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Holographic Entanglement, Pseudo Entropy and Wormholes

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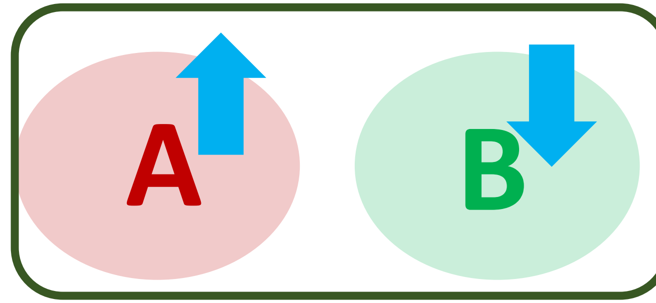
Grant-in-Aid for Transformative Research Areas A
Extreme Universe

① Introduction

Quantum Entanglement (QE)



Two parts (subsystems) A and B in a total system are quantum mechanically correlated.

e.g. Bell state: $|\Psi_{Bell}\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right]$ ⇒ **Minimal Unit of Entanglement**

Pure States: Non-zero QE ⇔ $|\Psi\rangle_{AB} \neq |\Psi_1\rangle_A \otimes |\Psi_2\rangle_B$.

Direct Product

The best (or only) measure of quantum entanglement for pure states is known to be **entanglement entropy (EE)**.

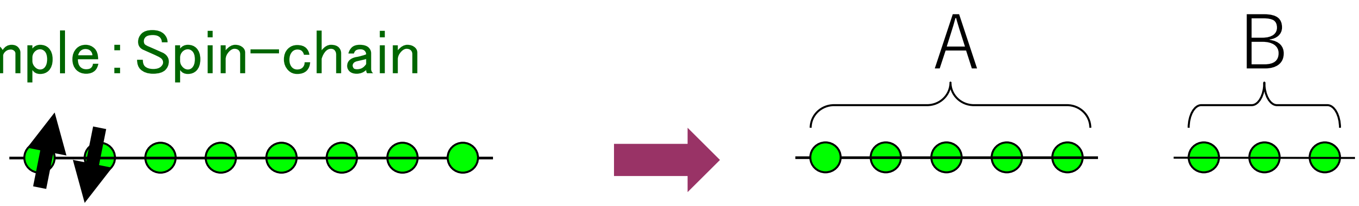
EE ＝ # of Bell Pairs between A and B

Entanglement Entropy (EE)

An amount of quantum entanglement (for pure states) is measured by **Entanglement Entropy (EE) .**

First we decompose the Hilbert space: $H_{tot} = H_A \otimes H_B$.

Example : Spin-chain



We introduce the reduced density matrix ρ_A

by tracing out B $\rho_A = \mathrm{Tr}_B\left[\,|\Psi_{tot}\rangle\langle\Psi_{tot}|\,\right].$

The entanglement entropy (EE) S_A is defined by

$$S_A = -\mathrm{Tr}[\rho_A \log \rho_A]$$ $\propto$ # of Bell Pairs between A and B

[Refer to e.g. Nilsen-Chuang's text book]

Measurement of EE in Experiments

Ex.1: Ultracold bosonic atoms in optical lattices

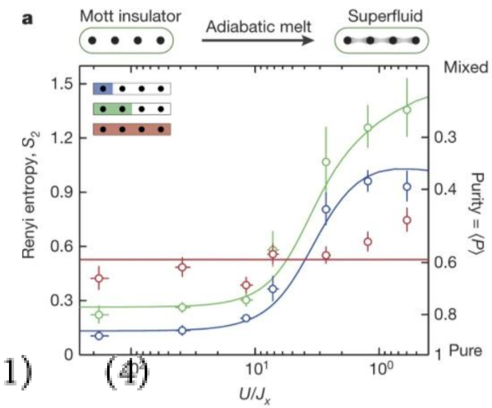
Published: 02 December 2015

Measuring entanglement entropy in a quantum many-body system

Rajibul Islam, Ruichao Ma, Philipp M. Preiss, M. Eric Tai, Alexander Lukin, Matthew Rispoli & Markus Greiner

Nature **528**, 77–83 (2015) | Cite this article

$$H = -J\sum_{\langle i,j\rangle} a_i^\dagger a_j + \frac{U}{2}\sum_i n_i(n_i - 1) \quad (4)$$



Ex2: Trapped-ion quantum simulator

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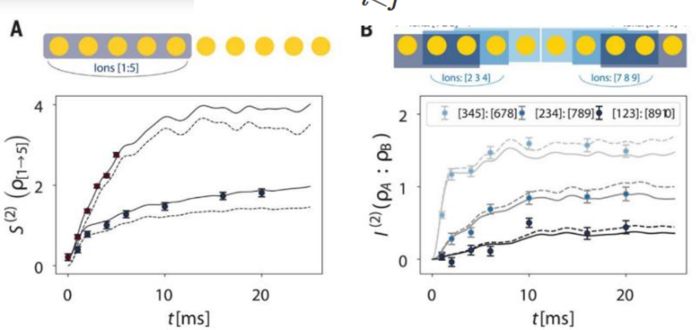
REPORT

Probing Rényi entanglement entropy via randomized measurements

TIFF BRYDGES, ANDREAS ELBEN, PETAR JURCEVIC, BENOÎT VERMERSCH, CHRISTINE MAIER, BEN P. LANYON, PETER ZOLLER, RAINER BLATT, AND CHRISTIAN F. ROOS
Authors Info & Affiliations

SCIENCE • 19 Apr 2019 • Vol 364, Issue 6437 • pp. 260-26

$$H_{XY} = \hbar\sum_{i<j} J_{ij}\left(\sigma_i^+\sigma_j^- + \sigma_i^-\sigma_j^+\right) + \hbar B\sum_j \sigma_j^z$$



Ex3. Topological EE in superconducting qubits

Science
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SCIENCE • 2 Dec 2021 • Vol 374, Issue 6572 • pp. 1237-1241 • DOI: 10.1126/science.abi8378
RESEARCH ARTICLE TOPOLOGICAL MATTER

Realizing topologically ordered states on a quantum processor

K. J. SATZINGER, Y.-J LIU, A. SMITH, C. KNAPP, M. NEWMAN, C. JONES, Z. CHEN, C. QUINTANA, X. MI, [...], AND P. ROUSHAN +88 authors

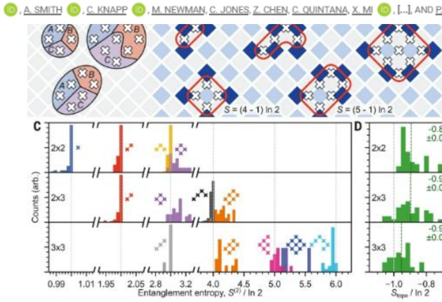


Fig. 2. Topological entanglement entropy.

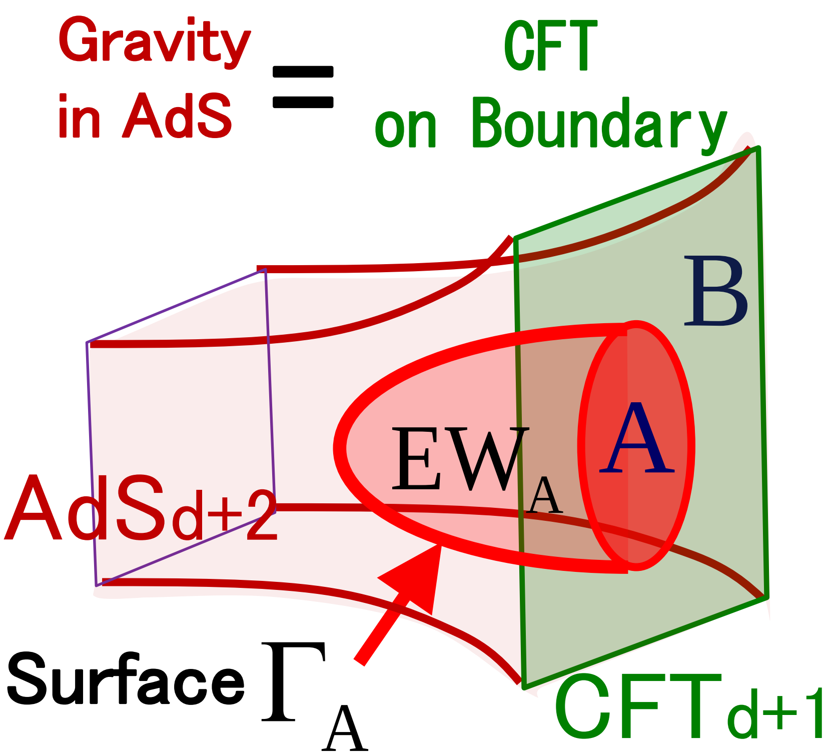
Holographic Entanglement Entropy [Ver.1:Static]

[Ryu-Takayanagi 2006]

Γ_A = Minimal Area Surface which surrounds A in AdS

$$S_A = \frac{\text{Area}(\Gamma_A)}{4G_N}$$

Entanglement Entropy between A and B



➡ **Information in A is encoded in the entanglement wedge EW_A !**

[Czech-Karczmarek-Nogueira-Raamsdonk, Wall 2012, Headrick-Hubeny-Lawrence-Rangamani 2014…]

Holographic Entanglement Entropy [Ver.2:Time-dep.]

[Hubeny−Rangamani−TT 07]

A generic Lorentzian asymptotic AdS spacetime is dual to a time dependent state $|\Psi(t)\rangle$ in the dual CFT.

