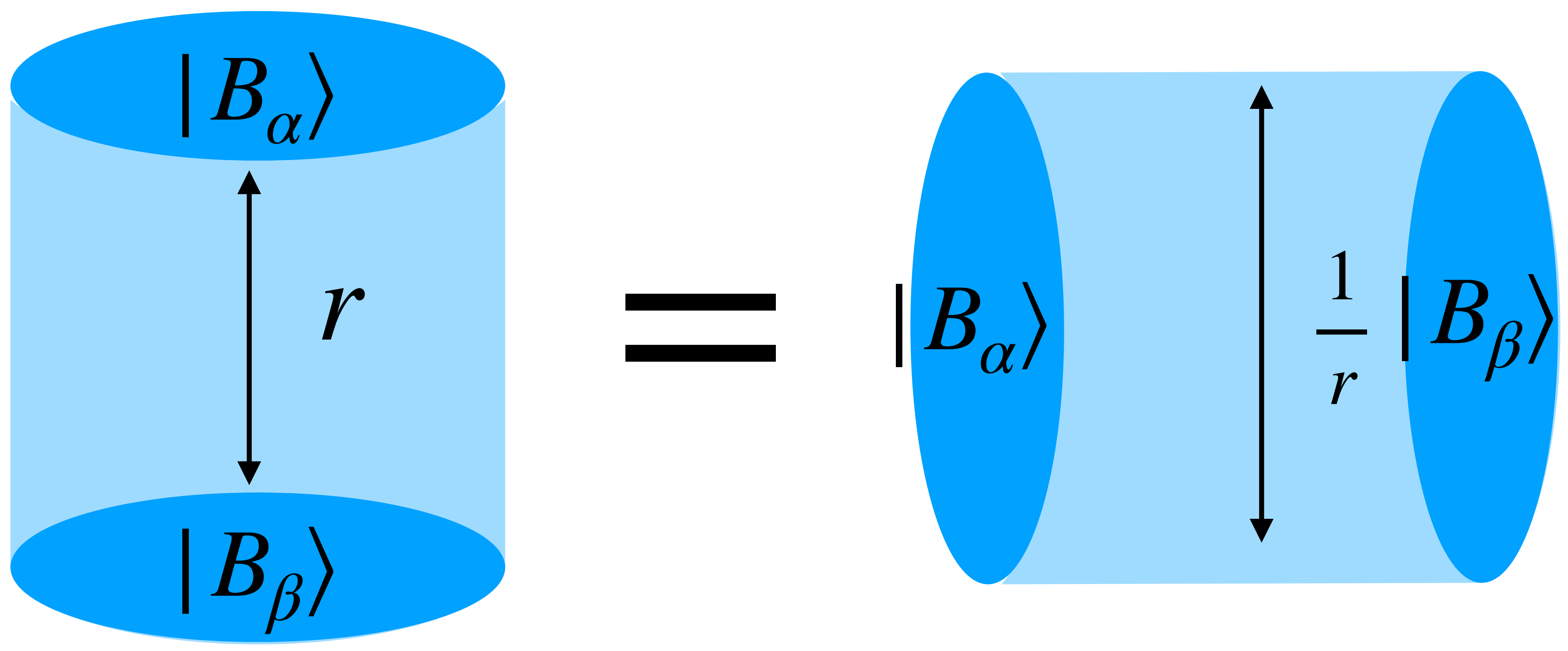
- Fusion of more general **non-topological** conformal defects?
    - Divergence from Casimir energy (e.g. $e^{-\frac{L\mathscr{E}}{r}}$ for line defects where $\mathscr{E}$ can be negative)
    - Nontrivial defect RG flow (integrated insertion of local operators on the defect)
    - Different properties of the fusion product compared to the topological case

[Diatlyk Khanchandani Popov YW 2024]

Cylinder/Annulus Bootstrap Equation from Fusion

- 2d Cardy Condition: powerful constraint on boundary states $|B_\alpha\rangle$
- Generalization to higher dimension?
- Universal (high energy) DCFT data from Defect EFT from integrating modes of mass $1/r$ in the fusion limit

Close-Open duality



“Close”-DEFT duality

Cardy-like formula for one-point functions in DCFT in general dimensions

Large Δ

$$\overline{\langle \mathcal{O}_\Delta \rangle_B^2} \sim \frac{((1-d)\mathcal{E}S_{d-1})^{\frac{d+1}{2d}}}{\Delta^{\frac{2d+1}{2d}}} e^{d\left(\frac{\Delta}{d-1}\right)^{\frac{d-1}{d}}(-\mathcal{E}S_{d-1})^{\frac{1}{d}}}$$

[Diatlyk Khanchandani Popov YW 24,
Kravchuk Radcliffe Sinha 24,
Cuomo He Komargodski 24]

Casimir energy in fusion limit

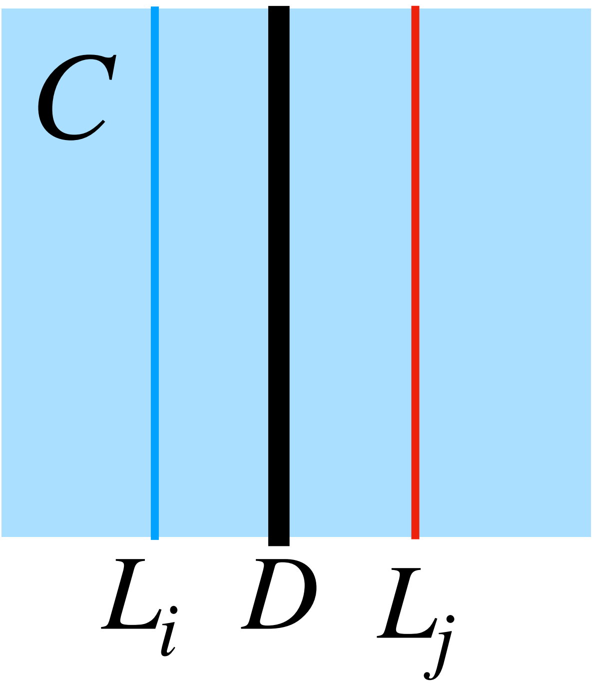
$$\lim_{r\to 0}\langle B|e^{-rH}|B\rangle \sim e^{-\frac{V}{r^{d-1}}\mathcal{E}}$$