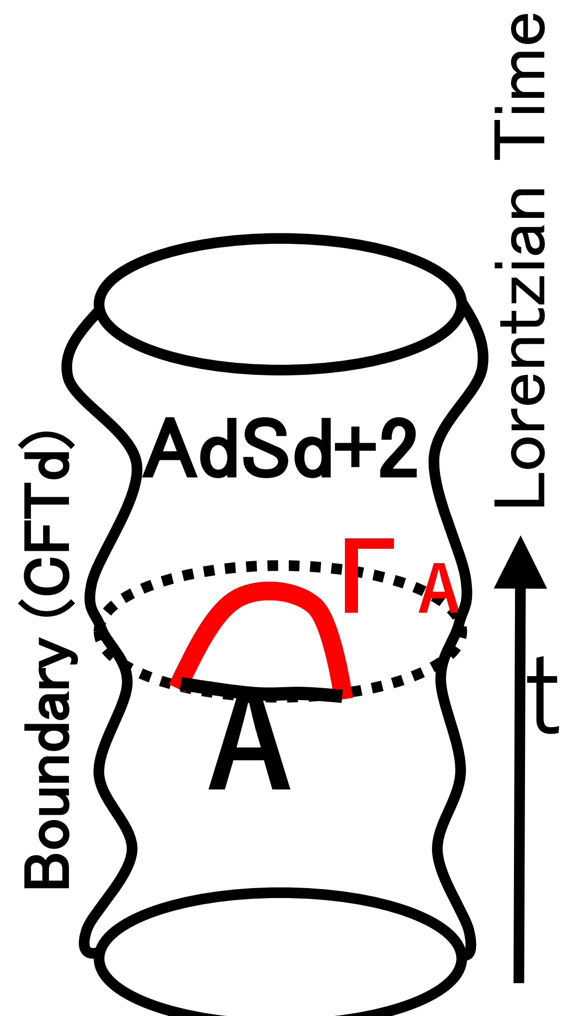
The time−dependent entanglement entropy

$$\rho_A(t) = \mathrm{Tr}_B[|\Psi(t)\rangle\langle\Psi(t)|] \Rightarrow S_A(t).$$

is computed from an extremal surface area:

$$S_A(t) = \mathrm{Min}_{\Gamma_A}\mathrm{Ext}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$

$$\partial A = \partial\gamma_A \text{ and } A \sim \gamma_A \,.$$



Question: Any Ver 3. Formula ?

Minimal areas in ***Euclidean time dependent***
asymptotically AdS spaces

= What kind of QI quantity in CFT ?

⇨ **The answer is Pseudo Entropy !**

[Nakata-Taki-Tamaoka-Wei-TT, 2020]

Contents

Based on arXiv:2502.03531 [to appear in JHEP]

with Taishi Kawamoto (YITP, Kyoto U.)
Ryota Maeda (YITP, Kyoto U.)
Nanami Nakamura (YITP, Kyoto U.)

② Pseudo Entropy and Holography

(2-1) Definition of Pseudo (Renyi) Entropy

Consider two quantum states $|\psi\rangle$ and $|\varphi\rangle$, and define the *transition matrix*:

$$\tau^{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}.$$

We decompose the Hilbert space as $H_{tot} = H_A \otimes H_B$.
and introduce the reduced transition matrix:

$$\tau_A^{\psi|\varphi} = \mathrm{Tr}_B\left[\tau^{\psi|\varphi}\right]$$

⇒ **Pseudo Entropy**

$$S\left(\tau_A^{\psi|\varphi}\right) = -\mathrm{Tr}\left[\tau_A^{\psi|\varphi}\log\tau_A^{\psi|\varphi}\right].$$

Renyi Pseudo Entropy

$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = \frac{1}{1-n}\log\mathrm{Tr}\left[\left(\tau_A^{\psi|\varphi}\right)^n\right].$$

(2-2) Basic Properties of Pseudo Entropy (PE)

- In general, $\tau_A^{\psi|\varphi}$ is not Hermitian. Thus PE is complex valued.

◆For thermal pseudo entropy, Kramers-Kronig relation relates the real part of PE to the imaginary part.
[Caputa-Chen-Tsuda-TT 2024]

$$\mathrm{Im}[f(t)] = \frac{1}{\pi} P \int_{-\infty}^{\infty} ds \frac{\mathrm{Re}[f(s)]}{s-t},$$

When does PE become real ? ➡
◆Real valued Euclidean PI= Holographic PE
◆Pseudo Hermiticity [Guo-He-Zhan 2022]

- If either $|\psi\rangle$ or $|\varphi\rangle$ has no entanglement (i.e. direct product state) , then

$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = 0.$$

- We can show $S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = \left[S^{(n)}\left(\tau_A^{\varphi|\psi}\right)\right]^{\dagger}$.

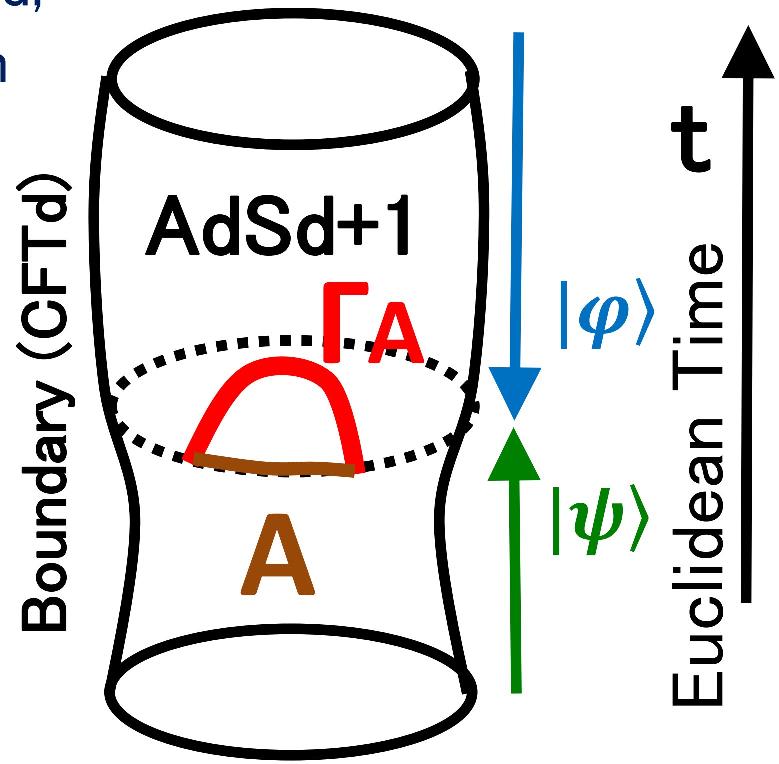
- We can show $S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = S^{(n)}\left(\tau_B^{\psi|\varphi}\right)$. →"SA=SB"

(2-3) Holographic Pseudo Entropy (HPE) Formula

[Nakata–Taki–Tamaoka–Wei–TT, 2020]

In Euclidean time dependent background, the minimal surface area coincides with the pseudo entropy.

$$S\left(\tau_A^{\psi|\varphi}\right) = \mathrm{Min}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$



As we will later see, we can apply HPE also to some Lorentzian spacetimes.

(2-4) Pseudo Entropy and Quantum Phases

[Mollabashi–Shiba–Tamaoka–Wei–TT 20, 21]

Basic Properties of Pseudo entropy in QFTs

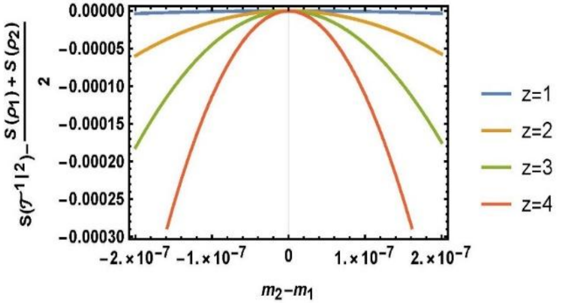
[1] Area law

$$S_A \sim \frac{\text{Area}(\partial A)}{\varepsilon^{d-1}} + (\text{subleading terms}),$$

[2] The difference

$$\Delta S = S\left(\tau_A^{1|2}\right) + S\left(\tau_A^{2|1}\right) - S(\rho_A^1) - S(\rho_A^2)$$

is negative if $|\psi_1\rangle$ and $|\psi_2\rangle$ are in a same phase.



PE in a 2 dim. free scalar when we change its mass.

What happen if they belong to different phases ?
Can ΔS be positive ?

Quantum Ising Chain with a transverse magnetic field

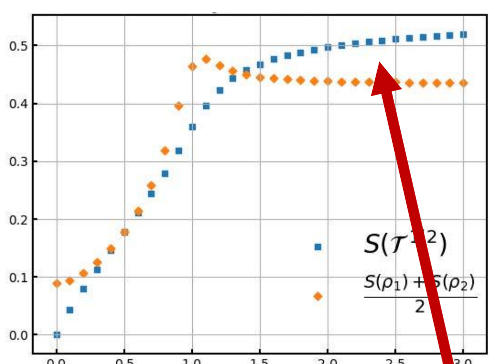
$$H = -J \sum_{i=0}^{N-1} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=0}^{N-1} \sigma_i^x ,$$

Ψ 1→ vacuum of H(J1)
Ψ 2→ vacuum of H(J2)
(We always set h=1)

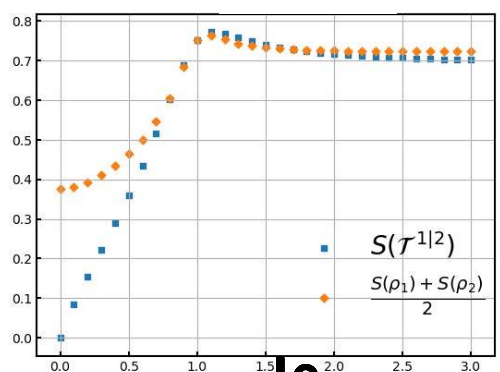
J<1 Paramagnetic Phase
J>1 Ferromagnetic Phase