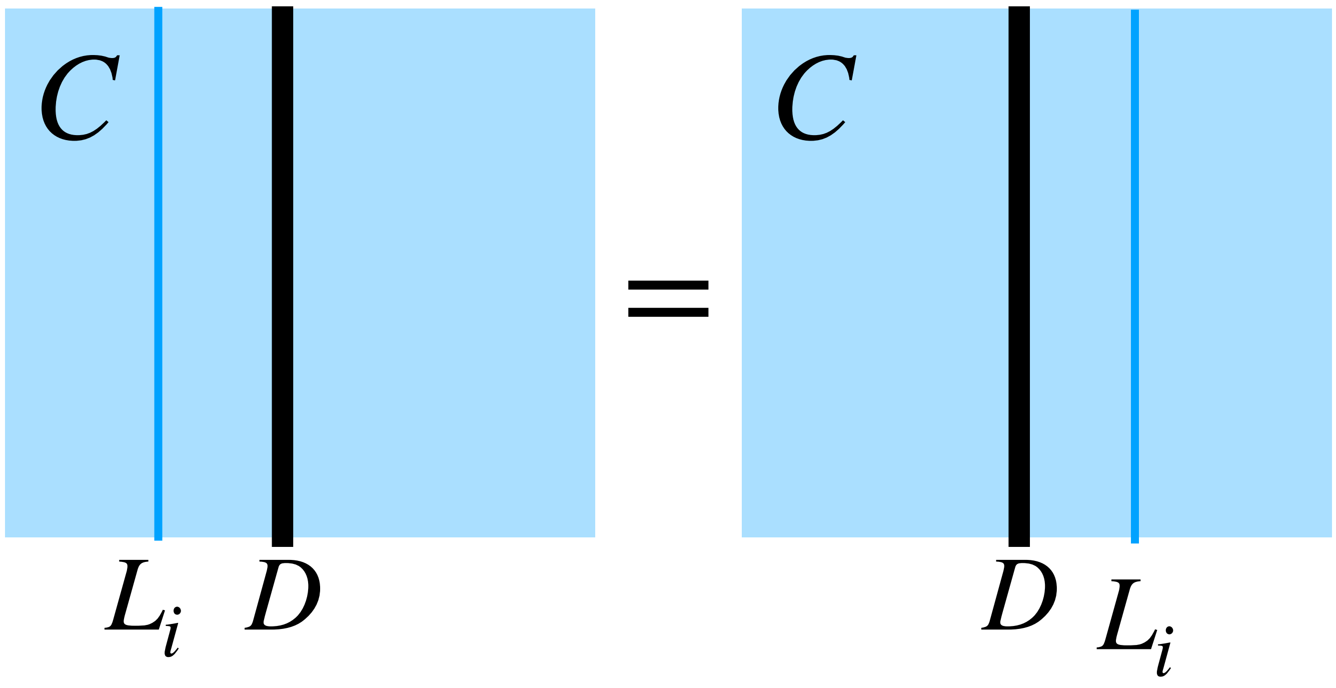
Example: One-point function with boundary in CFTd

Symmetry Constraints from Fusion

Focus on line defects in d=2

- Family of conformal defects related by fusion w/ topological defects for symmetry C
- Symmetry properties of conformal defects in theory with symmetry C captured by (C, C)-bimodule categories (fusion w/ topological defects)
- Defect fusion respects tensor structure of the bimodule categories
- C**-symmetric conformal defects**: commutes with C i.e. $D \circ L_i = L_i \circ D$

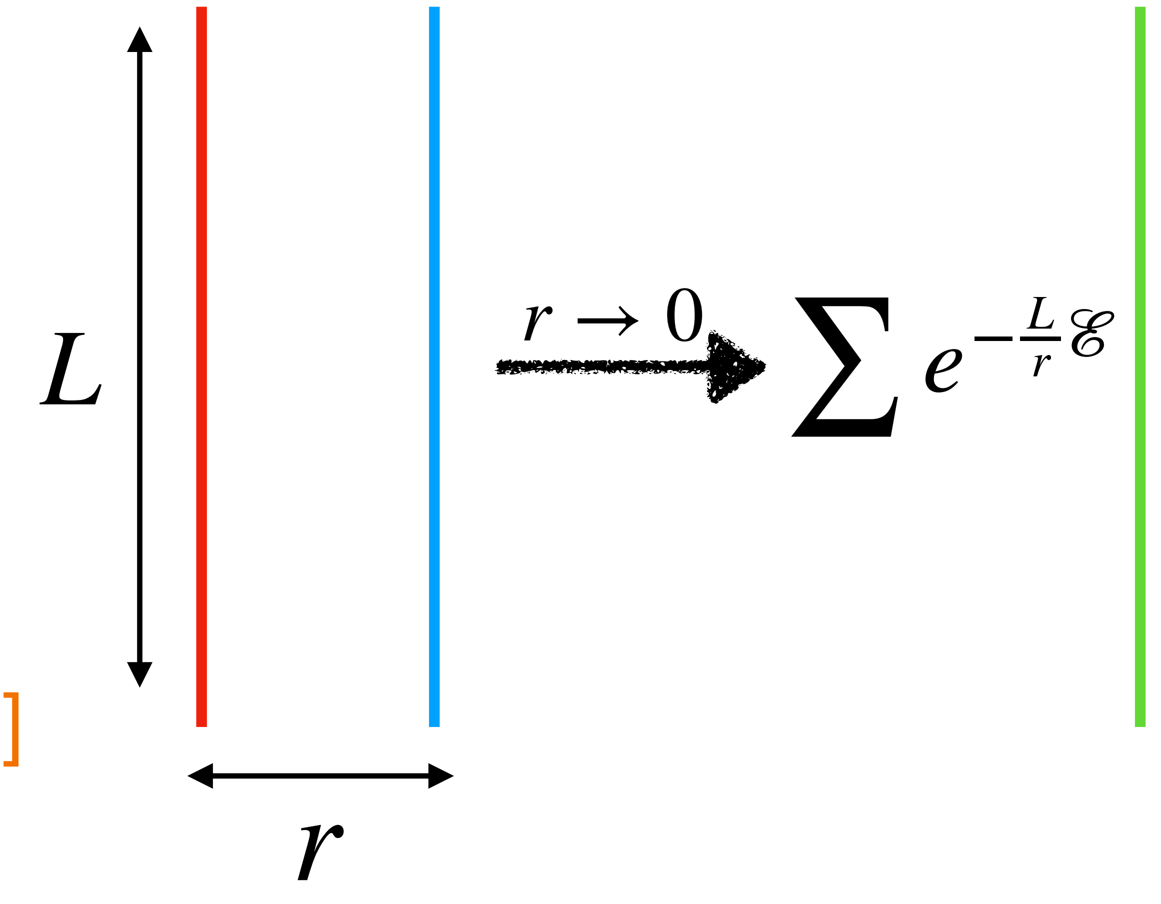




Constraints on RG flows $De^{-h\int dt\phi(x,t)}$ of C-symmetric defects

[Kormos Runkel Watts 09] also for boundaries [Konechny 19, Choi Rayhaun Sanghavi Shao 23] and recent works [Choi Rayhaun Zheng 24, Antinucci-Copetti Galati-Gizi 24]

Defect Fusion in 2d CFT



- Large family of conformal line defects (equivalently boundaries related by **folding trick**)

[Affleck Oshikawa 1997]

- Full classification **elusive** even for rational CFTs except for **Ising**
- Simple divergence from Casimir energy: $e^{-\frac{L}{r}\mathscr{E}}$ with $\mathscr{E} = h_{\min} - \frac{c}{24}$

Leading term in the defect OPE

- Fusion product defined by $\mathscr{D} \circ \mathscr{D}' \overset{\downarrow}{=} \lim_{r\to 0} e^{\frac{L\mathscr{E}}{r}} \mathscr{D}(r)\mathscr{D}'(0)$ **Hard!**

[Bachas Brunner 07, Bachas Brunner Roggenkamp 13, Konechny 15]

- More tractable case: many such line defects can be obtained by defect RG flows from topological defects

[Kormos Runkel Watts 09]

- Modest goal: deduce non-topological defect fusion from deformation

See also for fusion of non-conformal integrable defects [Manolopoulos Runkel 09]

Generalities: Defect Operator Algebra (∘ , ⊕) in General Dimension

$\mathscr{D}_{1,2}$ Conformal defects of dimension p in d-dimensional CFT $SO(p,2)\times SO(d-p)\subset SO(d,2)$

- Richer divergence structure for $p>1$

Curvature invariants on Σ

$$(\mathcal{D}_1\circ\mathcal{D}_2)(\Sigma)\equiv\lim_{r\to0}e^{\sum_{n=0}^{\lfloor p/2\rfloor}\int_\Sigma r^{2n-p}C^{(n)}_{\mathcal{D}_1\mathcal{D}_2}\mathcal{R}^n}\mathcal{D}_1(\Sigma_r)\mathcal{D}_2(\Sigma)$$

Leading term in the defect OPE

Generalized Casimir coefficients

transverse separation r

General properties of the fusion product $\mathcal{D}_1\circ\mathcal{D}_2(\Sigma)=\bigoplus_i\mathcal{C}_i(\Sigma)\mathcal{D}_i(\Sigma)$

(T)QFT fusion coefficients

[Roumpedakis Seifnashri Shao 22, Choi Cordova Hsin Lam Shao 22,…]

- Commutative for codimension >1
- Non-dualizable (i.e. ▯ $\notin\mathscr{D}\circ\overline{\mathscr{D}}$)

Orientation Reversal

In contrast to the fusion product of topological defects

- Non-associative (i.e. $\mathscr{D}_1\circ(\mathscr{D}_2\circ\mathscr{D}_3)\neq(\mathscr{D}_1\circ\mathscr{D}_2)\circ\mathscr{D}_3$) due to the truncation

[Diatlyk Khanchandani Popov YW 2024]

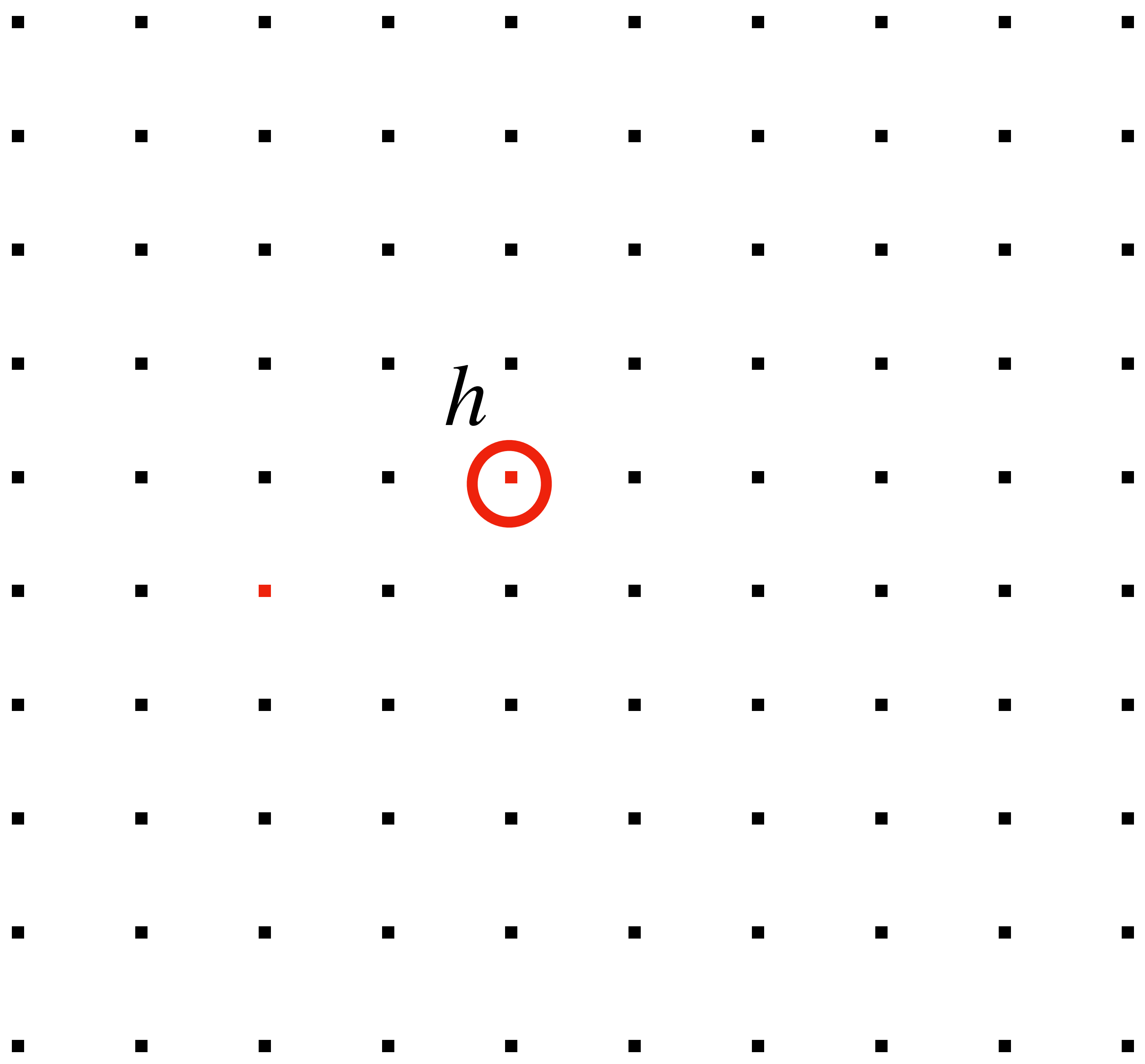
(Generalized) Pinning Field Defects

Simple UV definitions and nontrivial IR dynamics

Pinning Field Line Defects

拼音
pīnyīn Defects

$$S = \int d^d x\, (\partial\phi^I)^2 + \lambda(\phi^I)^4 + h\int dt\phi^I(t, \vec{x} = 0)$$



- Simple defects defined by local perturbation via the pinning field h which locally **pins the order** (also known as local magnetic defect)
- First introduced in lattice systems: efficient way to diagnose bulk phase by introduce local symmetry breaking [Assaad Herbut 13…]
- Rich family of nontrivial conformal line defects protected by g-theorem [Cuomo Komargodski Raviv Moshe 21] e.g. in 3d O(N) model […Cuomo Komargodski Mezei 21…]

Simple Examples of Defect RG flows

Pinning Field Defects in $d > 2$ $O(N)$ Model

- Wilson-Fisher ϕ^4 CFT with $O(N)$ global symmetry in $d = 4 - \epsilon < 4$

$\Delta_\phi = 1 - \epsilon/2 + \dots$

$$S = \int d^dx \left(\frac{1}{2}(\partial_\mu \phi^I)^2 + \frac{\lambda_b \Lambda_T^{4-d}}{4}(\phi^I\phi^I)^2 \right) \underline{+ h_{1,b} a_D^{\frac{d-4}{2}} \int dy \phi^1(y, z = 0, \vec{x} = 0)}$$