N=16, N_A=8

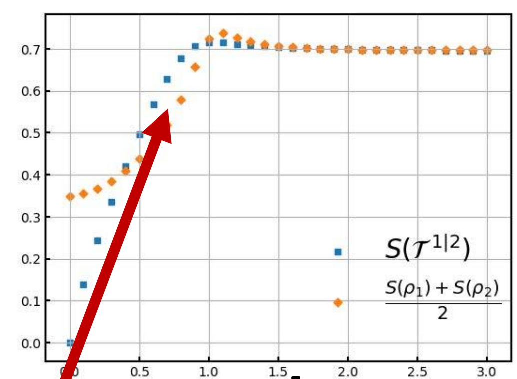
J1=1/2



J1=1



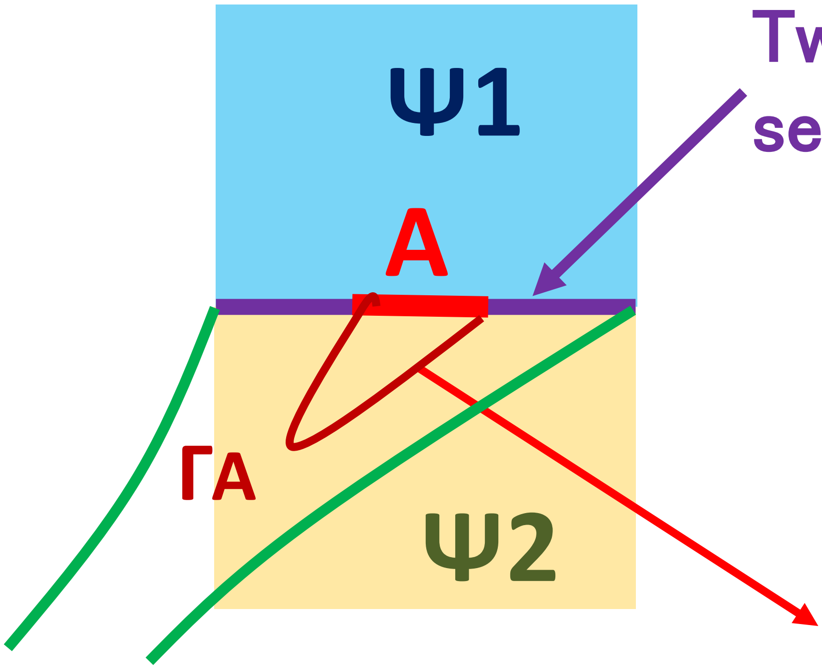
J1=2



$$\Delta S = S\left(\tau_A^{1|2}\right) + S\left(\tau_A^{2|1}\right) - S(\rho_A^1) - S(\rho_A^2) > 0$$

We find when Ψ 1 and Ψ 2 are in different phases !

Heuristic Interpretation



Two gapped phases are separated by a gapless phase.
CFT !

PE is enhanced ! $\Delta S > 0$

AdS Dual of Gapless Interface

The gapless interface (edge state) also occurs in topological orders.
➔Topological pseudo entropy
[Nishioka-Taki-TT 2021, Caputa-Purkayastha-Saha-Sułkowski 2024]

Question

Is the pseudo entropy relevant for holography in Lorentzian spacetimes ?

⇨ **Yes, it is !**

★ **Traversable wormholes !**

[Kawamoto–Maeda–Nakamura–TT, 2025]

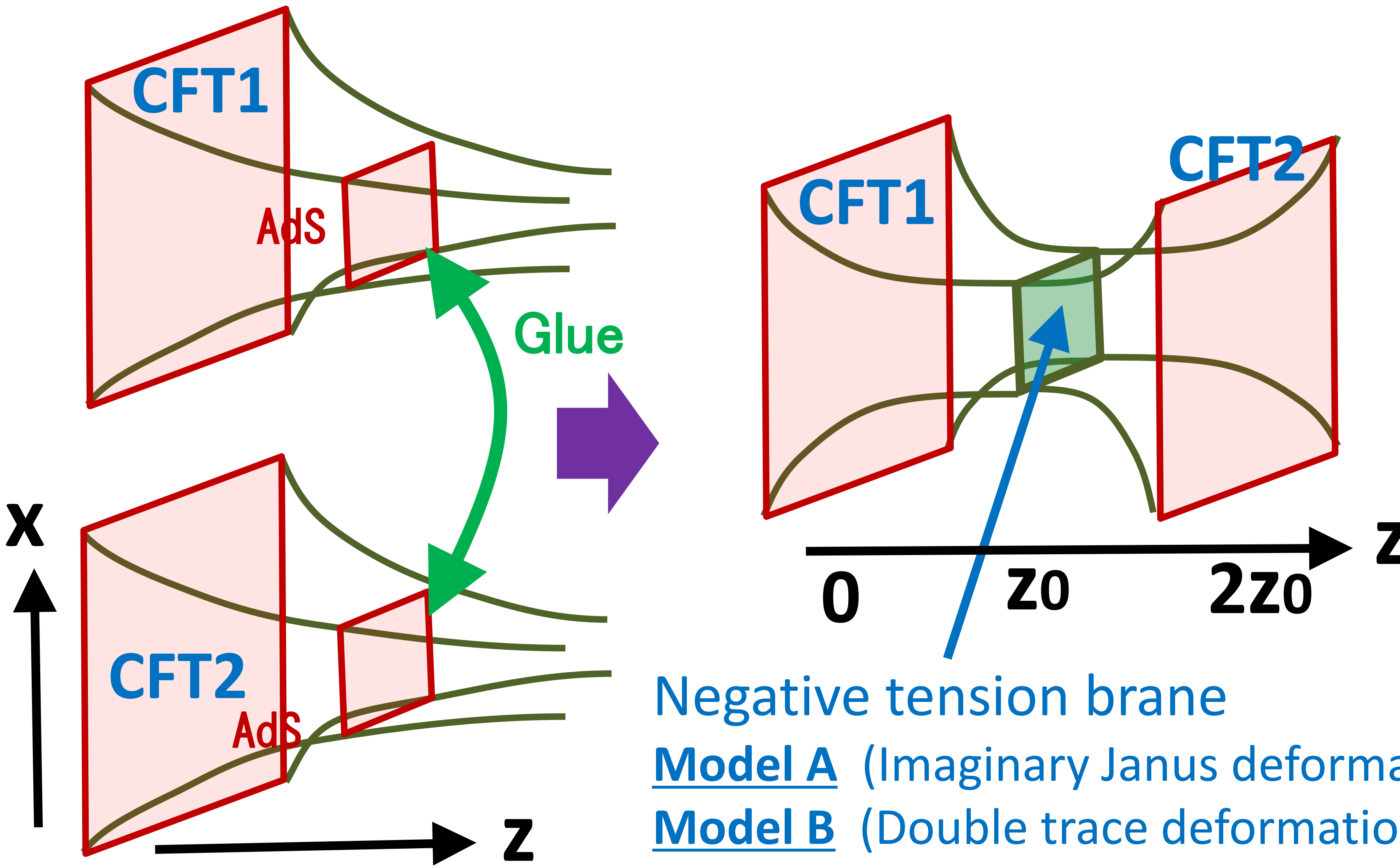
◆ Time-like EE

◆ dS Holography

[Doi–Harper–Mollabashi–Taki–TT 2022, Kawamoto–Ruan–Suzuki–TT 2023]

③ Simple model of traversable AdS wormhole

Consider a simple model of traversable AdS wormhole:



Model A (Imaginary Janus deformation)

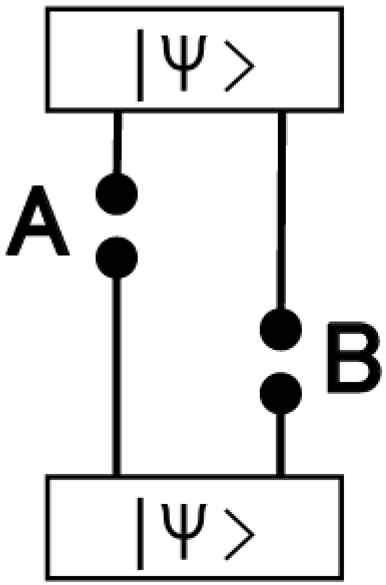
Model B (Double trace deformation)

Two constructions of CFT dual of Traversable AdS wormhole

Non-traversable

Traversable

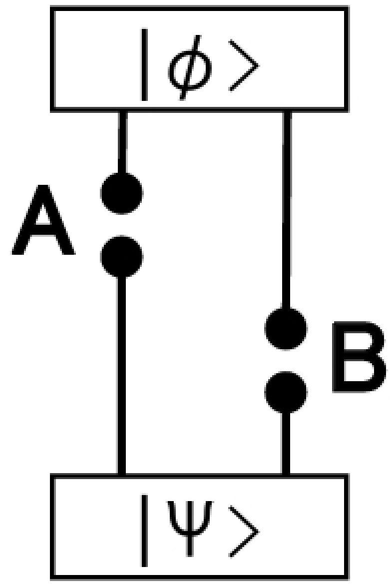
Thermofield double



$\rho_{AB}^{\dagger} = \rho_{AB}$

$S(\rho_{AB}) = 0$

Model A (Janus)



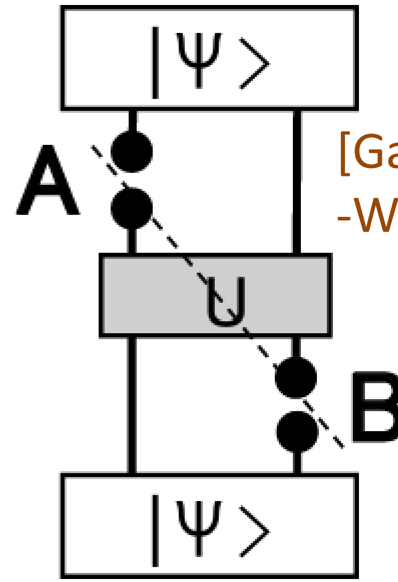
$\rho_{AB}^{\dagger} \neq \rho_{AB}$

$S(\rho_{AB}) = 0$

No interactions between A and B

[New !]