- Depends on an $O(N)$ orientation (breaking to $O(N-1)$)

Perturbative Fixed Points

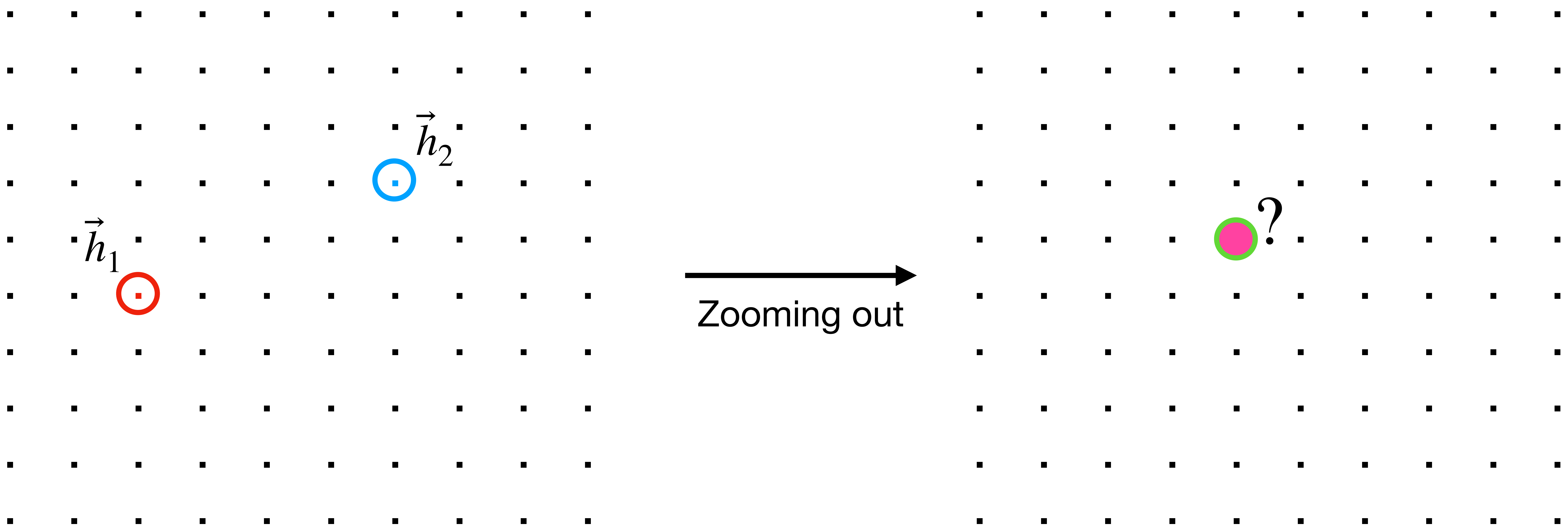
$$\lambda_* = \frac{8\pi^2}{(N+8)}\epsilon\,, \quad h_\pm = \pm h_*\,, \quad h_* = \left[\sqrt{N+8} + \frac{4N^2 + 45N + 170}{4(N+8)^{\frac{3}{2}}}\epsilon\right] + O(\epsilon^2)\,.$$

[Cuomo Komargodski Raviv-Moshe 2021]

- Family of conformal lines $\mathscr{D}(\hat{n})$ labelled by $\hat{n} \in S^{N-1}$
- Evidence they persist to $d = 3$ (by g-theorem, large N and lattice simulation)

Pinning Field Defects and Fusion

$$S = \int d^d x \, (\partial \phi^I)^2 + \lambda (\phi^I)^4 + \int dt \, \vec{h}_1 \cdot \phi(t, \vec{x} = 0) + \int dt \, \vec{h}_2 \cdot \phi(t, \vec{x} = \vec{r})$$



Fusing Pinning Line Defects in $O(N)$ Model

$$\mathscr{D}(\hat{n}) \circ \mathscr{D}(\hat{m}) = \mathscr{D}\left(\frac{\hat{n}+\hat{m}}{\sqrt{2(1+\hat{n}\cdot\hat{m})}}\right)$$

Symmetry enhancement to $O(N-1)$ upon fusion

Non-associative, non-dualizable

Idempotent fusion rule

$N = 1$
Ising

$$\boxed{\mathscr{D}_+ \circ \mathscr{D}_+ = \mathscr{D}_+} \qquad \mathscr{D}_- \circ \mathscr{D}_- = \mathscr{D}_- \qquad \mathscr{D}_+ \circ \mathscr{D}_- = 1$$

Casimir energy for a pair of magnetic line defects from similar computation

$$\langle \mathscr{D}_1(r)\mathscr{D}_2(0)\rangle \xrightarrow{\text{Small } r} e^{\frac{-L\mathscr{E}(\hat{n},\hat{m})}{r}}$$

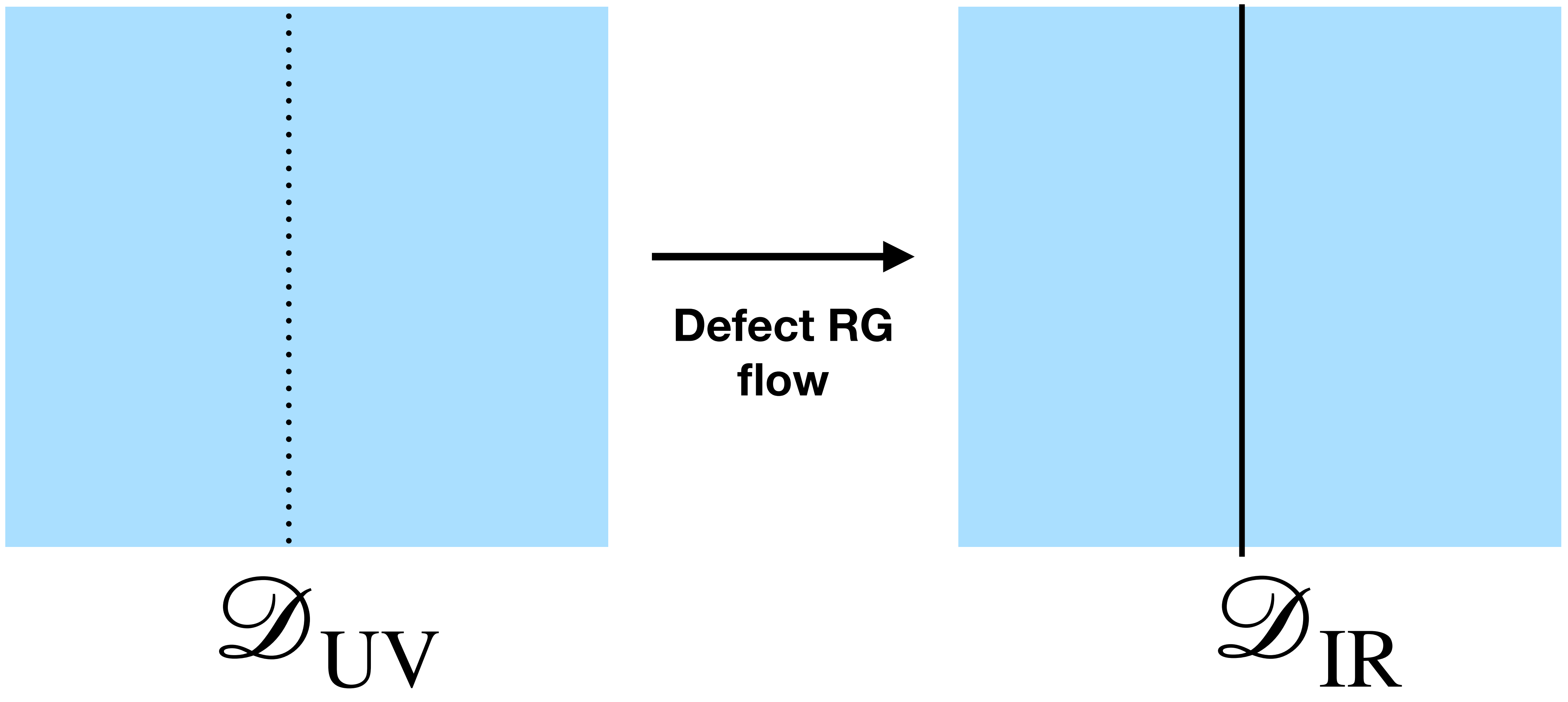
$$\mathcal{E}(\hat{n},\hat{m}) = -(\hat{n}\cdot\hat{m})\frac{N+8}{4\pi} - \frac{\epsilon}{4\pi}\left((\hat{n}\cdot\hat{m})\frac{N^2-3N-22}{2(N+8)} - \frac{(1+2(\hat{n}\cdot\hat{m})^2)\,\pi^2(N+8)}{16}\right)$$

[Diatlyk Khanchandani Popov YW 2024]

Generalized Pinning Field Defects

$$S_{\rm CFT} \to S_{\rm CFT} + h \int_{x_\perp = 0} d^{d-1}x\, \phi$$

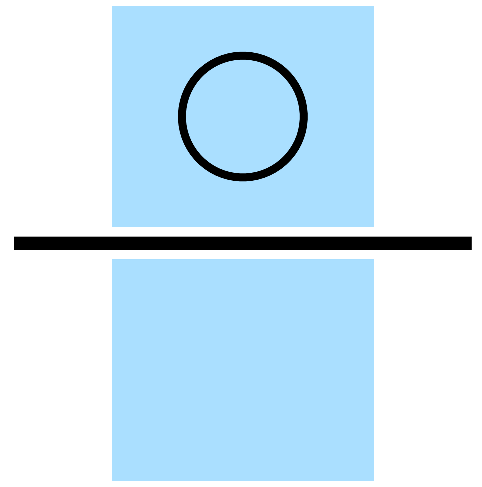
scalar $\Delta_\phi < d - 1$



- Pinning defect is conformal at large distance:
  g-theorem in d=2: $1 = g(\mathscr{D}_{\rm UV}) > g(\mathscr{D}_{\rm IR})$ [Affleck Ludwig 91, Friedan Konechny 03, Cuomo Komargodski Raviv Moshe 21, Casini Landea Torroba 22]

  b-theorem in d=3: $0 = b(\mathscr{D}_{\rm UV}) > b(\mathscr{D}_{\rm IR})$ [Jensen-O'Bannon 15, Casini Salazar Landea Torroba 18, Wang 20, Shachar Sinha Smolkin 22]

Defect free energy

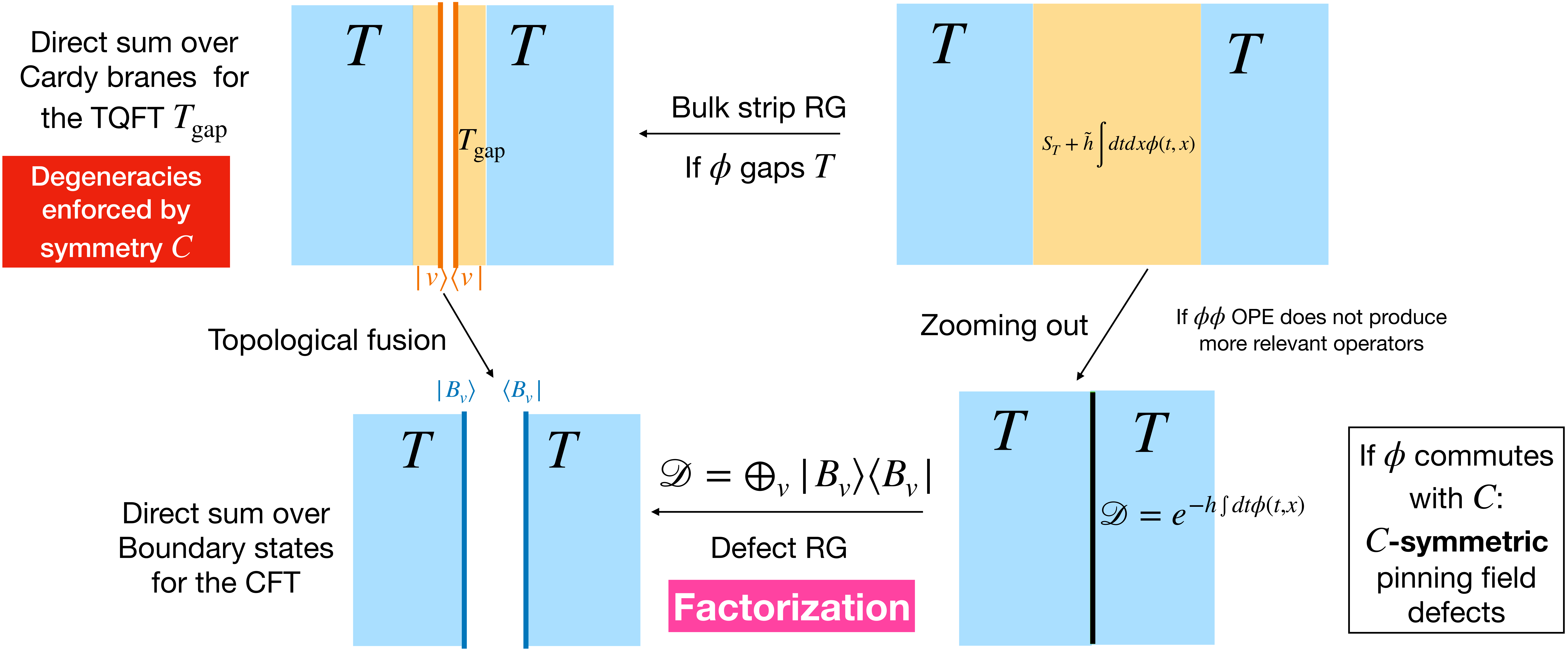


- Expect pinning defect to have simple fusion rule : $\mathscr{D}_{\rm IR} \circ \mathscr{D}_{\rm IR} = \mathscr{D}_{\rm IR}$

Pinning Field Defect in 2d

- Large zoo of (generalized) pinning field defects from turning on operator ϕ w/ $\Delta < 1$ along a line (e.g. all defects in Ising up to fusion with topological defects if also allow $\Delta = 1$)