Model B (Double trace)



[Gao-Jafferis -Wall 16]

$\rho_{AB}^{\dagger} \neq \rho_{AB}$

$S(\rho_{AB}) \neq 0$

Entanglement between A and B is Not necessary

Consider a scalar field Φ in the bulk:

$$I_{\rm scalar} = \int dz d^d x \left[\frac{1}{z^{d-1}} \left((\partial_z \Phi)^2 + (\partial_x \Phi)^2 \right) + \frac{m^2}{z^{d+1}} \Phi^2 \right].$$

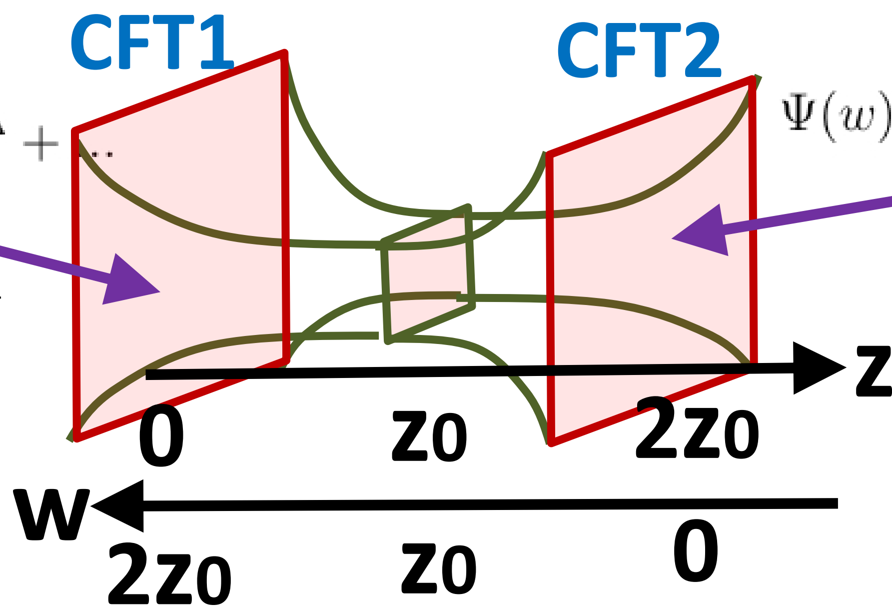
$$\Phi'' - \frac{d-1}{z}\Phi' - \left(k^2 + \frac{m^2}{z^2}\right)\Phi = 0.$$

Source J_1 ⟨O1⟩

$$\Phi(z) = \alpha_1 z^{d-\Delta} + \beta_1 z^{\Delta} + \dots$$

Source J_2 ⟨O2⟩

$$\Psi(w) = \alpha_2 w^{d-\Delta} + \beta_2 w^{\Delta} + \dots$$



$$\Delta = \frac{d}{2} + \nu, \quad \nu = \sqrt{m^2 + \frac{d^2}{4}}$$

$$ds^2 = R(z)\left(dz^2 + \sum_{i=0}^{d-1} dx_i^2\right), \quad R(z) = \frac{1}{z^2} \quad (0 < z < z_0), \qquad R(z) = \frac{1}{(2z_0 - z)^2} \quad (z_0 < z < 2z_0).$$

Two point functions read

$$P(\nu, k, z = z_0, d) := \langle \mathcal{O}_1(k)\mathcal{O}_1(-k)\rangle = -\frac{\beta_1}{\alpha_1}$$

$$= \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu} \frac{kz_0 I_{\nu-1}(kz_0)I_{-\nu}(kz_0) + (kz_0 I_{1-\nu}(kz_0) + (d-2\nu)I_{-\nu}(kz_0))\, I_\nu(kz_0)}{(d-2\nu)I_\nu(kz_0)^2 + 2kz_0 I_{\nu-1}(kz_0)I_\nu(kz_0)}.$$

$$Q(\nu, k, z = z_0, d) := \langle \mathcal{O}_1(k)\mathcal{O}_2(-k)\rangle = \frac{\beta_2}{\alpha_1}$$

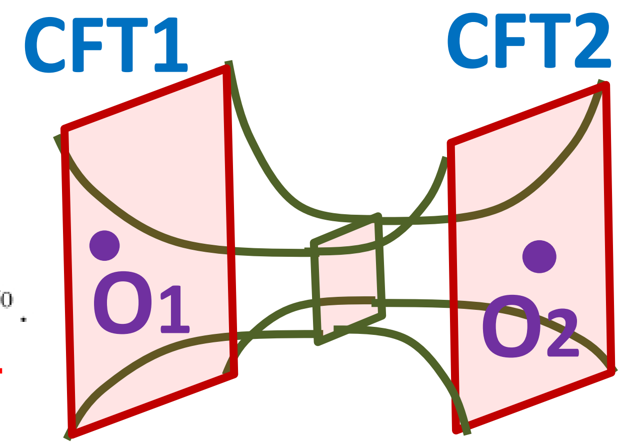
$$= \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu} \frac{2\sin\nu\pi}{\pi}\frac{1}{(d-2\nu)I_\nu(kz_0)^2 + 2kz_0 I_{\nu-1}(kz_0)I_\nu(kz_0)}.$$

In the UV limit ($kz_0 \gg 1$), we obtain

⟨O1O1⟩ $P(\nu, k, z = z_0, d) \simeq \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu}$

⟨O1O2⟩ $Q(\nu, k, z = z_0, d) \simeq \frac{2\sin\nu\pi\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu} e^{-2kz_0}.$

exp decay



In the IR limit ($kz_0 \ll 1$), we obtain

⟨O1O1⟩ $P(\nu, k, z = z_0, d) \simeq \frac{d}{d+2\nu}\frac{1}{z_0^{2\nu}} + O(kz_0)$

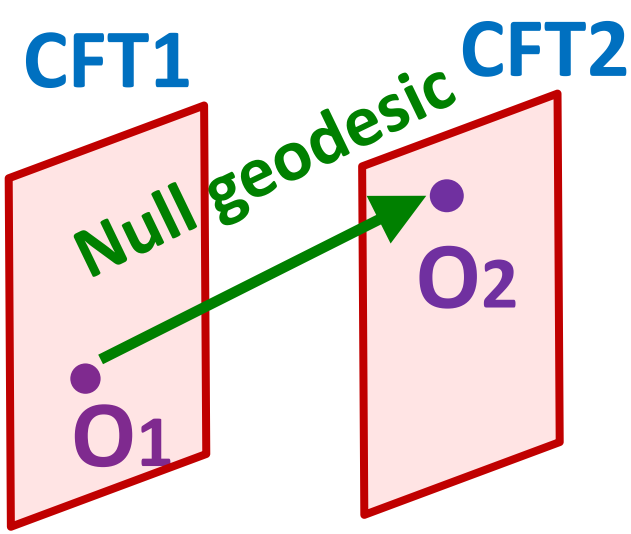
⟨O1O2⟩ $Q(\nu, k, z = z_0, d) \simeq \frac{2\nu}{d+2\nu}\frac{1}{z_0^{2\nu}} + O(kz_0).$

These results agree with the geodesic approximation when Δ>>1.

$$\langle \mathcal{O}_1(x_a)\mathcal{O}_1(x_b)\rangle \simeq e^{-\Delta D_{11}}, \quad \langle \mathcal{O}_1(x_a)\mathcal{O}_2(x_b)\rangle \simeq e^{-\Delta D_{12}},$$

In Lorentzian signature $x_0=it$, the two point function <O1O2> gets divergent at $-t^2+x^2+4z_0^2=0$ as the two points are null separated:

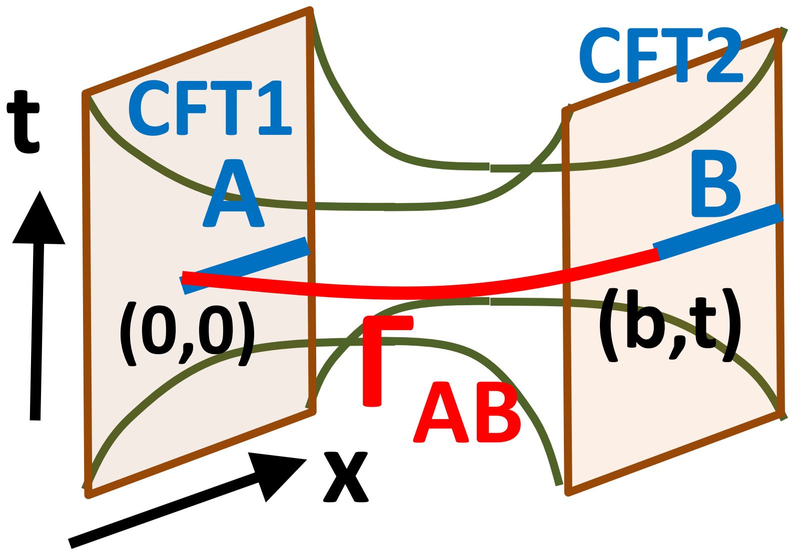
$$\langle \mathcal{O}_1(t,x)\mathcal{O}_2(0,0)\rangle \sim \frac{1}{(-t^2+x^2+4z_0^2)^{d+2\nu-\frac{1}{2}}}.$$



➡ A characteristic feature of traversable AdS black hole

Holographic Entanglement Entropy ?

How does S_{AB} look like ?



When $t^2 < b^2 + 4z_0^2$:

$$S_{AB} = \frac{c}{3}\log\frac{\frac{b^2-t^2}{4}+z_0^2}{\epsilon z_0}.$$

When $t^2 > b^2 + 4z_0^2$

$$S_{AB} = \frac{c}{3}\log\frac{\frac{t^2-b^2}{4}-z_0^2}{\epsilon z_0}+\frac{c}{3}\pi i.$$

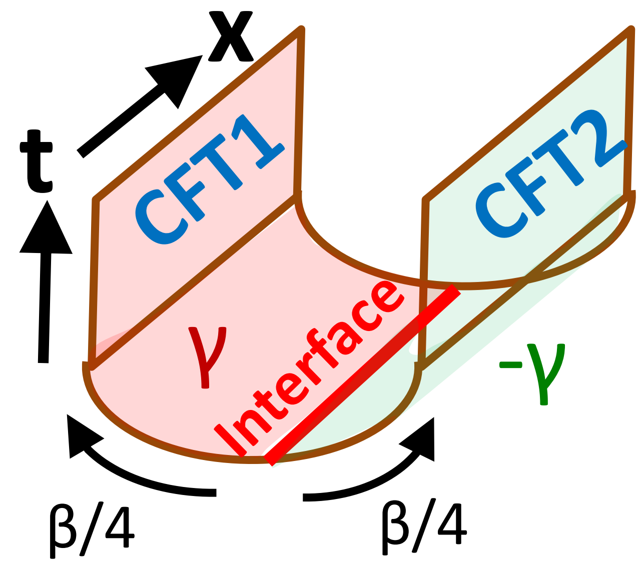
➡ S_{AB} becomes complex valued because $\rho^{\dagger}_{AB} \neq \rho_{AB}$.
Thus, S_{AB} should be regarded as pseudo entropy.