- Pinning field defect often gives rise to **factorized** interfaces in 2d CFT T



Direct sum over Cardy branes for the TQFT T_{gap}

Degeneracies enforced by symmetry C

Bulk strip RG

If ϕ gaps T

Zooming out

If $\phi\phi$ OPE does not produce more relevant operators

Topological fusion

$\mathscr{D} = \oplus_v |B_v\rangle\langle B_v|$

Defect RG

Factorization

Direct sum over Boundary states for the CFT

If ϕ commutes with C: **C-symmetric** pinning field defects

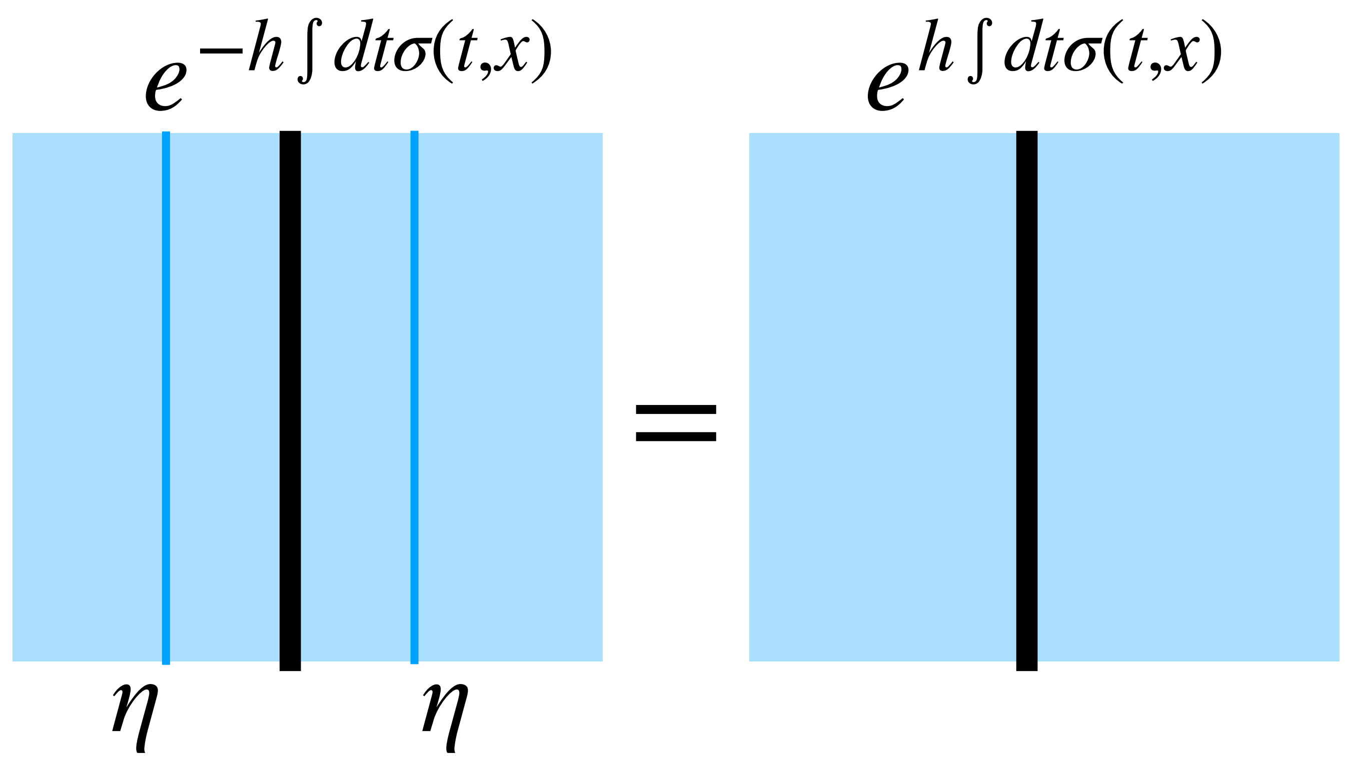
Pinning Field Defect in Ising CFT

$$S_{\text{Ising}} = \int d^2x\,(\partial\sigma)^2 + g\sigma^4$$

- Single operator of dimension $\Delta < 1$: σ
- IR defect is factorized: since the CFT under deformation is trivially gapped and $\sigma\sigma \to 1 + \epsilon$
- Boundary states of Ising: $|+\rangle, |-\rangle, |f\rangle$

Constraint from g-theorem
$g(\mathscr{D}) < 1$

$$\begin{tikzcd} & \mathscr{D}_+ = |+\rangle\langle+| \arrow[dd, leftrightarrow, "\mathbb{Z}_2\ \text{symmetry}"] \\ S_{\text{Ising}} + h\int dt\sigma(t,x) \arrow[ur, "h>0"] \arrow[dr, "h<0"'] \arrow[r, phantom, "\text{RG flow}"] & \\ & \mathscr{D}_- = |-\rangle\langle-| \end{tikzcd}$$



$$\mathscr{D}_+ = \eta\mathscr{D}_-\eta$$

- Simple Defect Fusion

Subtle defect RG flow

$$\mathscr{D}_+ \circ \mathscr{D}_+ = \mathscr{D}_+, \quad \mathscr{D}_- \circ \mathscr{D}_- = \mathscr{D}_-, \quad \mathscr{D}_+ \circ \mathscr{D}_- = |+\rangle\langle-|$$

Pinning Field Defect in Tricritical Ising CFT

$$\mathscr{L}_{\text{TIM}} = (\partial\sigma)^2 + g_1\sigma^6 + g_2\sigma^4$$

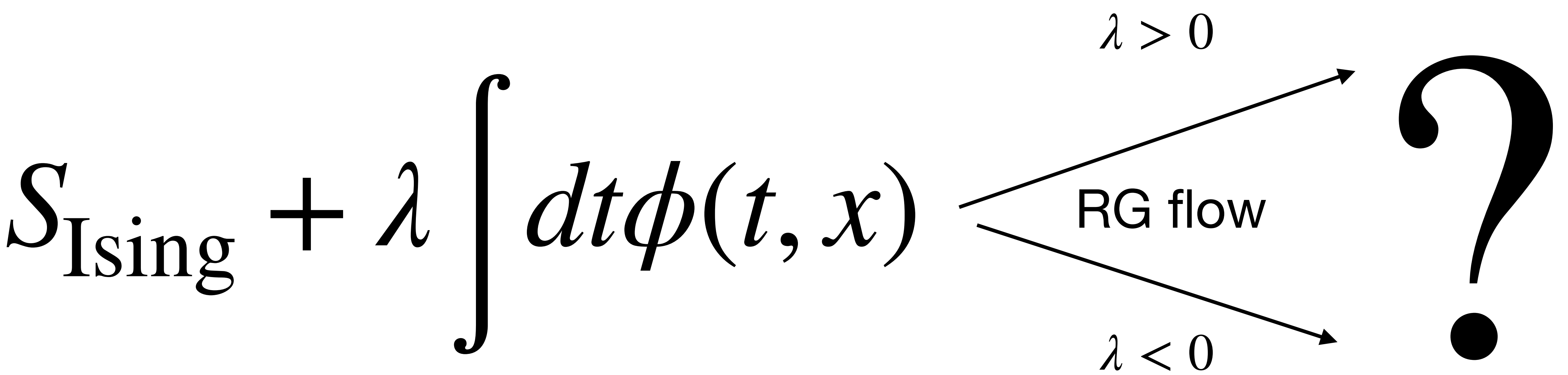
- Fixed point CFT has 6 primary operators and symmetry $\overset{\{1,\eta,\mathcal{N}\}}{\text{Ising}} \times \overset{\{1,W\}}{\text{Fib}}$
- 6 elementary boundary states: $|+\rangle, |-\rangle, |0\rangle, |d\rangle, |0-\rangle, |+0\rangle$

$$\begin{tikzcd}
{|+\rangle,} \arrow[r, bend right, "\eta"'] \arrow[rr, bend right, "\mathcal{N}"'] \arrow[rrrr, bend right, "W"'] & {|-\rangle,} & {|0\rangle,} \arrow[r, bend left, "W"] & {|d\rangle,} & {|0-\rangle,} \arrow[r, bend left, "\eta"] & {|+0\rangle}
\end{tikzcd}$$

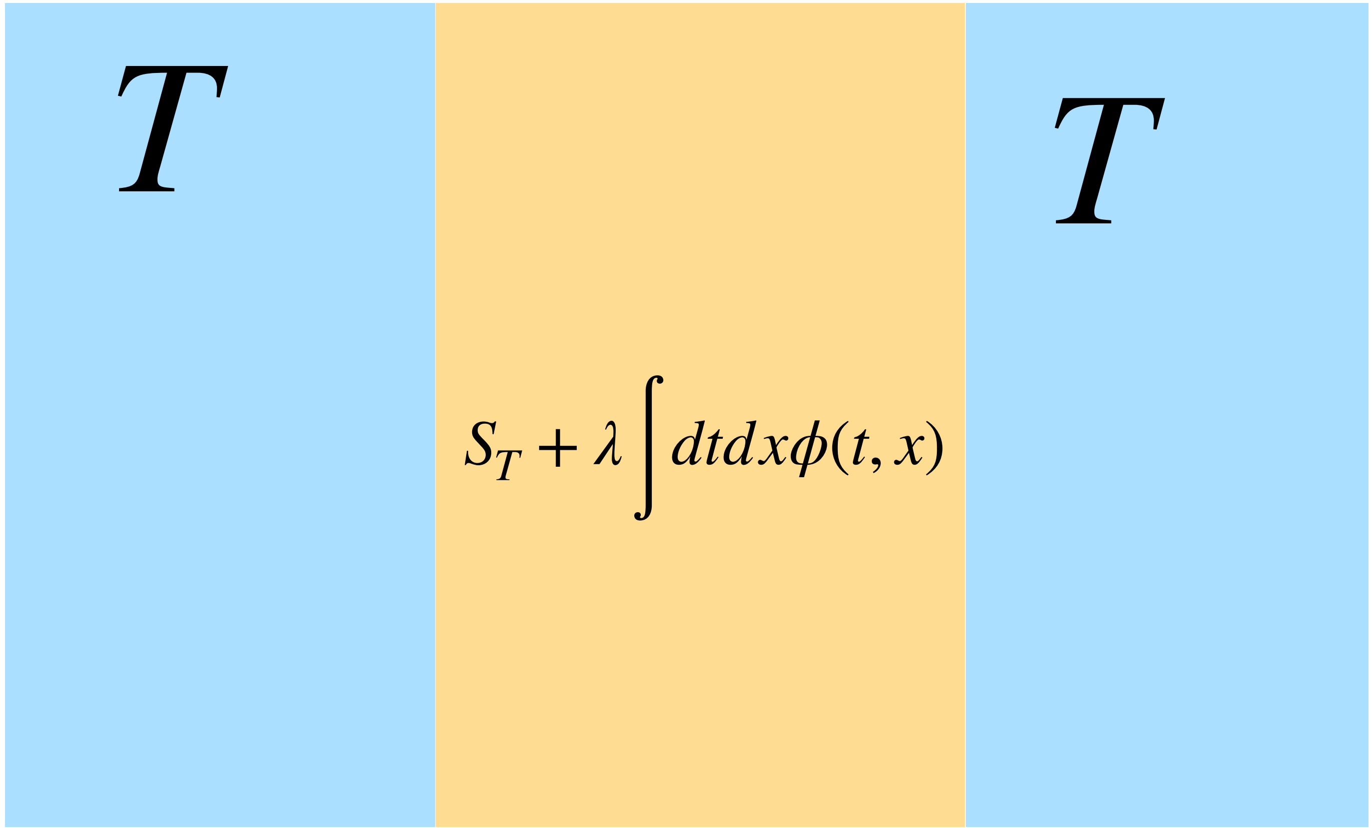
- Three operators with $\Delta = h + \bar{h} < 1$: $\epsilon_{1/10,1/10}$, $\sigma'_{7/16,7/16}$, $\sigma_{3/80,3/80}$

Pinning field	Symmetry	+ deformation in the bulk	- deformation in the bulk
ϵ	$\mathbb{Z}_2$	SSB	Trivial
σ	None	Trivial	Trivial
σ'	Fib	SSB w/ gsd=2	SSB w/ gsd=2

Pinning Field Defect in Tricritical Ising CFT



$$S_{\text{Ising}} + \lambda \int dt \phi(t,x)$$



Constraints from factorization, g-theorem and symmetries

Pinning field	Symmetry	+ deformation on the line	- deformation on the line
ϵ	$\mathbb{Z}_2$	$\vert+\rangle\langle+\vert \oplus \vert-\rangle\langle-\vert$	$\vert0\rangle\langle0\vert$
σ	None	$\vert+\rangle\langle+\vert$	$\vert-\rangle\langle-\vert$
σ'	Fib	$\vert+\rangle\langle+\vert \oplus \vert0-\rangle\langle0-\vert$	$\vert-\rangle\langle-\vert \oplus \vert+0\rangle\langle+0\vert$

Generalized Pinning Field Defects in d=3

$$S = \int d^3x \left[\frac{1}{2}\left(\partial\phi^I\right)^2 + \frac{\lambda}{4!}(\phi^I\phi^I)^2\right] + h\int_{x_\perp=0} d^2x(\phi^I)^2$$

- Localized mass deformation on a surface [Bray Moore 1977… Krishnan Metlitski 23, Trepanier 23, Raviv-Moshe Zhong 23, Giombi Liu 23, Cuomo Zhang 23]
- $\Delta_{\phi^2} < 2$ from bootstrap and leading O(N) singlet operator
- In the bulk: symmetry breaking for h<0 and trivially gapped for h>0

Dirichlet boundary condition for O(N)

$$S_{O(N)} + h\int_{x_\perp=0} d^2x(\phi^I)^2 \xrightarrow{h>0} \mathscr{D}_+ = |\mathrm{Ord}\rangle\langle\mathrm{Ord}|$$

- Conjecture based on large N analysis [Krishnan Metlitski 23]
- Evidence at finite N from ϵ expansion w/ comparison to numerics for Ord BCFT

[Oleksandr Sun Wang 24]

[Toldin 23, Zhou Zou 24]