④ CFT dual via Janus deformation (Model A)

Janus deformation = asymmetric exactly marginal perturbations of doubled CFTs

$$S_{\mathrm{CFT1}} = S^{(0)}_{\mathrm{CFT}} + \gamma \int dx^d O_1(x)$$

$$S_{\mathrm{CFT2}} = S^{(0)}_{\mathrm{CFT}} - \gamma \int dx^d O_2(x)$$



◆We consider the TFD state of the doubled CFT for d=2.

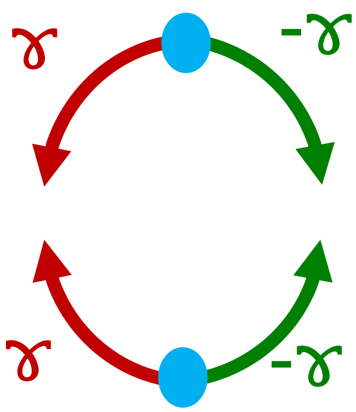
◆In the standard Janus deformation, γ is real valued.
We will extend γ to imaginary values.

When γ is real, the Janus deformed TFD state looks like

[Bak-Gutperle-Karch 07]

$$|TFD\rangle = \sum_n e^{-\frac{\beta(E_n^{(1)}+E_n^{(2)})}{4}} \left\langle E_n^{(1)},\gamma \middle| E_n^{(2)},-\gamma \right\rangle \; |E_n^{(1)},\gamma\rangle\, |E_n^{(2)},-\gamma\rangle$$

$$\langle TFD| = \sum_n e^{-\frac{\beta(E_n^{(1)}+E_n^{(2)})}{4}} \left\langle E_n^{(2)},-\gamma \middle| E_n^{(1)},\gamma \right\rangle \left\langle E_n^{(1)},\gamma \right| \left\langle E_n^{(2)},-\gamma \right|$$



The replica method leads to

$$\langle TFD|\sigma_n(a)\,\sigma_n(b)|TFD\rangle$$

Twist operator

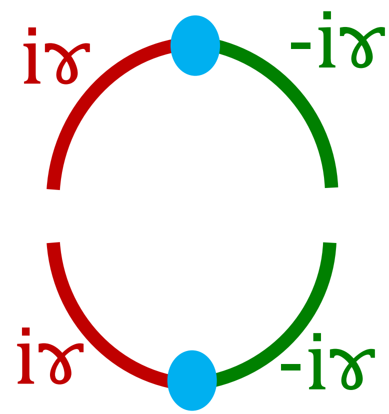
➡ We can compute the entanglement entropy.

When γ is imaginary, it is dual to an asymmetric TFD state:

$$|TFD\rangle = \sum_n e^{-\frac{\beta(E_n^{(1)}+E_n^{(2)})}{4}} \left\langle E_n^{(1)}, i\gamma \middle| E_n^{(2)}, -i\gamma \right\rangle \; |E_n^{(1)}, i\gamma\rangle\, |E_n^{(2)}, -i\gamma\rangle$$

$$\langle TFD'| = \sum_n e^{-\frac{\beta(E_n^{(1)}+E_n^{(2)})}{4}} \left\langle E_n^{(2)}, -i\gamma \middle| E_n^{(1)}, i\gamma \right\rangle \left\langle E_n^{(1)}, i\gamma \right| \left\langle E_n^{(2)}, -i\gamma \right|$$

Note: $(|TFD\rangle)^\dagger \equiv \langle TFD| \neq \langle TFD'|$.



The replica method leads to

$\langle TFD'|\sigma_n(a)\,\sigma_n(b)|TFD\rangle$ ➡ This gives pseudo entropy.

(→post selection)

Explicit construction from Janus deformation

We start with 3D Janus BH solutions in [Bak-Gutperle-Hirano 07].

The model is given by the 3d gravity action

$$I = \frac{1}{16\pi G_N}\int d^3x \left[R - g^{ab}\partial_a\phi\partial_b\phi + 2\right].$$

The solution ansatz looks like

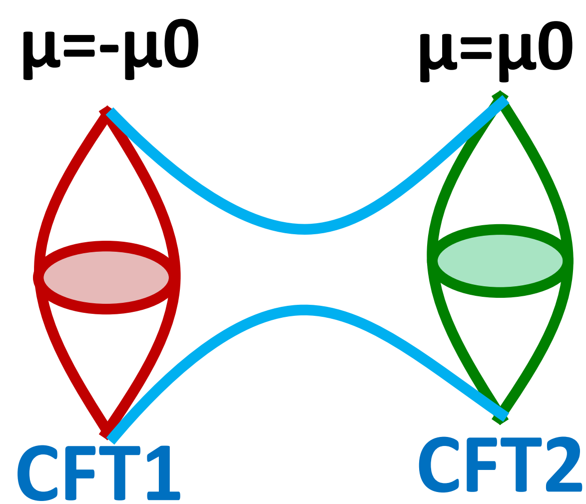
$$ds^2 = f(\mu)(d\mu^2 + ds^2_{AdS2}), \qquad \phi = \phi(\mu).$$

$$ds^2_{AdS2} = -d\tau^2 + r_0^2\cos^2\tau d\theta^2$$

γ is Janus deformation Parameter.

$$\frac{d\phi(\mu)}{d\mu} = \frac{\gamma}{\sqrt{f(\mu)}},$$

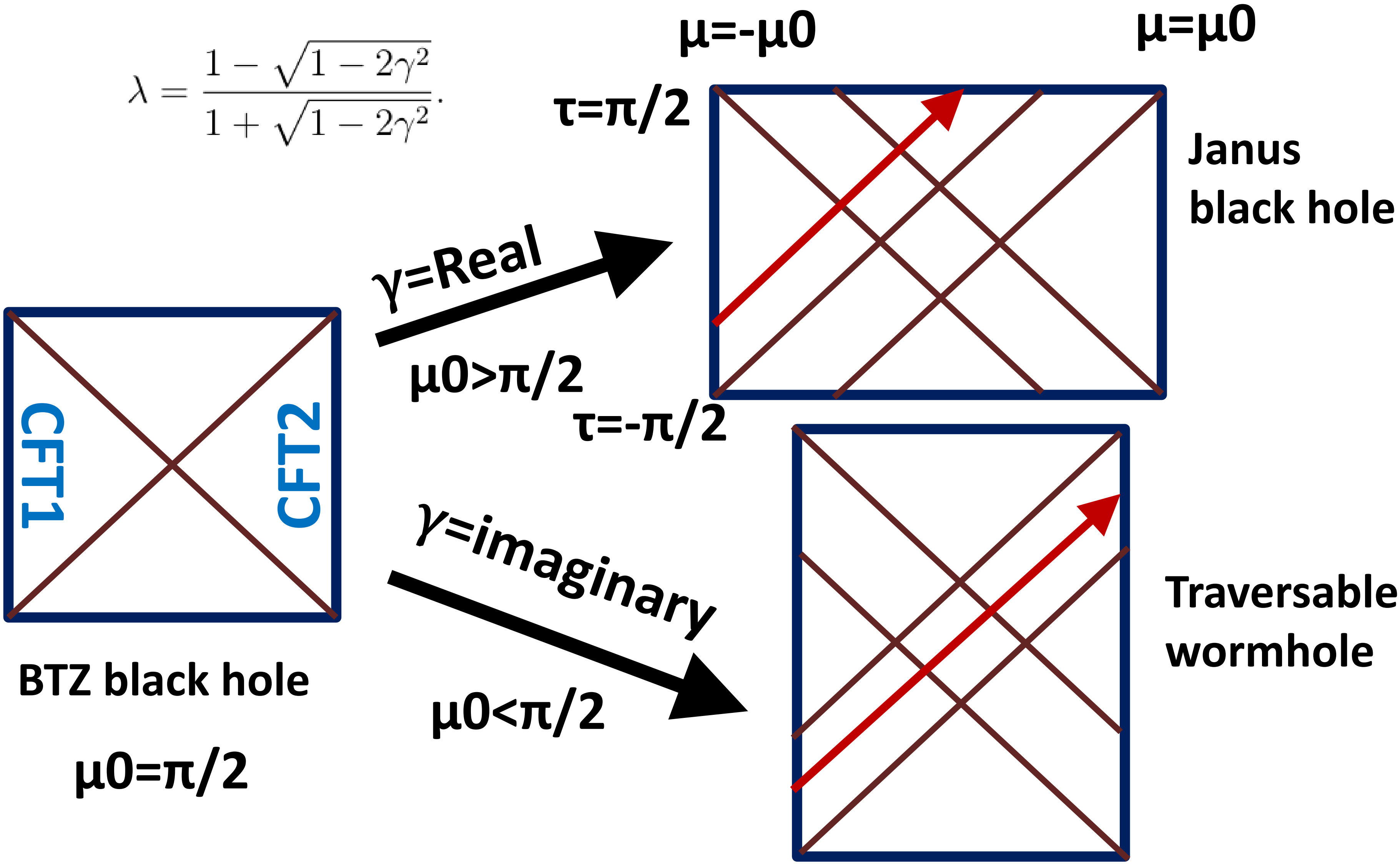
$$\frac{df(\mu)}{d\mu} = \sqrt{f(4f^2 - 4f + 2\gamma^2)}.$$