Factorization by RG

from codimension-one pinning field defects

Factorizing Interface

- Codimension-one defect (self-interface) that factorizes

$$I = \oplus_\nu |B_\nu\rangle\langle B'_\nu|$$

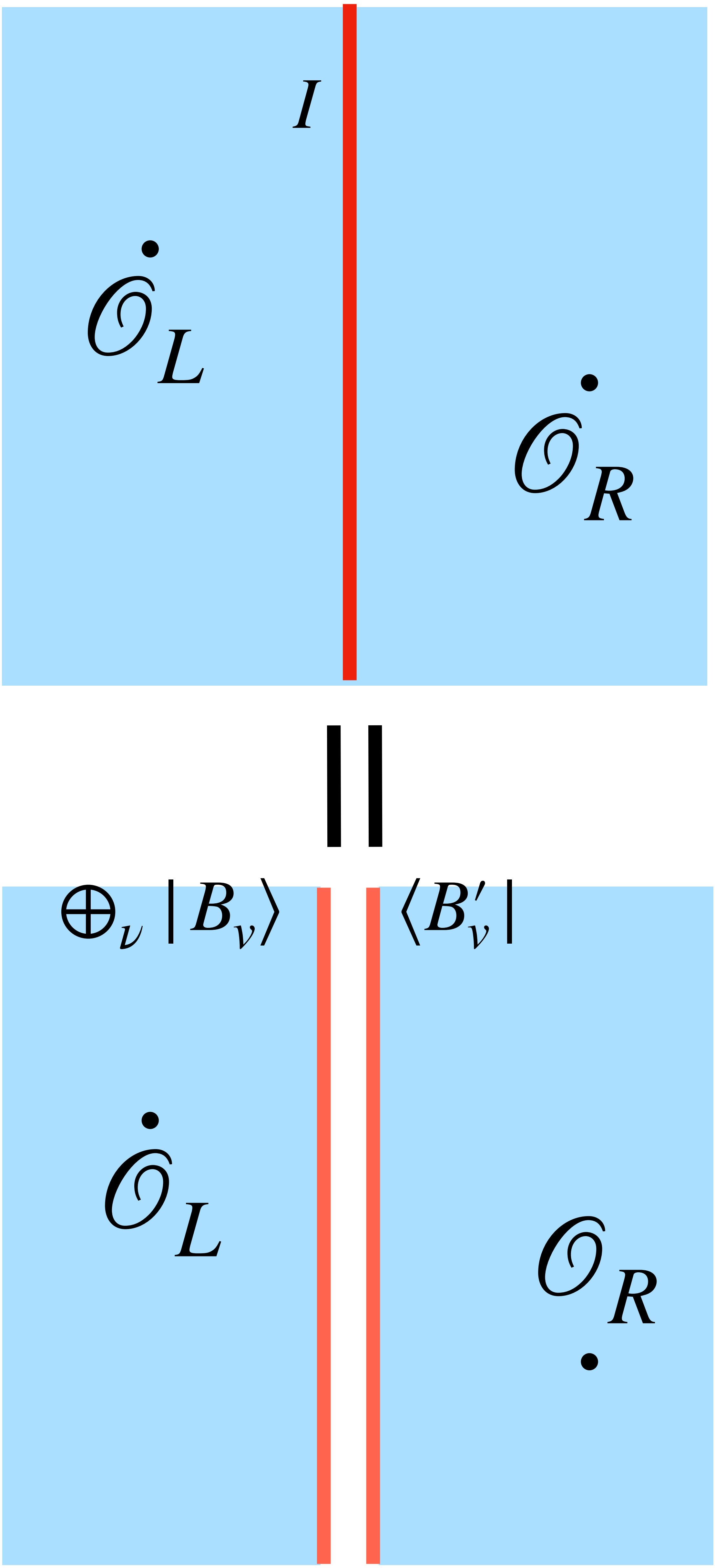
- Factorized correlation functions on the right and left

$$\langle \mathcal{O}_L \ldots \mathcal{O}_L \mathcal{O}_R \ldots \mathcal{O}_R \rangle_I = \sum_\nu \langle \mathcal{O}_L \ldots \mathcal{O}_L \rangle_{B_\nu} \langle \mathcal{O}_R \ldots \mathcal{O}_R \rangle_{B'_\nu}$$

- Total reflection of energy

$$\langle T_L^{\mu\nu} T_R^{\rho\sigma} \rangle = 0$$

Pinning field interface makes an Energy Shield!



Proving Factorization

$$D_h(\mathcal{O}) = [e^{-h\hat{\mathcal{O}}}]_{\text{ren}} \quad \hat{\mathcal{O}} \equiv \int d^{d-1}\mathcal{O}$$

- **Main idea:** the IR of $D_h(\mathcal{O})$ is dominated by the largest ($h < 0$) eigenvalue of $\hat{\mathcal{O}}$ acting on the CFT Hilbert space $\mathcal{H}$

- It suffices to show that the largest eigenvalue is realized by Ishibashi states

[Nakayama Ooguri 2015]

$$P_i|B\rangle = K_i|B\rangle = D|B\rangle = 0$$

Planar frame

- Then

$$D(\mathcal{O}) = \lim_{h\to\infty} D_h(\mathcal{O}) = \oplus_\alpha |B_\alpha\rangle\langle B_\alpha|$$

Ishibashi states

- **Caveats:** $\hat{\mathcal{O}}$ is unbounded operator, Ishibashi states are not in $\mathcal{H}$,…

- **Resolution:** Spectral theory of unbounded operators and Gelfand triple construction

CFT primary states

$$\Phi \subset \mathcal{H} \subset \Phi'$$

Ishibashi States

[Popov YW to appear]

Speculations

Further properties of pinning defects,
applications and beyond

Speculations

Stability

- Observation: Pinning defects are **stable** (symmetry enforced **self-organized criticalities**)
- Lower bound on g function of defects from symmetries (symmetry enriched g theorem)

[… Cordova García-Sepúlveda 2022]

Applications

- Pinning defects in 4d conformal gauge theories (Caswell-Banks-Zaks, SYM)
- Pinning defects as local quantum channels for weak measurement and decoherence

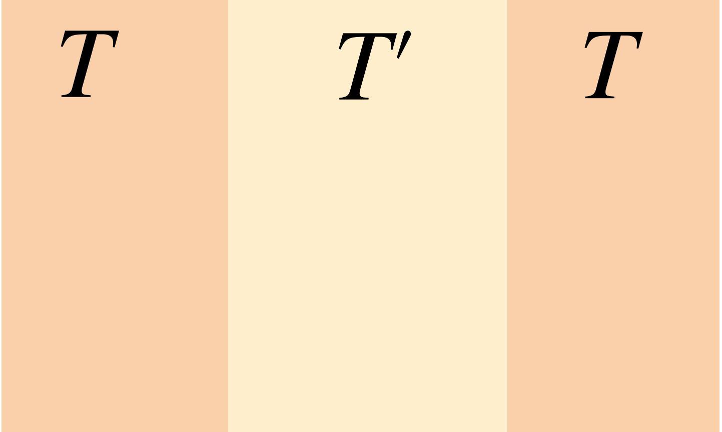
[… Lee Jian Xu 2023 …]

- Pinning defects dual to branes and geometries in the holographic dual

Generalizations

- Construction of defects from general RG flows on the strip and gauging (relatedly fusion of RG interfaces)

[Gaiotto 2012…]



Thank you!