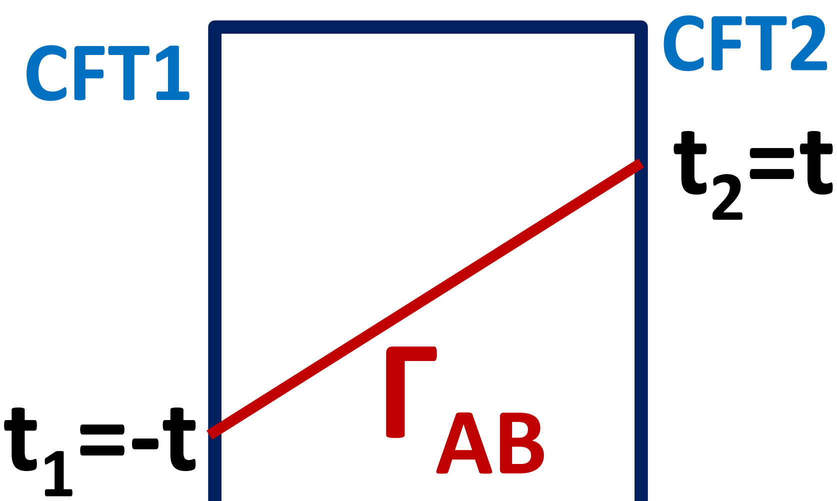
We now extend this solution to imaginary γ.

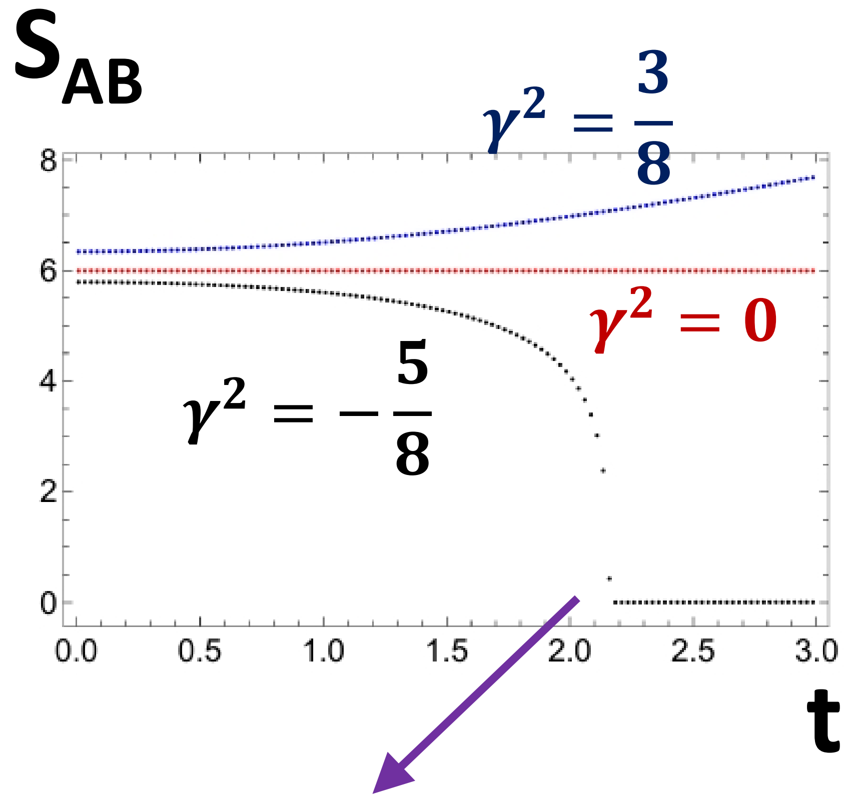
$$\mu_0 = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}},$$

$$\lambda = \frac{1-\sqrt{1-2\gamma^2}}{1+\sqrt{1-2\gamma^2}}.$$



Holographic pseudo entropy (=geodesic length)





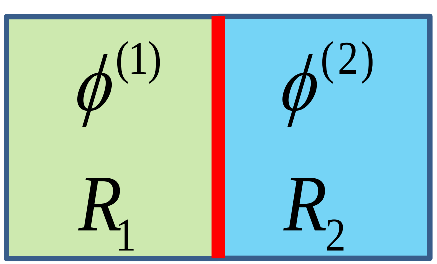
Γ_{AB} becomes light-like !
The characteristic feature of traversable wormhole.

A toy model of CFT dual

For a realization of AdS3/CFT2 Janus solution, consider AdS3 × S3 × 4 in IIB string theory, dual to the D1-D5 CFT given by the symmetric product CFT: $\mathrm{Sym}\left[(\mathrm{T}^4)^{Q_1 Q_5}\right]$.

The Janus deformation is performed by shifting the compactification radius R➔R1 in CFT1 and R➔R2 in CFT2.

Below we consider a toy model of Janus CFT based on the c=1 free compactified scalar ϕ (radius R).



$$\tan\theta = \frac{R_2}{R_1}.$$

Janus deformation

$$\theta = \frac{\pi}{4} + \gamma.$$

To probe its dual “geometry”, compute the two point function <V1V2>

$$V_1 = e^{i\lambda_+\phi_L^{(1)}(\tau_1)+i\lambda_-\phi_R^{(1)}(\tau_1)}, \qquad V_2 = e^{i\mu_+\phi_L^{(2)}(\tau_2)+i\mu_-\phi_R^{(2)}(\tau_2)},$$

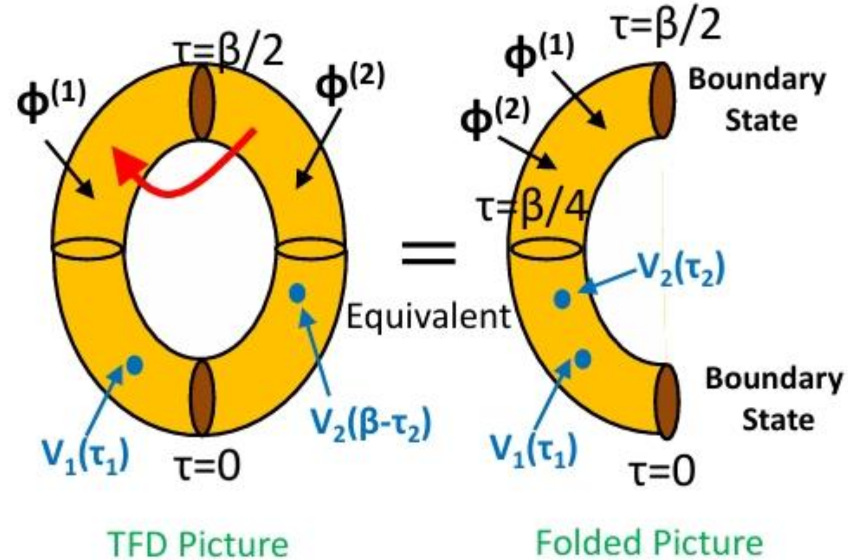
In the high temperature limit,

$$\lambda_\pm = \frac{n}{R_1} \pm \frac{wR_1}{2}, \qquad \mu_\pm = \frac{n}{R_2} \mp \frac{wR_2}{2}.$$

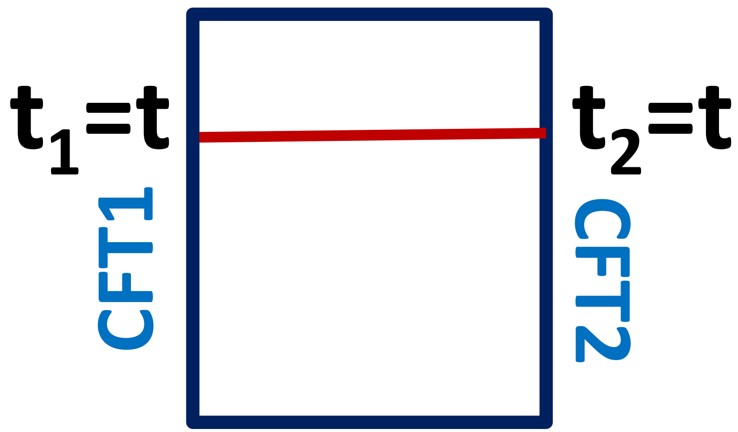
$$\langle V_1(\tau_1)V_2(\tau_2)\rangle$$
$$\simeq \left[\frac{\beta}{\pi}\cdot\sin\left(\frac{2\pi\tau_1}{\beta}\right)\right]^{\left[\left(\frac{n}{R_1}\right)^2-\left(\frac{wR_1}{2}\right)^2\right]\cos 2\theta} \cdot \left[\frac{\beta}{\pi}\cdot\sin\left(\frac{2\pi\tau_2}{\beta}\right)\right]^{\left[-\left(\frac{n}{R_2}\right)^2+\left(\frac{wR_2}{2}\right)^2\right]\cos 2\theta}$$
$$\cdot\left[\frac{\beta}{\pi}\cdot\sin\left(\frac{\pi(\tau_1+\tau_2)}{\beta}\right)\right]^{-2\left[\frac{n^2}{R_1R_2}+\frac{w^2R_1R_2}{4}\right]\sin 2\theta}$$

To evaluate the two point function, we employed the doubling trick of interface CFT.

[Bachas-de Boer-Dijkgraaf-Ooguri 2001, Sakai-Saoth 2008]

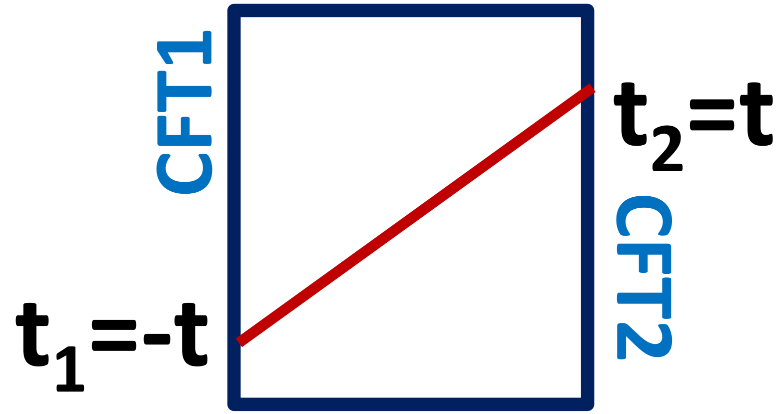


Case 1 $\tau_1 = \frac{\beta}{4} + it, \quad \tau_2 = \frac{\beta}{4} + it$



$$\langle V_1(\tau_1) V_2(\tau_2) \rangle \propto \left[\frac{\beta}{\pi} \cdot \cosh \frac{2\pi}{\beta} t \right]^{-\Delta_1 - \Delta_2}$$

Case 2 $\tau_1 = \frac{\beta}{4} + it, \quad \tau_2 = \frac{\beta}{4} - it$



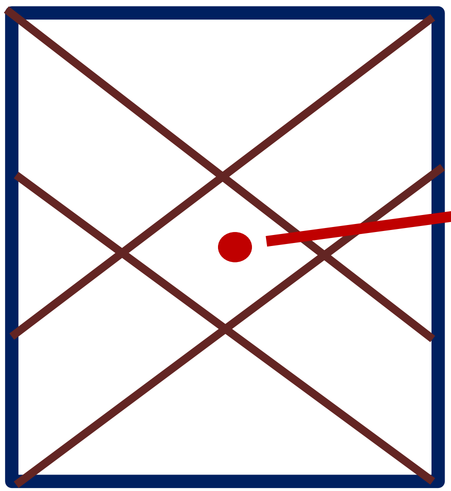
$$\langle V_1(t_1) V_2(t_2) \rangle \propto \left[\frac{\beta}{\pi} \cdot \cosh \frac{2\pi}{\beta} t \right]^{\eta}$$

$$\eta = -\frac{(R_1^2 - R_2^2)^2}{(R_1^2 + R_2^2) R_1 R_2} \cdot \left(\frac{n^2}{R_1 R_2} + \frac{w^2 R_1 R_2}{4} \right)$$

$\eta<0$ for real γ
$\eta>0$ for imaginary γ

Qualitatively agree with the gravity dual

Entanglement entropy between A=CFT1 and B=CFT2



$$S_A = S_B = \frac{2\pi r_0 f(0)}{4G_N} = \frac{\pi}{6} c r_0 \left(1 + \sqrt{1-\gamma^2}\right)$$

Area of throat

Monotonically decreasing function of γ^2

In the dual CFT , this corresponds to the EE in the deformed TFD state:

$$|\mathrm{TFD}(\beta,\gamma)\rangle = \tilde{\mathcal{N}} \exp\left[\sum_{i=1}^{\infty} e^{-\frac{\beta}{2}E_i} \left(\sin 2\theta \; a_i^\dagger b_i^\dagger + \cos 2\theta \left((a_i^\dagger)^2 - (b_i^\dagger)^2\right)\right)\right] |0\rangle$$

$$\langle \mathrm{TFD}(\beta,\gamma)| = \tilde{\mathcal{N}} \langle 0| \exp\left[\sum_{i=1}^{\infty} e^{-\frac{\beta}{2}E_i} \left(\sin 2\theta \; a_i b_i + \cos 2\theta \left((a_i)^2 - (b_i)^2\right)\right)\right]$$

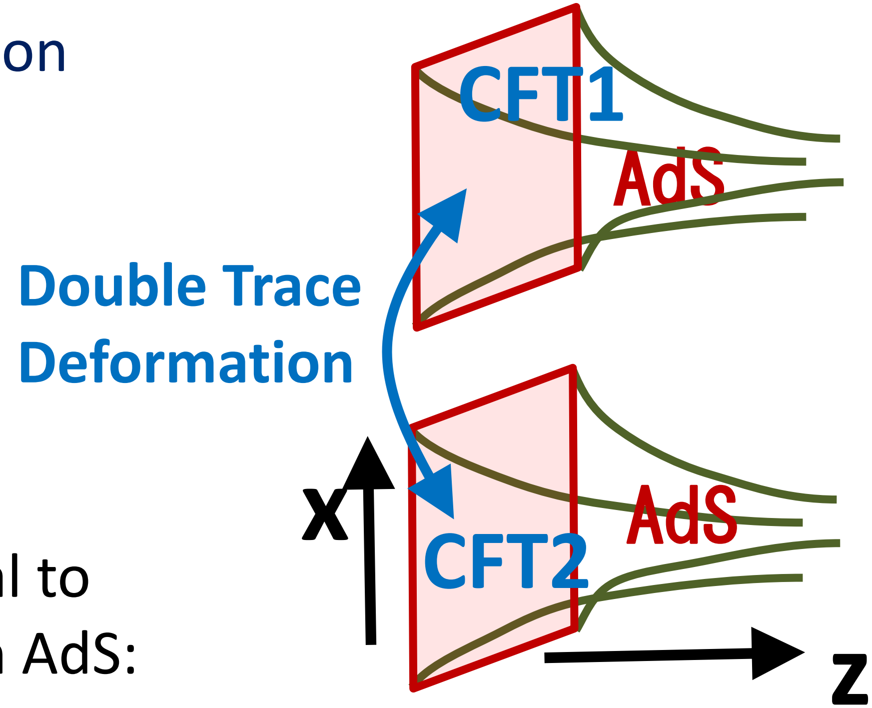
S_A becomes its maximum at θ=π/4 (i.e. undeformed) and decreases as γ^2 gets larger. This is consistent with the gravity dual.

⑤ CFT dual via double trace deformation (Model B)

Consider a double trace deformation
between CFT1 and CFT2

$$\int dxdy\lambda(x,y)O_1(x)O_2(y)$$

$$\lambda(x,y) = \int d^d k e^{ik(x-y)}\lambda(k)$$



The double trace deformation is dual to
the change of boundary condition in AdS:

$$J^{(1)} = \alpha^{(1)} - \lambda\beta^{(2)}, \quad J^{(2)} = \alpha^{(2)} - \lambda\beta^{(1)}$$ [Witten 2001]

Here the scalar field in each AdS is expanded as follows:

$$\Phi^{(i)} \simeq \alpha^{(i)} z_i^{d-\Delta} + \beta^{(i)} z_i^{\Delta} \quad (z_1, z_2 \to 0)$$

$$\frac{\beta^{(i)}}{\alpha^{(i)}} = -G(k), \quad G_p(k) \equiv \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu}$$

In this way we can compute the two point functions:

$$\langle \mathcal{O}_1(k)\mathcal{O}_1(-k)\rangle = \langle \mathcal{O}_2(k)\mathcal{O}_2(-k)\rangle = \frac{G}{1-\lambda^2 G^2},$$
$$\langle \mathcal{O}_1(k)\mathcal{O}_2(-k)\rangle = \frac{\lambda G^2}{1-\lambda^2 G^2}.$$

Two point functions in the simple model of traversable WH is reproduced by setting

$$G(k) = \frac{P(k)^2 - Q(k)^2}{P(k)} = \begin{cases} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu} & (kz_0 \gg 1) \\ \frac{d^2-4\nu^2}{(d+2\nu)d}\cdot\frac{1}{z_0^{2\nu}} & (kz_0 \ll 1). \end{cases}$$

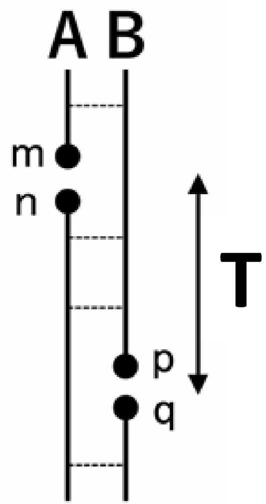
$$\lambda(k) = \frac{Q(k)}{P(k)^2 - Q(k)^2} = \begin{cases} \frac{2\sin\pi\nu\Gamma(1+\nu)}{\Gamma(1-\nu)}\left(\frac{k}{2}\right)^{-2\nu} e^{-2kz_0} & ((kz_0 \gg 1) \\ \frac{2(d+2\nu)\nu}{d^2-4\nu^2}\cdot z_0^{2\nu} & (kz_0 \ll 1). \end{cases}$$

UV regularized DT deformation

Note: In order to reproduce two point functions for all operators, we need to perform the double trace deformations for all primaries.

Quantum info. aspect: a toy model of coupled harmonic oscillators

$$(\rho_{AB})^{m\,p}_{n\,q} =$$



$$H = \frac{1}{\sqrt{1-\lambda^2}}\left[a^\dagger a + b^\dagger b + \lambda(a^\dagger b^\dagger + ab) + 1 - \sqrt{1-\lambda^2}\right].$$

Or equally,

$$H = \frac{1}{2}(p_1^2 + x_1^2) + \frac{1}{2}(p_2^2 + x_2^2) + \lambda(x_1 x_2 - p_1 p_2)$$

$$[\rho_{AB}]^{a_2,b_2}_{a_1,b_1} = \langle\Psi_0|_{12}\cdot(|b_2\rangle\langle b_1|)_2\cdot\mathcal{P}e^{-i\int_{t_1}^{t_2}dtH_{12}(t)}\cdot(|a_2\rangle\langle a_1|)_1\cdot|\Psi_0\rangle_{12},$$

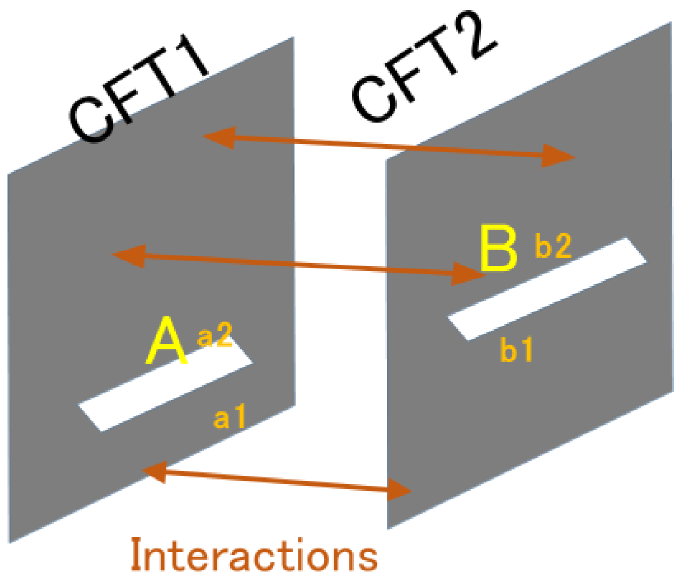
$$\begin{aligned}[\rho_{AB}]^{mp}_{np} &= \langle\Psi||m\rangle_A\langle n|e^{-iHT}|p\rangle_B\langle q||\Psi\rangle \\ &= \frac{1}{\cosh^2\theta}(-\tanh\theta)^{m+q}\langle n|_A\langle m|_B e^{-iHT}|q\rangle_A|p\rangle_B.\end{aligned}$$

$$\rho_{AB}^\dagger \neq \rho_{AB}$$

$$S^{(2)}_{AB} = \log\left[\frac{1+e^{-2iT}+(1-e^{-2iT})\cosh 4\theta}{2}\right].$$

$$S(\rho_{AB}) \neq 0$$

$$\rho_{AB} = \text{mixed}$$

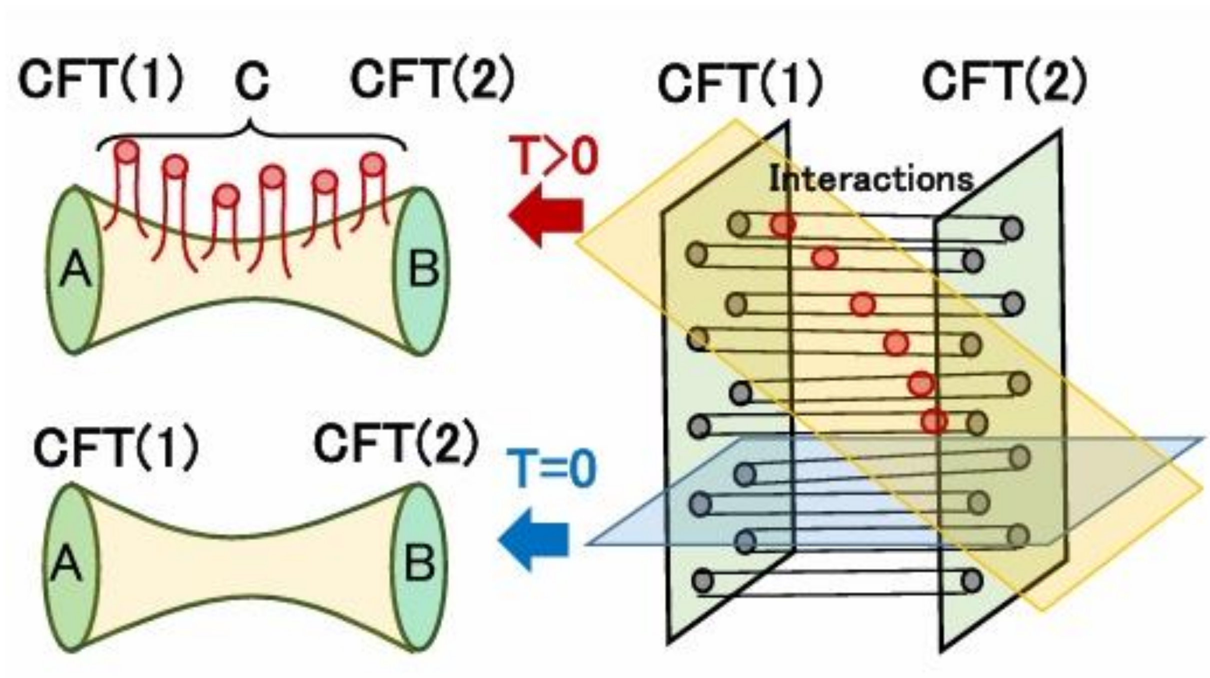


Indeed, we can easily find

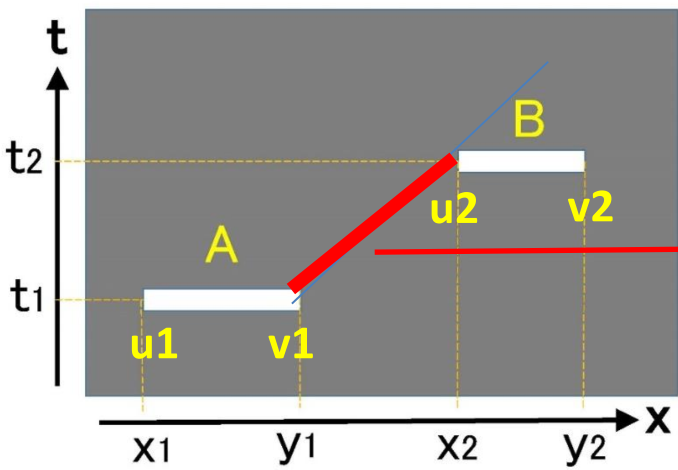
$$H_{tot} \neq H_{CFT1} \otimes H_{CFT2}$$

because A and B are causally connected.

$$S(\rho_{AB}) \neq 0$$



This is analogous to the following setup in a single CFT:



e.g. Free Dirac fermion CFT c=1

$$S_{AB} = \frac{c}{6}\log\frac{|v_1-u_1|^2}{\epsilon^2} + \frac{c}{6}\log\frac{|v_2-u_2|^2}{\epsilon^2} + \frac{c}{6}\log\frac{|v_1-u_2|^2}{\epsilon^2} + \frac{c}{6}\log\frac{|v_2-u_1|^2}{\epsilon^2} - \frac{c}{6}\log\frac{|u_1-u_2|^2}{\epsilon^2} - \frac{c}{6}\log\frac{|v_1-v_2|^2}{\epsilon^2}.$$

If this interval is time-like, entropy gets complex valued !

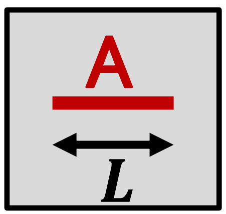
[See also Kusuki-Umemoto-TT 2017]

Cf. Time-like Entanglement Entropy in AdS/CFT

[Doi–Harper–Mollabashi–Taki–TT 22, 23,
Heller–Ori–Sereantes 23, Milekhin–Adamska–Preskill 25]

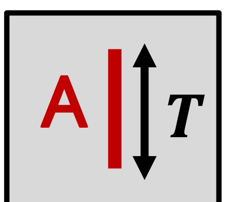
Consider a time–like version of entanglement entropy by rotating the subsystem A into a time–like one:

CFT on an infinite line



$$S_A = \frac{C}{3}\log\left[\frac{L}{\varepsilon}\right]$$

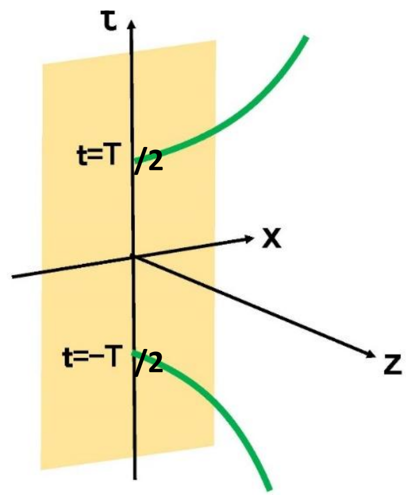
$$L \to iT$$



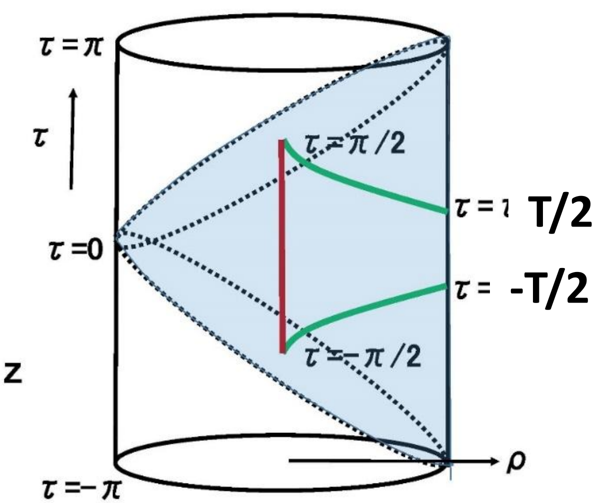
$$S_A = \frac{C}{3}\log\left[\frac{T}{\varepsilon}\right] + \frac{\pi}{6} i C_{CFT}$$

Holographic calculation

Poincare AdS



Global AdS



⑥ Conclusions

Pseudo entropy (PE) is a generalization of entanglement entropy.

- PE depends on both the initial and final state.
- PE is in general complex valued.
- ΔS for two states in different phases can be positive, while ΔS in the same phase is always non-positive ➡ New order parameter
- In AdS/CFT, PE is equal to the minimal area in Euclidean time-dependent asymptotically AdS. ➡ Emergent space from PE
- Traversable wormholes in AdS can be probed by PE. We point out that there are two different models of the CFT dual.

(i) Model A (Imaginary Janus deformation)

$$|\psi_I\rangle \neq |\psi_F\rangle, \quad \text{no interaction}, \quad S(\rho_{AB}) = 0$$

(ii) Model B (double trace deformation)

$$|\psi_I\rangle = |\psi_F\rangle, \quad \exists\text{interaction}, \quad S(\rho_{AB}) \neq 0$$

Future directions

- Quantum information meaning of the complex values of PE ?
- Applications to non-Hermitian cond-mat physics ?
- Implications to quantum gravity ? Emergent time ?
- Constraints on QFTs using PE ?

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Thank you !